A New class Closed Sets in Fuzzy Neutrosophic Topology

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Abstract

The goal of this study is to introduce fuzzy neutrosophic P^*-closed sets, a novel notion of collections in fuzzy neutrosophic topology. In this research, we use certain novel concepts, theories, and hypotheses to explore and analyze other innovative characteristics of these classes. In order to make clear the connections among the new research of P^*-closed sets and other sets, a collection of instances is given and explored.

Keywords: P^*-closed sets; Generalized P^*-closed sets; α^m-closed set

1. Introduction

Almost every branch of mathematics has been impacted by the concept of fuzziness since Zadeh [1] defined the theory. Utilize of fuzzy sets can be found in a wide range of fields. Chang [2] studied and developed the theory-concept for fuzzy topological spaces. A number of general topological concepts have gained popularity since then. Smarandche [3] established the definition for a neutrosophic set via membership, non-member ship, as well as indeterminacy levels, and Salama and Albowi [4-5] noticed topological spaces regarding neutrosophic sets, which, in turn, led to an expansion from fuzzy topological spaces in multiple directions.

Following then, a number of authors have published survey articles on the advanced domains for fuzzy neutrosophic topological environments (refer to [6-14,16]). Hajar et al. [15] explored a new concept, Q*. They studied some properties related to this concept. Through this concept, we will define a new concept that we called P^*-closed sets, and we will investigate some results and relationships between our new concept and some other concepts that were studied previously, such as Q* and the other concepts.

This survey's main goal is to give a thorough overview of the evolution for fuzzy neutrosophic collections and their topological applications, with an emphasis on how fuzzy set theory with neutrosophic logic are combined in topological structures.

Because of this, the goal of this work is to propose and explore the idea of P^*-closed sets in relation to fuzzy neutrosophic topology. By adding indeterminacy and uncertainty levels into the membership and non-membership functions, the phrase P^*-closed sets extend traditional ideas of closeness and nearness to characterize a particular kind of closeness relationship in fuzzy neutrosophic spaces. In order to give a more reliable model for comprehending the connections between points as well as collections within a fuzzy neutrosophic environment—where the ideas for membership, non-member, and indeterminacy coexist P^*-close sets were introduced. This method provides a broader framework over topological investigation in the presence of ambiguity and partial truth, enabling an expanded analysis of continuation and the compactness for fuzzy neutrosophic environments.

2. Preliminaries

Definition 2.1. [5]: Take $\mathcal{F}_{\mathcal{H}}$ be a fixed set that is not empty. An object h of shape

$\omega_{\mathcal{H}1} = \left\{< \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)\right\rangle : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$ is the fuzzy neutrosophic set (FNS),

$\omega_{\mathcal{H}1}$. Functions $\mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)$ and $v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) : \mathcal{F}_{\mathcal{H}} \to I$, where $I = [0,1]$, are used here. For each $\mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}$ to the set $\omega_{\mathcal{H}_1}$ as well as $0 \leq \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) + \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) + v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) \leq 3$, the

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degree of membership function (i.e., $\mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)$), the degree of indeterminacy function (i.e., $\sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)$), and the d egree of nonmembership function (i.e., $v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)$) were identified.

Remark 2. 2. [5]: The ordered triple $\left\{< \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)\right\rangle : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$ in I where $I = [0,1]$ on $\mathcal{F}_{\mathcal{H}}$ may be identified as follows: FNS $\omega_{\mathcal{H}_1} = \left\langle \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}, \sigma_{\omega_{\mathcal{H}_1}}, v_{\omega_{\mathcal{H}_1}} \right. >$.

Proposition 2.3. [8]: Given a non-empty set $\mathcal{F}_{\mathcal{H}}$, the FNSs $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$ should have the following form:

$\omega_{\mathcal{H}_1} = \left\{< \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)\right\rangle : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$ and

