Weighted Orlicz Spaces in the Context of Double Coset Spaces

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Abstract

In this paper, we introduce a class of weighted Orlicz spaces in the context of double coset spaces related to locally compact hypergroups in some way, which one can study that either these spaces are convolution algebras.

Keywords: Weighted Orlicz; Double Coset Spaces

1. Introduction

Orlicz spaces are famous generalizations of Lebesgue spaces, which are so much applicable in several branches of mathematics. See [1] for more details. In the last decades so many researches have been done on these important spaces; see [2–8]. Recently, in [9] the authors introduced and studied the weighted Orlicz spaces in the context of locally compact groups, and they investigate the cases which this spaces is a convolution Banach algebra. Similar approach has been done for the weighted Orlicz spaces in the context of locally compact hypergroups [10]. In this paper, we introduce a class of weighted Orlicz spaces in the context of double coset spaces related to locally compact hypergroups in some way, which one can study that either these spaces are convolution algebras. Next, p_x denotes the point mass measure at the point x. Here, we recall some notions regarding locally compact hypergroups. For knowing more refer to [11, 12].

Definition 1.1. Assume that $\mathcal{H}$ be a locally compact Hausdorff topological space, and assume that $M(\mathcal{H})$ is the space of all Radon measures on $\mathcal{H}$. Assume that the mappings $\circledast: M(\mathcal{H}) \times M(\mathcal{H}) \to M(\mathcal{H})$ and $\theta: \mathcal{H} \to \mathcal{H}$ satisfy the following properties:
(1) $M(\mathcal{H})$ equipped with $\circledast$ is an algebra;
(2) θ is a homeomorphism such that $\theta(\theta(x)) = x$ for all $x \in \mathcal{H}$;
(3) $p_x \circledast p_y$ is a probability measure which its support is compact;
(4) the mapping $(x, y) \mapsto \mathrm{supp}(p_x \circledast p_y)$ is continuous, where the family of all compact subsets of $\mathcal{H}$ is equipped with the Michael topology;

(5) $\theta(p_x \circledast p_y) = p_{\theta(y)} \circledast p_{\theta(x)}$ for all $x, y \in \mathcal{H}$. This menas that for every compact supported continuous function $f: \mathcal{H} \to \mathbb{C}$,

$$\int_{\mathcal{H}} f(t) d(p_{\theta(y)} \circledast p_{\theta(x)})(t) = \int_{\mathcal{H}} f(\theta(t)) d(p_x \circledast p_y)(t)$$

(6) for every $x, y \in \mathcal{H}, p_x \circledast p_y = p_e$ if and only if $y = \theta(x)$.
(7) The mapping $\mathcal{H} \times \mathcal{H} \to M^+(\mathcal{H}), (x, y) \to p_x \circledast p_y$ is continuous, where $M^+(\mathcal{H})$ is equipped with the cone topology.
Then, $\mathcal{H}$ equipped with the above two mappings is called a locally compact hypergroup.
Example 1.2. Assume that $\mathcal{H} := [0, \infty)$ equipped with the Euclidean topology. For every $x, y \in [0, \infty$ define

$$p_x \circledast p_y := \frac{1}{2}(p_{|x-y|} + p_{x+y}).$$

Then, $[0, \infty)$ equipped with this operator is a locally compact hypergroup.
Next, $\mathcal{H}$ is a locally compact hypergroup with the involution θ and convolution $\circledast$.
Definition 1.3. For every $A, B, C \subseteq \mathcal{H}$ we define

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$$A \circledast B \circledast C := \bigcup \left\{ \mathrm{supp}\left(p_x \circledast \left(p_y \circledast p_z\right)\right) : x \in A, y \in B, z \in C \right\}$$

An element $x \in \mathcal{H}$ is called a center element while $\mathrm{supp}(p_x \circledast p_{\theta(x)}) = \mathrm{supp}(p_{\theta(x)} \circledast p_x) = \{e\}$. The set of all center elements of $\mathcal{H}$ is denoted by $C(\mathcal{H})$.

Note that for every $t \in C(\mathcal{H})$ and $x \in \mathcal{H}$, the supportssupp$(p_x \circledast p_t)$ and $\mathrm{supp}(p_t \circledast p_x)$ are singletons. **Definition 1.4.** An element $x \in \mathcal{H}$ is called a commuting element while for every $y \in \mathcal{H}$,

$$\mathrm{supp}(p_x \circledast p_y) = \mathrm{supp}(p_y \circledast p_x)$$

The set of all commuting elements of $\mathcal{H}$ is denoted by $Z(\mathcal{H})$. **Definition 1.5.** Let $\mathcal{S}, \mathcal{C}$ be closed subhypergroups of $\mathcal{H}$. Then, we define

$$\mathcal{S} \setminus \mathcal{H}/\mathcal{C} := \{\mathcal{S} \circledast \{t\} \circledast \mathcal{C} : t \in \mathcal{H}\}$$

In this case, $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$ is called a double coset space. Note that in the case that $\mathcal{H} := G$ is a locally compact group,

$$Z(G) := \{x \in G : xy = yx \text{ for all } y \in G\}$$

For every $E \subseteq \mathcal{S} \setminus \mathcal{H}/\mathcal{C}$, we define that: E is open whenever the set

$$\{t \in \mathcal{H} : \mathcal{S} \circledast \{t\} \circledast \mathcal{C} \in E\}$$

is an open subset of $\mathcal{H}$. This topology is called the quotient topology. Recall that there is a modular function $\Delta_{\mathcal{H}} : \mathcal{H} \to (0, \infty)$ for $\mathcal{H}$. Note that $\mathcal{H}$ is called unimodular while $\Delta_{\mathcal{H}}(x) = 1$ for all $x \in \mathcal{H}$.

