LS-Extending Fuzzy Modules

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Abstract

The main aim of this paper is extend the notion of S-extending fz-modules into LS-extending fz-modules and study this new notion. This lead us introduce and study other notions such as: purely semisimple, purely extending and purely y-extending fz-modules. Moreover, the relationships LS-extending fz-module with the various types.

Keywords: LS-extending fz-module; Pure fz-sumodule; CS-fzmodules

1. Introduction

Throughout this paper $\mathcal{R}$ is commutative ring with unity, M is an $\mathcal{R}$-module and χ is fuzzy module of an $\mathcal{R}$-module, M denoted by ($\mathcal{R}$-fzmodule). A fz-submodule κ is called essential (briefly $\kappa \leq_e \chi$), if $\kappa \cap \rho \neq 0_1$, for any non-trivial fz-submodule ρ of χ [9] and a proper fz-submodule κ is called prime fz-submodule whenever $r_t a_\iota \subseteq \kappa$ for fz-singleton r_t of $\mathcal{R}$ and $a_\iota \subseteq \chi$ we have either $r_t \subseteq (\kappa :_R \chi)$ or $a_\iota \subseteq \kappa$ where $(\kappa :_R \chi) = \{ r_t : r_t\chi \subseteq \kappa, r_t$ fz singleton of $\mathcal{R})$ [9]. Let χ $\mathcal{R}$-fzmodule M, if ρ a fz-submodule of χ, then ρ is called a semi essential (in short S-essential) fz-submodule of χ, if for all prime fz-submodule η of χ and $\rho \cap \eta = 0_1$, then $\eta = 0_1$ [1]. A fz-submodule κ of fz-module χ is called closed (shortly $\kappa \leq_c \chi$), if κ has no proper essential, that is $\kappa \leq_e \rho \leq \chi$, then $\kappa = \rho$ [13]. Hasan in [13], studied extending fz-module (denoted by F-CS module), where χ is named extending fz-module if every closed fz-submodule of χ is a direct summand. In this paper, we introduce a new class of fz-modules named LS-extending fz-module. This class of fz-modules lies between F-CS modules and S-extending fz-module, where χ is named S-extending fz-module if every S-essential in direct summand of χ [15]. This research consists three sections, in the first section, we shall give some concepts and properties of fz-sets and fz-modules. In the second section, we define and we give some examples and we introduce characterizations and properties of this class. Section three we study relation LS-extending fz-module with some types fz-modules.

Next throughout this paper, (shortly fuzzy set, fuzzy submodule and fuzzy module is fz-set, fz-submodule and fz-module).

2. Preliminaries

This section contains some definitions and properties of fz-sets, fz-modules and fz-submodules, which will use in the next sections.

Definition 1.1 [21]:

Let S be a non-empty set and let I be the closed interval [0,1] of the real line (real numbers). An fz-set χ in S (a fz-subset χ of S) is a function from S into I.

"Definition 1.2 [22]:

Let $x_\ell : S \longrightarrow I$ be a fz-set in S, $x \in S$, $\ell \in [0,1]$, defined by:

$$x_\ell = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \quad \forall\ y \in S$$

Then x_ℓ is named a fz-singleton.

If $x = 0$ and $\ell = 1$ then :

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$$0_1(y) = \begin{cases} 1 & \text{if } \ y = 0 \\ 0 & \text{if } \ y \neq 0 \end{cases}$$

Definition 1.3 [22]:

Let κ, ρ be fz-sets in S, then :

1. $\kappa = \rho$ iff $\kappa(x) = \rho(x)$, $\forall\, x \in$ S.

2. $\kappa \subseteq \rho$ iff $\kappa(x) \leq \rho(x)$, $\forall\, x \in S$

3. $x_\ell \subseteq \nu$ iff $x_\ell(y) \leq \kappa(y)$, ∀ y ∈ S and if $\ell > 0$, then $\kappa(x) \geq \ell$. Thus $x_\ell \subseteq \kappa$ $(x \in \kappa_\ell)$, (that is $x \in \kappa_\ell$ iff $x_\ell \subseteq \kappa$).