$\varphi_{\mathcal{H}_1} = \left\{< \mathcal{J}_n, \mu_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right\rangle : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$. Thus, the following hold:

i. $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ iff $\mu_{\lambda_{\mathcal{H}_1}}(\mathcal{J}_n) \leq \mu_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) \leq \sigma_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)$ and $v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) \geq v_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\ \forall\ \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}$,

ii. $\omega_{\mathcal{H}_1} = \varphi_{\mathcal{H}_1}$ iff $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1} \subseteq \omega_{\mathcal{H}_1}$,

iii. $1_{\mathcal{H}_1} - \omega_{\mathcal{H}_1} = \left\{< \mathcal{J}_n, v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), 1_{\mathcal{H}_1} - \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) > \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$,

iv. $$\omega_{\mathcal{H}_1} \cap \varphi_{\mathcal{H}_1} = \begin{cases} < \mathcal{J}_n, \mathrm{Min}\left(\mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \mu_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right), \mathrm{Min}\left(\sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right), \\ \mathrm{Max}\left(v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right) > \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}, \end{cases}$$

v. $$\omega_{\mathcal{H}_1} \cup \varphi_{\mathcal{H}_1} = \begin{cases} < \mathcal{J}_n, \mathrm{Max}\left(\mu_{\lambda_{\mathcal{H}_1}}(\mathcal{J}_n), \mu_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right), \mathrm{Max}\left(\sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n)\right), \\ \mathrm{Min}\left(v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\varphi_{\mathcal{H}_1}}(\mathcal{J}_n) > \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\} \end{cases}$$

vi. $1_{\mathcal{H}_1} =< \mathcal{J}_n, 1,1,0 >$ and $0_{\mathcal{H}_1} =< \mathcal{J}_n, 0,1,1 >$.

Definition 2.4. [2]: Assume $\mathcal{F}_{\mathcal{H}}$ be a non-empty set, A fuzzy neutrosophic topologies (FNT) is a collection τ of fuzzy neutrosophic subsets that meet the subsequent criteria.

i. $0_{\mathcal{H}}, 1_{\mathcal{H}} \in \tau_{\mathcal{H}}$,
ii. $\omega_{\mathcal{H}_1} \cap \omega_{\mathcal{H}_2} \in \tau_{\mathcal{H}}$ for any $\omega_{\mathcal{H}_1}, \omega_{\mathcal{H}_2} \in \tau_{\mathcal{H}}$,
iii. $\cup\ \omega_{\mathcal{H}_i} \in \tau_{\mathcal{H}}, \forall \{\omega_{\mathcal{H}_i}; i \in J\} \subseteq \tau_{\mathcal{H}}$.

The FNTS represents the pair $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$. The fuzzy neutrosophic-open sets (FNOSs) include every one of the elements of τ. The fuzzy neutrosophic-closed set (FNCSs) is the counterpart of the FNOSs found in FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$.

Definition 2.5. [5]: Consider $\omega_{\mathcal{H}_1} = \left\{< \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n)\right\rangle : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\right\}$

is FNS in $\mathcal{F}_{\mathcal{H}}$, and let $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ be FNTS. Next, we construct the fuzzy neutrosophic-interior (FNInt) fuzzy neutrosophic-closure (FNCl) and fuzzy neutrosophic--boundary ($FNBd$) of $\omega_{\mathcal{H}_1}$as follows:

i. $FNInt\left(\omega_{\mathcal{H}_1}\right) = U\{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}$ is $FNOS$ -in $\mathcal{F}_{\mathcal{H}}$ and $\varphi_{\mathcal{H}_1} \subseteq \lambda_{N1}\}$.

ii. $FNCl\left(\omega_{\mathcal{H}_1}\right) = \cap \left\{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}\right.$ is FNCS in $\mathcal{F}_{\mathcal{H}}$ and $\left.\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}\right\}$,

iii. $FNBd(\omega_{\mathcal{H}_1}) = FNCl(\omega_{\mathcal{H}_1}) \setminus FNInt(\omega_{\mathcal{H}_1})$.

Moreover, if $FNInt(\omega_{\mathcal{H}})$ represents the FNOS and $FNCl(\omega_{\mathcal{H}})$ represents the FNCS in $\mathcal{F}_{\mathcal{H}}$ respectively, Additionally,

i. $\omega_{\mathcal{H}_1}$ is FNOS in F_N iff $FNInt\left(\omega_{\mathcal{H}_1}\right) = \omega_{\mathcal{H}_1}$.

ii. $\omega_{\mathcal{H}_1}$ is FNCS in F_N iff $FNCl\left(\omega_{\mathcal{H}_1}\right) = \omega_{\mathcal{H}_1}$,