2. Double Coset Spaces

In this section, we study some properties of some classes of double coset spaces related to locally compact hypergroups.

Theorem 2.1. Assume that $\mathcal{S}$ is a subgroup of $C(\mathcal{H}) \cap Z(\mathcal{H})$, and $\mathcal{C}$ is a compact subhypergroup of $\mathcal{H}$. Then, the mapping

$$C(\mathcal{H}) \times \mathcal{S} \setminus \mathcal{H}/\mathcal{C}, (a, \mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \mapsto \mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C}$$

is an action of $C(\mathcal{H})$ on $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$.

Proof. First, note that this mapping is well-defined. For this, consider elements $a \in C(\mathcal{H})$ and $x, y \in \mathcal{H}$, and assume that $\mathcal{S} \circledast \{x\} \circledast \mathcal{C} = \mathcal{S} \circledast \{y\} \circledast \mathcal{C}$. Since $\mathcal{S}, \mathcal{C}$ contain the identity element e, we have $x \in \mathcal{S} \circledast \{y\} \circledast \mathcal{C}$. Hence, there are $t \in \mathcal{S}$ and $s \in \mathcal{C}$ such that $x \in \{t\} \circledast \{y\} \circledast \{s\}$. Thus, since $t \in \mathcal{S} \subseteq Z(\mathcal{H})$,

$$\begin{aligned} \mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C} &\subseteq \mathcal{S} \circledast (\{a\} \circledast \{t\} \circledast \{y\} \circledast \{s\}) \circledast \mathcal{C} \\ &= (\mathcal{S} \circledast \{t\}) \circledast \{a\} \circledast \{y\} \circledast (\{s\} \circledast \mathcal{C}) \\ &\subseteq \mathcal{S} \circledast (\{a\} \circledast \{y\}) \circledast \mathcal{C}, \end{aligned}$$

so,

$$\mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C} \subseteq \mathcal{S} \circledast (\{a\} \circledast \{y\}) \circledast \mathcal{C} \tag{2.1}$$

Similarly, there are elements $u \in \mathcal{S}$ and $v \in \mathcal{C}$ such that $y \in \{u\} \circledast \{x\} \circledast \{v\}$, then

$$\begin{aligned} \mathcal{S} \circledast (\{a\} \circledast \{y\}) \circledast \mathcal{C} &\subseteq \mathcal{S} \circledast (\{a\} \circledast \{u\} \circledast \{x\} \circledast \{v\}) \circledast \mathcal{C} \\ &= (\mathcal{S} \circledast \{u\}) \circledast \{a\} \circledast \{x\} \circledast (\{v\} \circledast \mathcal{C}) \\ &\subseteq \mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C} \end{aligned}$$

thus,

$$\mathcal{S} \circledast (\{a\} \circledast \{y\}) \circledast \mathcal{C} \subseteq \mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C}. \tag{2.2}$$

By (2.1) and (2.2) we have

$$\mathcal{S} \circledast (\{a\} \circledast \{x\}) \circledast \mathcal{C} = \mathcal{S} \circledast (\{a\} \circledast \{y\}) \circledast \mathcal{C} \tag{2.3}$$

and so the mapping is well-defined. For associativity, assume that $a, b \in C(\mathcal{H})$, and $x \in \mathcal{H}$. Since a, b are center elements, the supportssupp$(p_a \circledast p_b)$ and $\mathrm{supp}(p_b \circledast p_x)$ are singletons. We denote $\mathrm{supp}(p_a \circledast p_b) = \{ab\}$ and $\mathrm{supp}(p_b \circledast p_x) = \{bx\}$. Then,

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$$\begin{aligned} ab \cdot \mathcal{S} \circledast \{x\} \circledast \mathcal{C} &= \mathcal{S} \circledast (\{ab\} \circledast \{x\}) \circledast \mathcal{C} \\ &= \mathcal{S} \circledast (\{a\} \circledast \{b\} \circledast \{x\}) \circledast \mathcal{C} \\ &= \mathcal{S} \circledast (\{a\} \circledast \{bx\}) \circledast \mathcal{C} \\ &= a \cdot \mathcal{S} \circledast \{bx\} \circledast \mathcal{C} \\ &= a \cdot [\mathcal{S} \circledast \{b\} \circledast \{x\} \circledast \mathcal{C}] \\ &= a \cdot [b \cdot \mathcal{S} \circledast \{x\} \circledast \mathcal{C}] \end{aligned}$$

For continuity, assume that $(a_\alpha)_\alpha \subseteq C(\mathcal{H}), a \in C(\mathcal{H}), (x_\alpha)_\alpha \subseteq \mathcal{H}$ and $x \in \mathcal{H}$ such that $a_\alpha \to a$ in $C(\mathcal{H})$, and $x_\alpha \to x$ in $\mathcal{H}$. Then, with Michael topology we have $a_\alpha \cdot \mathcal{S} \circledast \{x_\alpha\} \circledast \mathcal{C} = \mathcal{S} \circledast \{a_\alpha\} \circledast \{x_\alpha\} \circledast \mathcal{C} \to \mathcal{S} \circledast \{a\} \circledast \{x\} \circledast \mathcal{C} = a \cdot \mathcal{S} \circledast \{x\} \circledast \mathcal{C}$.