Definition 1.4 [22]:

Let κ, ρ be fz-sets in S, then:

1. $(\kappa \cup \rho)(x) = \max\{(\kappa(x), \rho(x)\}$, $\forall\, x \in$ S.

2. $(\kappa \cap \rho)(x) = \min\{(\kappa(x), \rho(x)\}$, $\forall\, x \in$ S.

$\kappa \cup \rho$ and $\kappa \cap \rho$ are fz-sets in S.

In general if $\{\nu_\alpha, \alpha \in \Lambda\}$, is κ family of fz-sets in S, then:

$$\left(\bigcap_{\alpha\in\Lambda} \kappa_\alpha\right)(x) = \inf\{\kappa_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in \text{S}.$$

$$\left(\bigcup_{\alpha\in\Lambda} \kappa_\alpha\right)(x) = \sup\{\kappa_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in \text{S}.$$

Definition 1.5 [16]:

Let κ be a fz-set in S, ∀ t ∈ [0,1], the set $\kappa_t = \{x \in S, \kappa(x) \geq t\}$ is named a level sub-set of κ.

Remark 1.6 [21]:

Assume κ, ρ are fz-subsets of a set S, then:

1. $(\kappa \cap \rho)_t = \kappa_t \cap \rho_t$ for any t ∈ [0,1] .

2. $(\kappa \cup \rho)_t = \kappa_t \cup \rho_t$ for any t ∈ [0,1] .

3. $\kappa = \rho$ iff $\kappa_t = \rho_t$, ∀ t ∈ [0,1].

Definition 1.7 [20]:

Let f : M ⟶ N. Let κ be a fz-set in M**,** the image of κ denoted by $f(\kappa)$ is the fz-set in N defined by:

$$f(\kappa)(y) = \begin{cases} sup\{\kappa(z) \mid z \in f^{-1}(y)\}\, if & f^{-1}(y) \neq \emptyset,\ for\ each\ y \in N \\ 0 & o.w \end{cases}$$

where $f^{-1}(y) = \{x : f(x) = y\}$

and let ρ be a fz-set in N, then the inverse image of ρ, denoted by $f^{-1}(\rho)$ is the fz-set in M defined by : $f^{-1}(\rho)(x) = \rho(f(x))$, $\forall\, x \in$ M.

Definition 1.8 [11]: If f : M ⟶ M$^\sim$ be any mapping . A fz-subset ν of M is called f-invariant if $\kappa(x) = \kappa(y)$, whenever $f(x) = f(y)$, where x, y ∈ M.

Definition 1.9 [21]:

A fz-set X of an $\mathcal{R}$ − module M is named fz-module ($\mathcal{R}$-fzmodule M) if :

1. X $(x - y) \geq \min\{X(x), X(y)\}$, ∀ x, y ∈. M

2. X $(r\chi) \geq$ X (χ), $\forall\, \chi \in$ M and r ∈ $\mathcal{R}$.

3. X (0) = 1.

Definition 1.10 [12]:

Let χ, ω be two an $\mathcal{R}$-fzmodule M (fz-module of an R-module M). ω is called a fz-submodule of χ if $\omega \subseteq \chi$.

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Definition 1.11 [22]:

If κ is fz-submodule of an $\mathcal{R}$-module M, then the submodule κ_t of M is called level submodule of M, where t ∈ [0,1].

Definition 1.12 [9]:

Let χ, γ be R-fzmodules M_1, M_2 individually, define $\chi\oplus\gamma$: $\mathrm{M}_1\oplus\mathrm{M}_2 \longrightarrow [0,1]$ $(\chi\oplus\gamma)(h,g) = \min\{\chi(h),\gamma(g)$, $\forall\ (h,g) \in \mathrm{M}_1\oplus\mathrm{M}_2\}$.

$\chi\oplus\gamma$ is named fz-external direct sum of χ and γ .

Definition 1.13 [11]:

Let χ be $\mathcal{R}$-fzmodule M is named semisimple if, χ is a sum of simple fz-submodules of χ, where χ $\mathcal{R}$-fzmodule M is named simple fz-module if χ has only one proper fz-submodule, which is 0_1.