Proposition 2.6. [6]: Suppose $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$ be FNSs in $\mathcal{F}_{\mathcal{H}}$ and $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ be FNTS. In that case, the next characteristics are valid:

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i. $\mathrm{FNInt}(1_{\mathcal{H}_1}) = 1_{\mathcal{H}_1}$ and $\mathrm{FNCl}(0_{\mathcal{H}_1}) = 0_{\mathcal{H}_1}$.

ii. $\mathrm{FNInt}\,(\omega_{\mathcal{H}_1}) \subseteq \omega_{\mathcal{H}_1}$ and $\omega_{\mathcal{H}_1} \subseteq \mathrm{FNCl}(\omega_{\mathcal{H}_1})$,

iii. $\mathrm{FNCl}\left(FNCl(\omega_{\mathcal{H}_1})\right) = FNCl(\omega_{\mathcal{H}_1})$, and $\mathrm{FNInt}\left(\mathrm{FNInt}(\omega_{\mathcal{H}_1})\right) = \mathrm{FNInt}(\omega_{\mathcal{H}_1})$,

iv. $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1} \Rightarrow \mathrm{FNCl}(\omega_{\mathcal{H}_1}) \subseteq \mathrm{FNCl}(\varphi_{\mathcal{H}_1})$ and, $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1} \Rightarrow \mathrm{FNInt}(\omega_{\mathcal{H}_1}) \subseteq \mathrm{FNInt}(\varphi_{\mathcal{H}_1})$,

v. $FNCl(\omega_{\mathcal{H}_1} \cup \varphi_{\mathcal{H}_1}) = FNCl(\omega_{\mathcal{H}_1}) \cup FNCl(\varphi_{\mathcal{H}_1})$, and $\mathrm{FNInt}\,(\omega_{\mathcal{H}_1} \cap \varphi_{\mathcal{H}_1}) = \mathrm{FNInt}(\omega_{\mathcal{H}_1}) \cap \mathrm{FNInt}(\varphi_{\mathcal{H}_1})$.

Definition 2.7. [7]: In FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$, FNS $\omega_{\mathcal{H}_1}$ is referred to as:

i. In the event when $\mathrm{FNCl}\left(\mathrm{FNInt}\left(\mathrm{FNCl}(\omega_{\mathcal{H}_1})\right)\right) \subseteq \omega_{\mathcal{H}_1}$, the fuzzy neutrosophic α-closed set (FN − α − CS) is created.

ii. The FN$-\alpha^g$ −CS is defined as $FNCl(\mathcal{D}) \subseteq$ U whenever $\mathcal{D} \subseteq$ U and U represents FN-α- OS.

iii. If (FNCl $(\omega_{\mathcal{H}_1})) \subseteq G_{\mathcal{H}}$, where $\omega_{\mathcal{H}_1} \subseteq G_{\mathcal{H}}$ and $G_{\mathcal{H}}$ is FNαOS, then $\omega_{\mathcal{H}_1}$ represents a FN$-\alpha^m$ −CS.

Definition 2.8. [15]: A $FNTS$ $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ has a subset φ, which is known as:

If $\mathrm{FN} - \mathrm{int}(\varphi) = 0_{\mathcal{H}}$, where φ is FNCS, then φ is FN-Q^*CS.

3. Fuzzy Neutrosophic P^*-closed sets

In this part, we will explore a new type of sets named fuzzy neutrosophic P^*-closed sets.

Definition 3. 1. A subset φ of a given $FNTS$ $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ is called P^*-closed sets

whenever the next two criteria exist:

1. $FNInt(\varphi) = FNCl(\varphi) \setminus FN - bd(\varphi)$
2. $FNCl(\varphi) = FNCl(\mathcal{M}) \cup FNCl(\mathcal{N})$.