Theorem 2.2. Assume that $\mathcal{H}$ is a unimodular locally compact hypergroup with the left Haar measure $\lambda, \mathcal{S}$ is a subgroup of $C(\mathcal{H}) \cap Z(\mathcal{H})$ with a Haar measure α, and $\mathcal{C}$ is a compact subhypergroup of $\mathcal{H}$ with a normalized left Haar measure β. Then, there exists a measure $\mu \in M(\mathcal{S} \setminus \mathcal{H}/\mathcal{C})$ satisfying

$$\int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} f(t) d(p_a \circledast p_x \circledast p_b)(t) d\alpha(a) d\beta(b) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) = \int_{\mathcal{H}} f(t) d\lambda(t). \tag{2.4}$$

Proof. Let $\eta \in C_c(\mathcal{S} \setminus \mathcal{H}/\mathcal{C})$. Then, $\mathrm{supp}(\eta)$ is a compact subset of $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$. There exists a continuous compact supported map $h: \mathcal{H} \to [0, \infty)$ such that for each $\mathcal{S} \circledast \{x\} \circledast \mathcal{C} \in \mathrm{supp}(\eta)$,

$$\int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} f(t) d(p_a \circledast p_x \circledast p_b)(t) d\alpha(a) d\beta(b) = 1.$$

Now, define a function $g \in C_c(\mathcal{H})$ by

$$g(x) := h(x) \eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}), (x \in \mathcal{H})$$

In this case,

$$\eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) = \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} g(t) d(p_a \circledast p_x \circledast p_b)(t) d\alpha(a) d\beta(b), (x \in \mathcal{H}).$$

Now, we define

$$\mu(\eta) := \int_{\mathcal{H}} g(t) d\lambda(t).$$

If also $j \in C_c(\mathcal{H})$ and

$$\eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) = \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} j(t) d(p_a \circledast p_x \circledast p_b)(t) d\alpha(a) d\beta(b), (x \in \mathcal{H})$$

then,

$$\begin{aligned} \int_{\mathcal{H}} j(x) \eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) d\lambda(x) &= \int_{\mathcal{H}} \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} j(x) g(t) d(p_a \circledast p_x \circledast p_b)(t) d\alpha(a) d\beta(b) d\lambda(x) \\ &= \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} \int_{\mathcal{H}} j(x) g(t) d(p_a \circledast p_x \circledast p_b)(t) d\lambda(x) d\beta(b) d\alpha(a) \\ &= \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} j(x) g(a \circledast x \circledast b) d\lambda(x) d\beta(b) d\alpha(a) \\ &= \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} j(\theta(a) \circledast x \circledast \theta(b)) g(x) d\lambda(x) d\beta(b) d\alpha(a) \\ &= \int_{\mathcal{S}} \int_{\mathcal{C}} \int_{\mathcal{H}} j(a \circledast x \circledast b) g(x) d\lambda(x) d\beta(b) d\alpha(a). \end{aligned}$$

On the other hand, since $\mathcal{S}, \mathcal{H}, \mathcal{C}$ are unimodular,

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$$\begin{aligned}\int_{\mathcal{H}} g(x)\eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})d\lambda(x) &= \int_{\mathcal{H}}\int_{\mathcal{S}}\int_{\mathcal{C}}\int_{\mathcal{H}} g(x)j(t)d(p_a \circledast p_x \circledast p_b)(t)d\alpha(a)d\beta(b)d\lambda(x)\\ &= \int_{\mathcal{S}}\int_{\mathcal{C}}\int_{\mathcal{H}}\int_{\mathcal{H}} g(x)j(t)d(p_a \circledast p_x \circledast p_b)(t)d\lambda(x)d\beta(b)d\alpha(a)\\ &= \int_{\mathcal{S}}\int_{\mathcal{C}}\int_{\mathcal{H}} g(x)j(a \circledast x \circledast b)d\lambda(x)d\beta(b)d\alpha(a)\\ &= \int_{\mathcal{S}}\int_{\mathcal{C}}\int_{\mathcal{H}} j(\theta(a) \circledast x \circledast \theta(b))g(x)d\lambda(x)d\beta(b)d\alpha(a).\end{aligned}$$

This implies that

$$\int_{\mathcal{H}} j(x)\eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})d\lambda(x) = \int_{\mathcal{H}} g(x)\eta(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})d\lambda(x). \tag{2.5}$$

By this relation we obtain that

$$\int_{\mathcal{H}} g(t)d\lambda(t) = \int_{\mathcal{H}} j(t)d\lambda(t),$$

so, μ is linear and well-defined. Finally, note that

$$\begin{aligned}|\mu(\eta)| &= \left|\int_{\mathcal{H}} g(t)d\lambda(t)\right|\\ &\leq \int_{\mathcal{H}} |g(t)|d\lambda(t)\\ &\leq \int_{D} |g(t)|d\lambda(t)\\ &\leq \int_{D} \|g\|_{\sup} d\lambda(t)\\ &= \lambda(D)\|g\|_{\sup} \leq C\|\eta\|_{\sup}\end{aligned}$$

where $C > 0$ is a constant number, and g is supported on a compact set D. This completes the proof.