Definition 1.14 [9]:

A fz-module χ is named uniform fz-module if $\kappa \cap \rho \neq 0_1$, for any non-trivial fz-submodule κ and ρ of χ.

Definition 1.15 [7]:

Let χ be $\mathcal{R}$-fzmodle M, χ is named semiuniform (in short S-uniform) fz-module, if every non-trivial fz-submodule κ of χ is a semiessential fz-submodule of χ.

Definition 1.16 [22]:

Suppose that κ and ρ be two fz-submodules of $\mathcal{R}$-fzmodule M. The residual quotient of κ and ρ. We define $(\kappa$: ρ) by: $(\kappa$: ρ) = $\{r_t : r_t$ is a fz-singleton of R such that $r_t\rho \subseteq \kappa\}$"

Definition 1.17: [9]

Let χ be $\mathcal{R}$-fz module M is named fully prime fz-module, if every proper fzsubmodule of χ is prime fz-submodule.

Definition 1.18: [9]:

Afz-sumodule ρ of χ is named pure fz-sumodule of χ, if for each fz-ideal ϖ of $\mathcal{R}$ $\varpi\chi \cap \rho = \varpi\rho$.

3. LS-extending FZ-Modules:

In section two, we introduced the notion of a LS-extending $\mathcal{R}$-fzmodule M (fz-module of an $\mathcal{R}$-module M), by extended (ordinary), almost semi-extending module. We also state and prove same basic results a but this concept.

Definition 2.1: [2]

An $\mathcal{R}$-module K is named almost semiextending module, for each sub-module of K is a semi essential in a pure sub-module ofK.

First, we fuzzify as the follows the definition:

Definition 2.2:

A sound fz-module χ of an $\mathcal{R}$-module M is named almost semi-extending fz-module (borfily LS-extending), if every S-essential fz-submodule in a pure of χ.

The following proposition gives the relationship between LS-extending fz-module and its level.

Proposition 2.3:

Let χ be a fz-module of an $\mathcal{R}$-module M, then χ_* is a LS-extending module iff χ is LS-extending fz-module .

Proof:

(⇒) Impose χ_* be LS-extending module, we have to show that X is LS-extending fz-module.

Let κ be fz-submodule of X, then $\kappa_* \leq \chi_*$ by proposition (1.6). Since χ_* is a LS-extending module, so there exists a pure N of χ_* such that $\kappa_* \leq_{sem}$ N. Define ρ : M ⟶[0,1], by: $\rho(a) = \begin{cases} 1 & if\ a\ \in N \\ 0 & otherwise \end{cases}$

Clearly ρ a fz-submodule of χ, ρ_t = N. Since N is pure of χ_* imply that ρ is a pure fz-submodule in X (See [11, Proposition (2.1.3)]). On the other hand $\kappa_* \leq_{sem} \rho_*$ = ℕ, therefore $\kappa \leq_{sem} \rho$ (See [1, Proposition (3.4)]).That is χ is LS-extending fz-module .

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(⇐) To prove that χ_* is LS-extending module .

Let N be non-zero submodule of X_* and let ρ : M ⟶[0,1], define by $\rho(a) = \begin{cases} 1 & if\ a\ \in N \\ 0 & otherwise \end{cases}$

It is clear that $\rho \leq X$ and $\rho_* =$ N. Since X is LS-extending fz-module, then $\rho \leq_{sem} \kappa$, where κ is a pure in X, by [11, Proposition.(2.1.3)], ρ_* is a pure in κ_* and $\rho_* \leq_{sem} \kappa_*$, (See [1, Proposition(3.4)]), therefore X_* is SL-extending module.

Remarks and Examples 2.4:

1. Whole semisimple fz-module is LS-extending fz-module.

Proof:

Since every fz-sumodule of semisimple fz-module a direct summand, and pure fz-submodule (see [11], Proposition (2.2.7)) so by define it breakpoint LS-extending fz-module.

2. Every S-extending fz-module is LS-extending fz-module.

Proof:

Impose X is S-extending fz-module and let $\rho \leq_{sem} \chi$, and a direct summand of X. Because every direct summand is pure (See[11, Proposition.(2.2.7)]), Therefore ρ is S-essential in a pure fz-submodule.