Example 3.2. Letting $\mathcal{K} = \{\mathcal{J}_n\}$ construct the FNSs, and the sets $\mathcal{M}$ and $\mathcal{N}$ subsequently:

$$\mathcal{M} = \{\mathcal{J}_n(0.7,0.6,0.5): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}\ \mathcal{N} = \{\mathcal{J}_n(0.8,0.9,0.4): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$$

Considering the family $\tau_{\mathcal{H}} = \{0_{\mathcal{H}}, 1_{\mathcal{H}}, \mathcal{M}, \mathcal{N}\}$ and $\mathcal{M}$ with $\mathcal{N}$ are subsets of FNS $\mathcal{F}_{\mathcal{H}}$.Then, Set $\mathcal{M}$ has $FNInt(\mathcal{M}) = 0_{\mathcal{H}}$ and $\mathrm{FNCl}(\mathcal{M}) = 1_{\mathcal{H}}$. As a result, the distinction between the closure with the boundary yields the interior for $\mathcal{M}$. Also, $\mathrm{FNCl}(\mathcal{M}) = 1_{\mathcal{H}} = \mathrm{FNCl}(\mathcal{N})$ Since $\mathcal{M} \cup \mathcal{N} = \varphi$, thus: $FNCl(\varphi) = FNCl(\mathcal{M} \cup \mathcal{N}) = FNCl(\mathcal{M}) \cup FNCl(\mathcal{N}) = 1_{\mathcal{H}} \cup 1_{\mathcal{H}} = 1_{\mathcal{H}}$. Therefore, the above definition conditions are met.

Definition 3.3. Assume $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ is FNTS and $\omega_{\mathcal{H}_1} = \langle \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) \rangle$ is FNS in $\mathcal{F}_{\mathcal{H}}$. Then a FN-$P^*Cl(\omega_{\mathcal{H}_1})$ and FN-$P^*Int(\omega_{\mathcal{H}_1})$ is expressed as:

i. $\mathrm{FN} - P^*\mathrm{Cl}(\omega_{\mathcal{H}_1}) = \cap\{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}$ is $\mathrm{FN} - P^*CS$ in $\mathcal{F}_{\mathcal{H}}$ and $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}\}$,

ii. $\mathrm{FN}-P^*\mathrm{Int}(\omega_{\mathcal{H}_1}) = U\{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}$ is FN- P^*OS in $\mathcal{F}_{\mathcal{H}}$ and $\varphi_{\mathcal{H}_1} \subseteq \omega_{\mathcal{H}_1}\}$.

Theorem 3.4. Consider a FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$, and let $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$ be FNSs.The characteristics listed below are true:

1. $FN - P^*\mathrm{Cl}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$ and $FN - P^*\mathrm{Cl}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$.
2. For $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$, subsequently $\mathrm{FN}-P^*\mathrm{Cl}(\omega_{\mathcal{H}_1}) \subseteq FN - P^*\mathrm{Cl}(\varphi_{\mathcal{H}_1})$.
3. $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ is FN-P^*CS in $\mathcal{F}_{\mathcal{H}}$ if and only if $\mathrm{FN}-P^*Cl\,(\omega_{\mathcal{H}_1}) = FN - P^*\mathrm{Cl}(\varphi_{\mathcal{H}_1})$.
4. FN- $P^*\mathrm{Cl}(\omega_{\mathcal{H}_1}) = \mathrm{FN} - \mathrm{Cl}\left(\mathrm{FN} - P^*\mathrm{Cl}(\omega_{\mathcal{H}_1})\right)$.

Proof:

1. By Definition 3.1 we know the following:

$\mathrm{FN}-P^*\mathrm{Cl}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$ and $\mathrm{FN}-P^*\mathrm{Cl}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$.In fuzzy neutrosophic structures, we can determine from the closures attributes that: $\mathrm{FN}-P^*\mathrm{Cl}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$, because $0_{\mathcal{H}}$ represents neutrosophic space's minimum value and $\mathrm{FN}-P^*\mathrm{Cl}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$because $1_{\mathcal{H}}$ represents neutrosophic space's maximum value. Consequently, the first criteria are met.