Setting $\mathcal{S} := \{e\}$ in the above theorem, we can conclude the following fact.

Corollary 2.3. Assume that $\mathcal{H}$ is a unimodular locally compact hypergroup with the left Haar measure λ, and $\mathcal{C}$ is a compact subhypergroup of $\mathcal{H}$ with a normalized left Haar measure β. Then, there exists a measure $\mu \in M(\mathcal{H}/\mathcal{C})$ satisfying

$$\int_{\mathcal{H}/\mathcal{C}}\int_{\mathcal{C}}\int_{\mathcal{H}} f(t)d(p_x \circledast p_b)(t)d\beta(b)d\mu(\{x\} \circledast \mathcal{C}) = \int_{\mathcal{H}} f(t)d\lambda(t). \tag{2.6}$$

Corollary 2.4. Assume that G is a unimodular locally compact group with the left Haar measure λ, and $\mathcal{C}$ is a compact subgroup of $\mathcal{H}$ with a normalized left Haar measure β. Then, there exists a measure $\mu \in M(\mathcal{H}/\mathcal{C})$ satisfying

$$\int_{G/\mathcal{C}}\int_{\mathcal{C}} f(xb)d\beta(b)d\mu(x\mathcal{C}) = \int_{\mathcal{H}} f(t)d\lambda(t) \tag{2.7}$$

Corollary 2.5. Assume that $\mathcal{H}$ is a unimodular locally compact hypergroup with the left Haar measure $\lambda, \mathcal{S}$ is a subgroup of $C(\mathcal{H}) \cap Z(\mathcal{H})$ with a Haar measure α, and $\mathcal{C}$ is a compact subhypergroup of $\mathcal{H}$ with a normalized left Haar measure β. Then, there exists a measure $\mu \in M(\mathcal{S} \setminus \mathcal{H}/\mathcal{C})$ such that for every $a \in C(\mathcal{H})$ and every Borel $E \subseteq \mathcal{S} \setminus \mathcal{H}/\mathcal{C}$,

$$\mu(E) = \mu(\{\mathcal{S} \circledast \{ax\} \circledast \mathcal{C} : \mathcal{S} \circledast \{x\} \circledast \mathcal{C} \in E\}) \tag{2.8}$$

Corollary 2.6. Assume that G is a unimodular locally compact group, $\mathcal{S}$ is a subgroup of $Z(G)$, and $\mathcal{C}$ is a compact subgroup of G. Then, there exists a measure $\mu \in M(\mathcal{S} \setminus G/\mathcal{C})$ such that for every $a \in G$ and every Borel $E \subseteq \mathcal{S} \setminus G/\mathcal{C}$,

$$\mu(E) = \mu(\{\mathcal{S}ax\mathcal{C} : \mathcal{S}x\mathcal{C} \in E\}) \tag{2.9}$$

Definition 2.7. The measure given in the above corollary is called the $C(E)$ invariant measure on $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$.

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In the sequel, $\mathcal{H}$ is a unimodular locally compact hypergroup with the left Haar measure $\lambda, \mathcal{S}$ is a subgroup of $C(\mathcal{H}) \cap Z(\mathcal{H})$ with a Haar measure α, and $\mathcal{C}$ is a compact subhypergroup of $\mathcal{H}$ with a normalized left Haar measure β. Also, we assume that μ is the $C(E)$-invariant measure on $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$.

3. Weighted Orlicz Spaces on Double Coset Spaces

Definition 3.1. A continuous convex function $\Phi: [0, \infty) \to \mathbb{R}$ is called a nice function whenever $\lim_{a\to 0} \frac{\Phi(a)}{a} = 0$ and $\lim_{a\to\infty} \frac{\Phi(a)}{a} = \infty$, and $\Phi(x) = 0$ if and only if $x = 0$.

In this section, always Φ is a nice function, and ϕ is its complementary. Also, $u, v: \mathcal{S} \setminus \mathcal{H}/\mathcal{C} \to (0, \infty)$ are continuous.

Definition 3.2. The set of all Borel subsets K of $\mathcal{S} \setminus \mathcal{H}/\mathcal{C}$ such that $\mu(K)$ is finite, is denoted by $F(\mathcal{S} \setminus \mathcal{H}/\mathcal{C})$.

Next, we assume that the supremum

$$\sup_{y\in C(\mathcal{H}), K\in F(\mathcal{S}\setminus\mathcal{H}/\mathcal{C})} \int_K u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

is finite.