3. Every F-CS module is LS-extending fz-module.

Proof:

Since X is F-CS module imply that X is S-extending fz-module by [15, Remarks and Examples. 2.5 (2)] , so by Remark (2) X is LS-extending fz-module.

4. The converse of (3) is not true, for example:

Example:

Let M be Z-module $Z_8 \oplus Z_2$ and let X ⟶ [0,1], define by :

$X(\alpha,\beta) = 1, \forall\ (\alpha,\beta) \in\ Z_8 \oplus Z_2$.

X is S-extending fz-module by [15, Example (3.1)], and by (2), X is LS-extending fz-module but not F-CS module.

5. Whole uniform fz-module is LS-extending fz-module. The converse is not true, for example :

Let M be Z-module Z_{36} and let X : M ⟶[0,1], define by :

$X(\alpha) = 1, \forall\ \alpha \in Z_{36}$. For any $\rho \leq X$ such that :

$X_* = Z_{36}$ is S-extending (See [3]), imply that X_* is LS-extending module by [2, Remarks and Examples 2.3 (4)), so X is LS-extnding fz-module by Proposition (2.3). All the same X_* contains submodule ρ_* which isnt essential, so that ρ is not essential fz-submodule by [13, Example (2.1)] **.**

The next Proposition show that the converse of Remark (2), is not true. However, it is true certain conditions. First we recall some basic properties of these concepts.

Definition 2.5: [17]

A fz-module X in M is called divisible fz-module iff for each $\alpha_t \subseteq X$ with t> 0 and for each r ∈ R, r ≠ 0, there exist fz-point $\beta_t \subseteq$ X such that $r(\beta_t) = \alpha_t$, where $r(\beta_t) = (r\beta)_t$.

Definition 2.6: [8]

The R-module K is named Noetherian, if every submodule of K are finitely generated.

Proposition 2.7:

Assume that χ be a divisible fz-module over principle ideal domain R. Imply χ is LS-extending fz-module iff X is S-extending fz-module.

Proof:

Suppose that χ is LS-extending fz-module and $\rho \leq_{sem} \chi$, with the pure fz-submodule say κ. Since χ is a divisible, then by [17, proposition (2.1.3)], X_t is a divisible over principle ideal domain R. Thus $\kappa_t \leq^{\oplus} X_t$[4,Corollary. (2.9)], so $\kappa \leq^{\oplus}$ X, by [13, Lemma. (2.1.14)], and X is S-extending fz-module.

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The converse it is clear.

Proposition 2.8:

Let X be a finitely generated fz-module over Noetherain ring R. Then X is LS-extending fz-module iff X is S-extending fz-module.

Proof:

Assume that X is LS-extending fz-module and $\rho \leq \boldsymbol{X}$**,** then there exists a pure fz-submodule ν such that $\rho \leq_{sem} \nu$, so ν_t is a pure (See[11, Proposition(2.1.3)) and X is finitely generated, then X_t finitely generated by [9]. But R Noetherain ring imply that ν_t is a direct summand by [4, proposition. (2.10)], thus ν a direct summand by [13, Proposition (2.1.14)].

Let (*) means the following: For a fz-module X and $\boldsymbol{\rho}, \boldsymbol{\nu}$ be non-empty fz-submodules of X, if $\boldsymbol{\rho}_* \subseteq \boldsymbol{\nu}_*$ implies that $\boldsymbol{\rho} \subseteq \boldsymbol{\nu}$.

The following Proposition gives characterization of LS-extending fz-module

Proposition 2.9:

A fz-module χ of an R-module M is LS-extending fz-module iff every S-closed fz-submodule is a pure of χ.