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2. For $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$, the characteristics of closures then give us: $\text{FN}-P^*\text{Cl}(\omega_{\mathcal{H}_1}) \subseteq FN - P^*\text{Cl}(\varphi_{\mathcal{H}_1})$. This is due to the fact that the closure for a bigger set always contains the closures of the smaller set (especially in regard to the neutrosophic space).Consequently, Requirement 2 is met.
3. We have to demonstrate that when $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ is $\text{FN}-P^*\text{CS}$, subsequently their closures have to be matching: Assuming that $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ is FN-P^*CS in $\mathcal{F}_{\mathcal{H}}$, we can say that: $\text{FNInt}(\omega_{\mathcal{H}_1}) = 0_{\mathcal{H}}$ and $\text{FNCl}(\omega_{\mathcal{H}_1}) = 1_{\mathcal{H}}$. Likewise, for $\varphi_{\mathcal{H}_1}$ it is known that: $\text{FNInt}(\varphi_{\mathcal{H}_1}) = 0_{\mathcal{H}}$ and $\text{FNCl}(\varphi_{\mathcal{H}_1}) = 1_{\mathcal{H}}$.Consequently, because the whole space $1_{\mathcal{H}}$ is the closure for both sets, we have $\text{FNCl}(\omega_{\mathcal{H}_1}) = \text{FNCl}(\varphi_{\mathcal{H}_1})$. Thus, requirement 3 is met as the closures are equivalent.

In the other direction:

Claim now that $\text{FNCl}(\omega_{\mathcal{H}_1}) = \text{FNCl}(\varphi_{\mathcal{H}_1})$.Given that the closures are the same, the following is true: $\text{FNInt}(\omega_{\mathcal{H}_1}) = \text{FNInt}(\varphi_{\mathcal{H}_1})$. As a result, both sets are by condition FN-P^*CS, and the whole space is their closure. As a result, requirement 3 is also met in the opposite way.

4. Via, (3), we obtain, $\omega_{\mathcal{H}_1} = \text{FN} - P^*\text{Cl}(\omega_{\mathcal{H}_1})$. Therefore, FN- $P^*\text{Cl}(\omega_{\mathcal{H}_1}) = \text{FN} - \text{Cl}\left(\text{FN} - P^*\text{Cl}(\omega_{\mathcal{H}_1})\right)$. $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$.

Theorem 3.5. Consider a FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$, and let $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$ be FNSs.The characteristics listed below are true:

1. $\text{FN} - P^*\text{Int}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$ and $\text{FN} - P^*\text{Int}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$.
2. For $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$, subsequently $\text{FN}-P^*\text{Int}(\omega_{\mathcal{H}_1}) \subseteq FN - P^*\text{Int}(\varphi_{\mathcal{H}_1})$.
3. $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ is FN-P^*OS in $\mathcal{F}_{\mathcal{H}}$ if and only if $\text{FN}-P^*\text{Int}\left(\omega_{\mathcal{H}_1}\right) = \text{FN} - \text{P}^*\text{Int}(\text{FN} - P^*(\omega_{\mathcal{H}_1}))$.
4. FN- $P^*\text{Int}(\omega_{\mathcal{H}_1}) = \text{FN} - P^*\text{Int}(\text{FN} - P^*\text{Int}(\omega_{\mathcal{H}_1}))$.

Proof:

1. Definition 3.1 gives us:$\text{FN} - P^*\text{Int}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$ with $\text{FN} - P^*\text{Int}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$ are the following:

Since $0_{\mathcal{H}}$ is the minimum item found in fuzzy neutrosophic space, it has no open sets within it save itself, so it's the interior of $\text{FN} - P^*$ is $0_{\mathcal{H}}$. Likewise, since $1_{\mathcal{H}}$ is the largest item found in fuzzy neutrosophic space that is, it is capable of containing itself as an open set its the interior of $\text{FN} - P^*$interior is $1_{\mathcal{H}}$.

2. Considering that $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$ and that set inclusion is preserved by the interior of $\text{FN} - P^*$ operation, we obtain $\text{FN}-P^*\text{Int}(\omega_{\mathcal{H}_1}) \subseteq FN - P^*\text{Int}(\varphi_{\mathcal{H}_1})$. This is the result of the fact that the interior for a larger set always contains the interior for a smaller set.
3. It is assumed that $\omega_{\mathcal{H}_1}$ is $\text{FN} - \text{P}^*\text{OS}$.It is known that the interior of $\text{FN}-P^*$ represents the biggest $\text{FN}-P^*\text{OS}$ inside $\omega_{\mathcal{H}_1}$; hence, if $\omega_{\mathcal{H}_1}$is open, the interior must represent the complete set $\omega_{\mathcal{H}_1}$, and $\text{FN}-P^*\text{Int}(\omega_{\mathcal{H}_1}) = \omega_{\mathcal{H}_1}$ and $FN - P^*\text{Int}(\omega_{\mathcal{H}_1}) = \omega_{\mathcal{H}_1}$. Therefore, $\text{FN} - \text{P}^*\text{Int}(\text{FN} - P^*(\omega_{\mathcal{H}_1}))$, must likewise be $\omega_{\mathcal{H}_1}$. We deduce that:

$$\text{FN}-P^*\text{Int}\left(\omega_{\mathcal{H}_1}\right) = \text{FN} - \text{P}^*\text{Int}(\text{FN} - P^*(\omega_{\mathcal{H}_1})).$$