Example 3.3. Assume that $u, v: \mathcal{S} \setminus \mathcal{H}/\mathcal{C} \to (0, \infty)$ are measurable functions such that

$$C := \int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) < \infty$$

Then, since μ is a $C(\mathcal{H})$-invariant measure, for every $y \in C(\mathcal{H})$ and $K \in F(\mathcal{H})$,

$$\begin{aligned}
&\int_K u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \\
&\leq \int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \\
&= \int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{\theta(y)\} \circledast \{x\} \circledast \mathcal{C}) \\
&= \int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) = C,
\end{aligned}$$

So,

$$\sup_{y\in C(\mathcal{H}), K\in F(\mathcal{S}\setminus\mathcal{H}/\mathcal{C})} \int_K u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \leq C < \infty.$$

Lemma 3.4. By the above assumption, there exists some measurable function $f: \mathcal{S} \setminus \mathcal{H}/\mathcal{C} \to \mathbb{R}$ such that

$$\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\setminus\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \phi(|f(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})|) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \leq 1.$$

Proof. Denoting
$\Lambda := \sup_{y\in C(\mathcal{H}), K\in F(\mathcal{S}\setminus\mathcal{H}/\mathcal{C})} \int_K u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$,
we have $0 < \Lambda < \infty$. Fix some set $K_0 \in F(\mathcal{S} \setminus \mathcal{H}/\mathcal{C})$. Define the function $f: \mathcal{S} \setminus \mathcal{H}/\mathcal{C} \to \mathbb{R}$ as

$$f_0 := \phi^{-1}\left(\frac{\chi_{K_0}}{\Lambda}\right).$$

Then, for every $x \in \mathcal{H}$,

$$\phi(|f_0(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})|) = \frac{\chi_{K_0}(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})}{\Lambda},$$

so, for every $y \in C(\mathcal{H})$,

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$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|f_0(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$= \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{\chi_{K_0}(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})}{\Lambda}d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$= \int_{K_0} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{1}{\Lambda}d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$= \frac{1}{\Lambda}\int_{K_0} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$\leq \frac{1}{\Lambda}\Lambda = 1.$$

Note that since μ is a $C(\mathcal{H})$-invariant measure and for every $z \in C(\mathcal{H})$, $C(\mathcal{H}) = \{zx: x \in C(\mathcal{H})\}$, we have

$$\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1 \qquad (3.1)$$

if and only if for each $z \in C(\mathcal{H})$,

$$\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|f(\mathcal{S}\circledast\{zx\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1. \qquad (3.2)$$

Definition 3.5. We say that a Borel measurable function $\xi: \mathcal{S}\setminus\mathcal{H}/\mathcal{C} \to \mathbb{R}$ belongs to $\mathcal{W} := \mathcal{W}(\Phi; u, v)(\mathcal{S}\setminus\mathcal{H}/\mathcal{C})$ whenever

$$\|\xi\|_{\mathcal{W}} := \sup_{f\in\Omega} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} |\xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \qquad (3.3)$$

is finite, where $f \in \Omega$ means that

$$\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1.$$

Remark 3.6. In the case that $v := \frac{1}{u}$, we have $f \in \Omega$ whenever

$$\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} \phi(|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1.$$

This means that in the special case $v := \frac{1}{u}$, $\mathcal{W}$ is same as the weighted Orlicz space $L_u^{\Phi}(\mathcal{S}\setminus\mathcal{H}/\mathcal{C})$.

Theorem 3.7. By the above notations, $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a Banach space.

Proof. Clearly, $\|0\|_{\mathcal{W}} = 0$. Assume that $\xi: \mathcal{S}\setminus\mathcal{H}/\mathcal{C} \to [0,\infty)$ is measurable and let $\|\xi\|_{\mathcal{W}} = 0$. This means that $\sup_{f\in\Omega} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} \xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) = 0$.
So, for every f with $\sup_{y\in C(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1$,
we have

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} \xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) = 0.$$

Let

$$\sigma(\xi) := \{\mathcal{S}\circledast\{x\}\circledast\mathcal{C} \in \mathcal{S}\setminus\mathcal{H}/\mathcal{C} : 0 < \xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\}.$$

We have

$$\mu(\sigma(\xi)) = \sup\{\mu(K): K \subseteq \sigma(\xi) \text{ and } K \text{ is compact }\}.$$

In contrast, assume that $\mu(\sigma(\xi)) > 0$. Hence, there is some compact $K \subseteq \sigma(\xi)$ with $0 < \mu(K) < \infty$. Choose some $c \geq 1$ such that c is also greater than the amount

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$$\sup_{y\in C(\mathcal{H})}\int_K u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\Big\}$$

Define $\eta := \phi^{-1}(c^{-1}\chi_K)$. Then,

$$\begin{aligned}&\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|\eta(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\\ &=\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} w(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi\phi^{-1}(c^{-1}\chi_K(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}))d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\\ &\le\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})c^{-1}\chi_K(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\\ &=c^{-1}\sup_{y\in C(\mathcal{H})}\int_K u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\end{aligned}$$

≤ 1.