Proof:

Assume that X is LS-extending fz-module and $\rho \leq_{sc} \chi$. By define LS-extending fz-module, there exists a pure fz-submodule ν of X such that $\rho \leq_{sem} \nu$, so by [1, proposition(2.3)], $\rho_* \leq_{sem} \nu_*$. But $\rho \leq_{sc} \chi$, then by [14, proposition (2.19)], $\rho_* \leq_{sc} \chi_*$, therefore $\rho_* = \nu_*$, by condition(*), $\rho = \nu$. Conversely, let $\rho \leq X$, if $\rho = 0_1$, clearly $0_1 \leq_{sem} 0_1$ and pure (See [11, Remark.(2.1.2)]. If $\rho \neq 0_1$, then there exists S-closed fz-submodule κ such that $\rho \leq_{sem} \kappa$ by [14, proposition (2.8)]. Since $\rho \leq_{sc} \chi$, and κ is a pure in χ. We have χ is LS-extending fz-module.

In [19] various writer provides that the definition of purely semisimple. We modify definition as follows:

Definition 2.10:

A fz-module χ of an $\mathcal{R}$-module M is named pure semisemple (borfily P-semisemple), if for every pure fz-submodule ρ of χ there exists a direct summand κ s.t $\kappa \leq_e \rho$.

We needed the following lemma

Lemma 2.11:

A fz-module X is purely semisimple iff every pure fz-submodule of X is a direct summand.

Proof:

Let ρ pure fz-submodule of X, so there exists $\kappa \leq^{\oplus}$ X such that $\kappa \leq_e \rho$. But κ is a direct summand imply that $\kappa \leq_c$ X [13], hence $\rho = \kappa$ and so κ is a direct summand.

Proposition 2.12:

Assume that χ be a P-semisimple fz-module, then χ is LS-extending fz-module iff χ is S-extending fz-module

Proof:

(⇒) Impost $\rho \leq_{sc}$ X. Since $\chi$is LS-extending fz-module, then by Proposition (2.9), ρ is pure fz-module. But X is P-semisimple fz-module, so $\rho \leq^{\oplus} \chi$ (See lemma (2.11), imply that $\chi$is S-extending fz-module.

(⇐) It follows that by Remark and Examples (2.4)(2)

Theorem 2.13:

If $\chi$is a P-semisimple fz-module, consequently the following :

1. If $\chi$is F-CS, we have χ is S-extending fz-module.
2. If $\chi$is S-extending fz-module, then $\chi$is LS-extending fz-module.
3. If $\chi\chi$is fully prime fz-module, we have the following two case :

i-Ifχ is LS-extending fz-module, then $\chi$is F-CS.

ii-Ifχ is S-extending fz-module, imply χ is F-CS.

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Proof:

1- It follows directly from [15, Remarks and Examples (2.4) (2)].

2- It follows by Remark (2.4) (2).

3- (i) Let $\rho\chi$, we have if $\rho = 0_1$, imply ρ is a direct summand. On the other hand χ is fully prime fz-module, then by [14, Remark (2.13)], $\rho \leq_{sc} \chi$. Since χ is LS-extending fz-module, then ρ is a pure. But χ is P-semisimple, $\rho \leq^{\oplus} \chi$, by lemma (2.11).

(ii) Since X is fully prime fz-module, so the result follows [15, proposition (3.4)].

Proposition 2.14:

Let X be LS-extending fz-module and ρ, κ be two any fz-modules of X, if $\rho \cap \kappa \leq_{sc}$ X, then $\rho \cap \kappa$ is a pure fz-module in ρ *and* κ.

Proof:

Since X is SL-extending fz-module, then by Proposition (2.2), X_* is SL-extending and $(\rho \cap \kappa)_* \leq_{sc} X_*$ (See[14, proposition(2.19)) implies that $(\rho \cap \kappa)_*$ pure in ρ_* *and* κ_* by [2, proposition (2.13)] , so $\rho \cap \kappa$ is a pure fz-module in ρ *and* κ by [11, proposition (2.1.3)].

Recall that a fz-module is a chained fz-module, if for each fz-submodules ρ, κ, then $\rho \leq \kappa$ or $\kappa \leq \rho$. [18].

Proposition 2.15:

If for each S-closed fz-submodule of χ and χ a chained is LS-extending fz-module.