Hence,

$$\begin{aligned}0&=\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}}\xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\eta(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\\ &=\phi^{-1}(c^{-1})\int_K\xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}),\end{aligned}$$

so, $\mu(K)=0$, which contradicts the previous facts. Therefore, $\mu(\sigma(\xi))=0$ i.e. $\xi=0$ almost everywhere.

Easily, one can verify that for every $d>0$ and non-negative $\xi,\xi_1,\xi_2\in\mathcal{W}$, $\|d\xi\|_{\mathcal{W}}=d\|\xi\|_{\mathcal{W}}$, and

$$\|\xi_1+\xi_2\|_{\mathcal{W}}\le\|\xi_1\|_{\mathcal{W}}+\|\xi_2\|_{\mathcal{W}}.$$

So far, we have shown that $\|\cdot\|_{\mathcal{W}}$ is a norm on $\mathcal{W}$. Finally, thanks to the Monotone Convergence Theorem, for every sequence $(\xi_n)_n$ of non-negative elements of $\mathcal{W}$, and $\xi\in\mathcal{W}$, if $\xi_n\uparrow\xi$, then $\|\xi_n\|_{\mathcal{W}}\to\|\xi\|_{\mathcal{W}}$. This completes the proof.

Definition 3.8. For each $\xi\in\mathcal{W}$ we define $\|\xi\|^{\circ}_{\mathcal{W}}$ as

$$\begin{aligned}&\|\xi\|^{\circ}_{\mathcal{W}}\\ &:=\inf\Big\{\zeta>0:\ \sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\\ &\Phi\left(\left|\frac{f(\mathcal{S}\circledast\{\theta(y)x\}\circledast\mathcal{C})}{\zeta}\right|\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\le 1\Big\}.\end{aligned}$$

Theorem 3.9. For each Borel measurable functions ξ,η on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$ and y in $C(\mathcal{H})$, $\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|\xi\eta(\mathcal{S}\circledast\{\theta(y)x\}\circledast\mathcal{C})|d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\le 2\|\xi\|^{\circ}_{\mathcal{W}}\|\eta\|_{\mathcal{W}}$.

Proof. Assume that ξ,η are Borel measurable functions on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$ and $y\in C(\mathcal{H})$. In the cases $\|\xi\|^{\circ}_{\mathcal{W}}=0$ or $\|\eta\|^{\circ}_{\mathcal{W}}=0$, then the conclusion is clear because both sides are zero. Let $\|\xi\|_{\mathcal{W}}\neq 0$ and $\|\eta\|_{\mathcal{W}}\neq 0$. Then, for every $x\in\mathcal{H}$,

$$\begin{aligned}&u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{|f|}{\|\xi\|^{\circ}_{\mathcal{W}}}\frac{|\eta|}{\|\eta\|^{\circ}_{\mathcal{W}}}\\ &\le u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\left[\Phi\left(\frac{|f|}{\|\xi\|^{\circ}_{\mathcal{W}}}\right)+\phi\left(\frac{|\eta|}{\|\eta\|^{\circ}_{\mathcal{W}}}\right)\right].\end{aligned}$$

So since for each $y\in C(\mathcal{H})$

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi\left(\frac{|\xi|}{\|\xi\|^{\circ}_{\mathcal{W}}}\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\le 1$$

we have

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{|\xi|}{\|\xi\|^{\circ}_{\mathcal{W}}}\frac{|\eta|}{\|\eta\|^{\circ}_{\mathcal{W}}}d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\le 2$$

therefore,

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$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})|\xi\eta|d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 2\|\xi\|_{\mathcal{W}}^{\circ}\|\eta\|_{\mathcal{W}}^{\circ}.$$

Theorem 3.10. If $v \geq 1$, then for each Borel measurable function ξ on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$, $\|\xi\|_{\mathcal{W}} \leq 2\|\xi\|_{\mathcal{W}}$.
Proof. Note that $\|\xi\|_{\mathcal{W}}^{\circ} \leq 1$ if and only if

$$\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\phi(|\xi(\mathcal{S}\circledast\{\theta(y)\}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1.$$

Hence, by the hypothesis $v \geq 1$ and Theorem 3.9,

$$\|\xi\|_{\mathcal{W}} = \sup\left\{\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} |\xi g|wd\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}):\right.$$
$$\left.\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\phi(|g(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1\right\}$$
$$\leq 2\sup\{\|\xi\|_{\mathcal{W}}\|g\|_{\mathcal{W}}: \|g\|_{\mathcal{W}} \leq 1\} \leq 2\|\xi\|_{\mathcal{W}}.$$

Definition 3.11. For every measurable function $f: \mathcal{S}\setminus\mathcal{H}/\mathcal{C} \to \mathbb{C}$, we write $f \in \mathcal{Z} := \mathcal{Z}(\Phi; u, v)$ whenever $\|f\|_{\mathcal{W}} < \infty$.

Theorem 3.12. By the above notations, $\mathcal{Z}$ equipped with $\|\cdot\|_{\mathcal{W}}$ is a Banach space.
Proof. (i) We have

$$\|0\|_{\mathcal{W}}^{\circ} = \inf\left\{\zeta > 0: \sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi(0)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1\right\}$$

$$= \inf\left\{\zeta > 0: \sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C}) \times 0 d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1\right\} = 0.$$

Now, consider some non-negative Borel measurable function f on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$ with $\|f\|_{\mathcal{W}} = 0$. Denote

$$\sigma(f) := \{\mathcal{S}\circledast\{x\}\circledast\mathcal{C} \in \mathcal{S}\setminus\mathcal{H}/\mathcal{C}: f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \neq 0\}$$

In contrast, let $\mu(\sigma(f)) > 0$. So, for some compact $D \subseteq \sigma(f)$ s.t. $0 < \mu(D) < \infty$. By $\|f\|_{\mathcal{W}} = 0$, for every $0 < \epsilon < 1$ and $y \in C(\mathcal{H})$,

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi(f\chi_D)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$= \int_D u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi(f)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$\leq \epsilon\int_D u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi\left(\frac{f}{\epsilon}\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$\leq \epsilon\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi\left(\frac{f}{\epsilon}\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq \epsilon$$

So, $\mu(D) = 0$, which contracts some previous facts. Hence, $f = 0$ almost everywhere.
(ii) Let ξ_1, ξ_2 be non-negative measurable functions on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$. Assume that $\epsilon_1, \epsilon_2 > 0$ and for $i = 1,2$, $\sup_{y\in C(\mathcal{H})}\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\Phi\left(\frac{\xi_i(\mathcal{S}\circledast\{\theta(y)x\}\circledast\mathcal{C})}{\epsilon_i}\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \leq 1$.
By convexity of Φ, for every $y \in C(\mathcal{H})$,

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi\left(\frac{\xi_1+\xi_2}{\epsilon_1+\epsilon_2}\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) \tag{3.4}$$

$$= \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\Phi\left(\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\left(\frac{\xi_1}{\epsilon_1}\right)+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\left(\frac{\xi_2}{\epsilon_2}\right)\right)d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$\leq \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})v(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\left[\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\Phi\left(\frac{\xi_1}{\epsilon_1}\right)+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}\Phi\left(\frac{\xi_2}{\epsilon_2}\right)\right]d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})$$
$$\leq \frac{\epsilon_1}{\epsilon_1+\epsilon_2}+\frac{\epsilon_2}{\epsilon_1+\epsilon_2} = 1.$$

Then, $\|\xi_1+\xi_2\|_{\mathcal{W}}^{\circ} \leq \epsilon_1+\epsilon_2$. Now, by taking infimum on these ϵ_1 and ϵ_2 we conclude that

$$\|\xi_1+\xi_2\|_{\mathcal{W}}^{\circ} \leq \|\xi_1\|_{\mathcal{W}}^{\circ}+\|\xi_2\|_{\mathcal{W}}^{\circ}.$$

(iii) Assume that $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of non-negative measurable functions on $\mathcal{S}\setminus\mathcal{H}/\mathcal{C}$ and let f be a function in this kind, and $\xi_n \longrightarrow f$ μ-almost everywhere. Denote

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$$\eta := \liminf_{n} \|\xi_n\|^{\circ}_{\mathcal{W}}.$$

Assume that $f \neq 0$ and $\eta < \infty$. Without lossing the generality, we assume that for every $n \in \mathbb{N}$, $\|\xi_n\|^{\circ}_{\mathcal{W}} > 0$. There exists $\epsilon_0 > 0$ that for any $\epsilon \leq \epsilon_0$,

$$\sup_{y \in \mathcal{C}(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{f}{\epsilon}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) > 1$$

Hence,

$$1 \leq \liminf_{n} \left(\sup_{y \in \mathcal{C}(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_n}{\epsilon}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \right).$$

So, for every $n_0 \in \mathbb{N}$ there is $n \geq n_0$ s.t.

$$1 < \sup_{y \in \mathcal{C}(\mathcal{H})} \int_{G} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_n}{\epsilon}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}).$$

So, $\|\xi_n\|_{\mathcal{W}} \geq \epsilon_0 > 0$.

In contrast, if $\eta = 0$, there exists $\{\xi_{n_k}\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \|\xi_{n_k}\|_{\mathcal{W}} = 0$. There is $k_0 \in \mathbb{N}$ with $\|\xi_{n_k}\|_{\mathcal{W}} \leq 1$ for all $k \geq k_0$. Then,

$$\frac{1}{\|\xi_{n_k}\|^{\circ}_{W}} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi(\xi_{n_k}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

$$\leq \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_{n_k}}{\|\xi_{n_k}\|_{\mathcal{W}}}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \leq 1$$

for all $k \geq k_0$ and $y \in \mathcal{C}(\mathcal{H})$. Then, for each $y \in \mathcal{C}(\mathcal{H})$,

$$0 \leq \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi(f) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

$$\leq \liminf_{k\to\infty} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi(\xi_{n_k}) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

$$\leq \liminf_{k\to\infty} \|\xi_{n_k}\|_{\mathcal{W}} = 0.$$

Therefore,

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} wv\Phi(f) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) = 0.$$

Hence, $\Phi(f) = 0$ almost everywhere. So, $f = 0$ almost everywhere, which contracts the previous facts. Therefore, $0 < \eta < \infty$. Then, for each $\gamma > \eta$, $\|\xi_n\|_{\mathcal{W}} < \eta$ for enough large n. So, for every $y \in \mathcal{C}(\mathcal{H})$,

$$\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_n}{\gamma}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

$$\leq \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_n}{\|\xi_n\|_{\mathcal{W}}}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \leq 1.$$