Proof:

Let $\rho \leq_{sc}$ X and assume $\kappa \leq_{sc} \rho$. Since X is a chained, then by [14, corollary (2.20)], $\kappa \leq_{sc}$ X. But X is LS-extending, then κ is a pure fz-submodule and since $\kappa \leq \rho$, then κ is a pure in ρ (See [11, Proposition (2.1.13)); that is ρ is SL-extending fz-module.

3. Various Types of FZ-modules with The LS-Extending.

Section three we studying certain types of fz-modules such as Semuniform , regular, CLS and purely y-extending fz-modules which are related with the type LS-extending and we check some condition under which LS-extending fz-module are equivalent.

Proposition 3.1:

If X is S-uniform fz-module, thence X is LS-extending fz-module.

Proof:

Since χ S-uniform fz-module, then χ_* is S-uniform module by [7, Theorem(3.15)], so χ_* is SL-extending module (See [2, proposition (3.1)), imply that χ is SL-extending fz-module by Proposition (2.3).

The converse of Proposition (3.1), is not true, for example:

Example:

Let M = Z_{24} as Z-module and let χ : M ⟶ [0,1], defined by:

$\chi(m) = 1, \forall \ m \in Z_{24}$.

It is easy that χ is a fz-module and $\chi_* = Z_{24}$ is a LS-extending module by [2, Proposition (3.1)], so χ is LS-extending fz-module by Proposition (2.3).

But not S-uniform fz-module, since there exists fz-submodule κ: M ⟶ [0,1], defined by: $\kappa(n) = \begin{cases} 1 & if \quad n \in \langle \overline{8} \rangle \\ 0 & otherwise \end{cases}$, is not S-uniform of χ.

Before giving our next result, we have the following definition.

Definition 3.2 [11]:

A fz-module χ of an $\mathcal{R}$-module M is named is pure simple fz-module if we have $0_1, \chi$ are the only pure fz-submodule of χ.

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Theorem 3.3:

Assume that χ be pure simple fz-module, then χ is S-uniform fz-module iff χ is LS-extending fz-module.

Proof:

(⇒) By Propostion (3.1)

(⇐) Since χ is LS-extending fz-module and pure simple fz-module, then by Proposition (2.3) χ_* is LS-extending and by [11, Proposition (3.3.17)] χ_* is pure simple, so χ_* is S-uniform module [2 proposition (3.2)]. Thus X is S-uniform fz-module (See [7, Theorem (3.15).

The condition pure simple from Theorem (3.3),con't drop for example:

Example 3.4:

Let M = Z_{12} as Z-module and $\mu = \chi_{Z_{12}}$ is n't pure simple fz-module because $\chi_{(4)} \leq \chi_{Z_{12}}$ which is pure fz-submodule. However $\chi_{Z_{12}}$is S-extending fz-module, so it is LS-extending fz-module, but not S-uniform since $\chi_{(4)}$ is n't S-essential fz-module of that $\chi_{Z_{12}}$.

Recall that a fz-module X is called F-regular if every fz-submodule of X is pure [11].

Proposition 3.5:

If χ be F-regular fz-module, then χ is LS-extending fz-module.

Proof:

Since χ F-regular fz-module, then by [11, proposition(3.2.1)], χ_t is F-regular, so χ_* is F-regular imply that χ_* is LS-extending module (See[2, proposition(3.4)). Thus χ is LS-extending fz-module by Proposition (2.3).

First we recall some basic properties of these concepts.

Definition 3.6 [13]:

Let χ be a fz-module such that : Z(χ) = $\{x_t \subseteq \chi$: F-ann(x_t) is an essential fz-ideal of R$\}$ is named fz-singular submodule of χ. If Z(χ) = χ and χ is named non-singular if Z(χ) = 0_1.

Definition 3.7 [13]:

A fz-module χ of an $\mathcal{R}$-module M is named CLS-fzmodule if for each $\mathcal{y}$-closed fz-submodule a direct summand in χ, where a fz-submodule ν is named $\mathcal{y}$-closed fz-submodule, if χ/ν is non-singular fz-module.

Proposition 3.8:

Let X be non-singular fz-module. If X is a CLS-fzmodule, then X is LS-extending fz-module.