This implies that

$$\sup_{y \in \mathcal{C}(\mathcal{H})} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{f}{\gamma}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C})$$

$$\leq \sup_{y \in \mathcal{C}(\mathcal{H})} \left(\liminf_{n\to\infty} \int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} u(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) v(\mathcal{S} \circledast \{yx\} \circledast \mathcal{C}) \Phi\left(\frac{\xi_n}{\gamma}\right) d\mu(\mathcal{S} \circledast \{x\} \circledast \mathcal{C}) \right) \leq 1.$$

Hence, $\|f\|_{\mathcal{W}} \leq \gamma$. Therefore,

$$\|f\|^{\circ}_{\mathcal{W}} \leq \eta = \liminf_{n\to\infty} \|\xi_n\|^{\circ}_{\mathcal{W}}.$$

The other conditions for that $\mathcal{Z}$ to be a Banach space is routin.
Definition 3.13. Fix some $p > 0$. Then, we define

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$$\|f\|^{\circ}_{\mathcal{W},p} := \sup_{y\in C(\mathcal{H})}\left(\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})|f(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})|^p d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right)^{\frac{1}{p}}$$

Theorem 3.14. With the above notations, the below conditions are equivalent.
(i) we have

$$0 < \inf\left\{\sup_{y\in C(\mathcal{H})} \int_E (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) : \mu(E) > 0\right\}.$$

(ii) for every $p > 0$,

$$\mathcal{W}_p \subseteq L^{\infty}(\mathcal{S}\setminus\mathcal{H}/\mathcal{C},\mu)$$

(iii) for every $p, q > 0$ that $p < q, \mathcal{W}_p \subseteq \mathcal{W}_q$.

Proof. $(i) \Longrightarrow (ii)$: Assume

$$0 < \inf\left\{\sup_{y\in C(\mathcal{H})} \int_E (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) : E \subseteq \mathcal{S}\setminus\mathcal{H}/\mathcal{C}, \mu(E) > 0\right\}.$$

Let $p > 0$ and $\xi \in \mathcal{W}_p$. For every $n \in \mathbb{N}$, denote

$$D_n := \{\mathcal{S}\circledast\{x\}\circledast\mathcal{C} \in \mathcal{S}\setminus\mathcal{H}/\mathcal{C} : |\xi(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})| > n\}.$$

Then, $n\chi_F D_n \le |\xi|$, and so

$$n \sup_{y\in C(\mathcal{H})}\left(\int_{D_n} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right)^{\frac{1}{p}} \le \|\xi\|^{\circ}_{\mathcal{W},p}$$

This implies that

$$\lim_{n\to\infty}\sup_{y\in C(\mathcal{H})}\left(\int_{D_n} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right) = 0.$$

So, there is $n \in \mathbb{N}$ s.t. $\mu(D_n) = 0$, hence $\xi \in L^{\infty}(\mathcal{S}\setminus\mathcal{H}/\mathcal{C},\mu)$.
$(ii) \Longrightarrow (iii)$: Let for each $p > 0$,

$$\mathcal{W}_p \subseteq L^{\infty}(\mathcal{S}\setminus\mathcal{H}/\mathcal{C},\mu)$$

Let $0 < p < q$ and $\xi \in \mathcal{W}_p$. So, there is $k > 0$ that $|\xi| \le k$ almost everywhere, and

$$\begin{aligned}&\sup_{y\in C(\mathcal{H})}\left(\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{|\xi|^q}{k^q}d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right)\\ &\le \sup_{y\in C(\mathcal{H})}\left(\int_{\mathcal{S}\backslash\mathcal{H}/\mathcal{C}} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})\frac{|\xi|^p}{k^p}d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right) < \infty,\end{aligned}$$

so, $\xi \in \mathcal{W}_q$.

$(iii) \Longrightarrow (i)$: Let $0 < p < q$ and $\mathcal{W}_p \subseteq \mathcal{W}_q$. Then, for a constant $C > 0$ and every $\xi \in \mathcal{W}_p$,

$$\|\xi\|_{\mathcal{W},q} \le C\|\xi\|_{\mathcal{W},p}. \tag{3.5}$$

Let $\mu(E) > 0$ and $\sup_{y\in C(\mathcal{H})} (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C}) < \infty$. So, $\chi_E \in \mathcal{W}_p$, and by (3.5),

$$0 < C^{\frac{pq}{p-q}} \le \sup_{y\in C(\mathcal{H})}\left(\int_E (uv)(\mathcal{S}\circledast\{yx\}\circledast\mathcal{C})d\mu(\mathcal{S}\circledast\{x\}\circledast\mathcal{C})\right).$$

4. Conclusion

In this paper, we studied the weighted Orlicz spaces on double coset spaces, which induce some invariant measure regarding the centre of hypergroup. In addition, we introduce two novel norms that are versions of the Orlicz and Luxemburg norms. The results of this research article cover so many new function spaces. They have this capacity to study them as convolution modules and convolution algebras.

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