Suppose that $\rho \leq_{sc}$ X. Since X is non-singular, then ρ $\mathcal{y}$-closed by [14, proposition(2.21)]. But X is CLS-fzmodule, therefore $\rho \leq^{\oplus}$ X, hence ρ is pure by [11, Proposition(2.2.7)], so that X is LS-extending fz-module.

Recall that, an $\mathcal{R}$-module M is called purely $\mathcal{y}$-extending if every $\mathcal{y}$-closed of M is pure [5].

We shall fuzziy this notion and study its relationship with LS-extending fz-module.

Definition 3.9:

A fz-module χ of an $\mathcal{R}$-module M is named purely $\mathcal{y}$-extending fz-module (borfily $p\mathcal{y}$-extending), if every $\mathcal{y}$-closed fz-submodule is pure..

Proposition 3.10:

Let χ non-singular fz-module. If X $p\mathcal{y}$-extending fz-module, then χ is LS-extending fz-module.

Proof:

Assume $\rho \leq_{sc} \chi$. Since χ non-singular, so that ρ $\mathcal{y}$-closed fz-submodule by [14, proposition(2.21)]. But χ is $p\mathcal{y}$-extending fz-module, therefore ρ is pure imply χ is LS-extending fz-module.

Proposition 3.11:

Every LS-extending fz-module is $p\mathcal{y}$-extending fz-module. if χ be a fully prime fz-module

Proof:

Impose that χ is LS-extending fz-module and fully prime, ρ be y-closed in χ. If $\rho = 0_1$, then ρ is pure by [11, Remark (2.1.2)]. Other wise $\rho \neq 0_1$, since χ is fully prime then $\rho \leq_{sc} \chi$ (See[14, Proposition(2.22)). But ρ is pure of χ, that is χ is py-extending fz-module.

Theorem 3.12:

Let χ is non-singular fz-module, consequently the following:

1- If χ is a CLS-fzmodule, so that χ is py-extending fz-module.

2- If χ is py-extending fz-module, we have χ is LS-extending fz-module.

3-Providing χ fully prime and P-semisimple with χ is LS-extending fz-module, we have χ is CLS-fzmodule.

Proof:

1- Let ρ be y-closed of X. But χ is CLS-fzmodule , then $\rho \leq^{\oplus} \chi$. That is ρ is pure by [11, Proposition(2.2.7)], and we are done.

2- Clearly by Proposition (3.10).

3- Assume that ρ be y-closed fz-submodule of X. If $\rho = 0_1$, then $\rho \leq^{\oplus}$ X. On the other hand since χ is fully prime, then $\rho \leq_{sc} \chi$ by [14, proposition(2.22)], but χ is LS-extending imply ρ is a pure. By assumption χ is P-semisimple, so $\rho \leq^{\oplus} \chi$ (See Lemma (2.11), thus χ is CLS-fzmodule.

Theorem 3.13:

Suppose χ be non-singular with the fully prime fz-module, then we have the following:

1- If χ is S-extending fz-module, then χ is LS-extending fz-module.

2- If χ is LS-extending fz-module iff χ py-extending fz-module.

3- If χ is P-semisimple and py-extending fz-module, then χ is S-extending.

Proof:

1. It is clear.
2. It follows by Proposition (3.11)
3. Let $\rho \leq_{sc} \chi$. Since X is non-singular then by [14, proposition (2.21)], ρ is y-closed. But χ is py-extending fz-module therefore, ρ is a pure and χ is P-semisimple so by lemma (2.11), $\rho \leq^{\oplus} \chi$. Thus χ is S-extending fz-module.

4. Conclusion and Discussion

Our work aims: firstly, we give the definitions of LS-extending fz-modules, we shall study the relationship between LS-extending f-module, and its level modules (see Proposition (2.3). In addition, we give the relation between SL-extending fz-module and S-extending fz-module (See Remark (2.4) (2). Some equivalent statements for LS-extending fz-module under sufficient condition are given Proposition (2.7) and Proposition (2.8), and relationships of LS-extending fz-modules with the certain types such as: Semuniform, F-regular, CLS and py-extending fz-module

(See: Theorem (3.3), Proposition (3.5), Proposition (3.8) and Proposition (3.10)

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