In the other direction:

Assume that $\text{FN}-P^*\text{Int}\left(\omega_{\mathcal{H}_1}\right) = \text{FN} - \text{P}^*\text{Int}(\text{FN} - P^*(\omega_{\mathcal{H}_1}))$. It follows that $\text{FN} - \text{P}^*\text{Int}(\omega_{\mathcal{H}_1}) = \omega_{\mathcal{H}_1}$, because the interior of a $\text{FN}-P^*$set equals the largest $\text{FN}-P^*\text{OS}$-inside the set, so the interiors of its $\text{FN}-P^*$ closure equals the set itself. Consequently, $\omega_{\mathcal{H}_1}$ needs to be $\text{FN}-P^*\text{OS}$.

4. Via, (3). We obtain, $\omega_{\mathcal{H}_1} = FN - P^*\text{Int}(\omega_{\mathcal{H}_1})$. Therefore, FN- $P^*\text{Int}(\omega_{\mathcal{H}_1}) = \text{FN} - P^*\text{Int}(\text{FN} - P^*\text{Int}(\omega_{\mathcal{H}_1}))$.

Proposition 3.6. Each $\text{FN} - \text{P}^*\text{CS}$ are FNCS.

Proof: Suppose φ be a $\text{FN} - \text{P}^*\text{CS}$. Thus, from definition:

$\text{FNInt}(\varphi) = \text{FNCl}(\varphi) \setminus \text{FN} - \text{bd}(\varphi)$ fulfills criterion 1. According to this formula the portion of $\omega_{\mathcal{H}_1}$ that

doesn't fall on its boundary represents its interior. The second requirement is that $FNCl(\varphi) = FNCl(\omega) \cup FNCl(\omega)$, indicating that φ is constructed from a combination of all the closure of other sets ω and ω.

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Given that $\mathrm{FNCl}(\varphi) = 1_{\mathcal{H}}$, we may conclude that φ must contain all of its limit points because no point in the fuzzy neutrosophic space may be found outside of its closure. As a result, φ is FNCS.

Remark 3.7. The revers above does not true. Here's an illustration of how this situation works.

Example 3.8. Consider $\mathcal{F}_{\mathcal{H}} = \{\mathcal{J}_n\}$. Characterize FNSs $\omega_{1\mathcal{H}}, \omega_{2\mathcal{H}}, \omega_{3\mathcal{H}}, \omega_{4\mathcal{H}}$

In $\mathcal{F}_{\mathcal{H}}$ as follows:

- $\omega_{1\mathcal{H}} = \{\mathcal{J}_n(0.2,0.5,0.7): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\omega_{2\mathcal{H}} = \{\mathcal{J}_n(0.4,0.3,0.6): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\omega_{3\mathcal{H}} = \{\mathcal{J}_n(0.1,0.4,0.8): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\omega_{4\mathcal{H}} = \{\mathcal{J}_n(0.3,0.6,0.9): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$

And the family $\tau_{\mathcal{H}} = \{0_{\mathcal{H}}, 1_{\mathcal{H}}, \omega_{1\mathcal{H}}, \omega_{2\mathcal{H}}, \omega_{3\mathcal{H}}, \omega_{4\mathcal{H}}\}$ be FNTS. Thus, $\mathrm{Int}({\omega_{1\mathcal{H}}}^c) = \{\mathcal{J}_n(0.3,0.5,0.6)\} \neq$

$0_{\mathcal{H}}$. Therefore, ${\omega_{1\mathcal{H}}}^c$is a FNCS but not $\mathrm{FN} - \mathrm{P}^*\mathrm{CS}$.

Proposition 3.9. For any FNTS, the following statements satisfy:

i. Each FN-P^*CS is FN$-\alpha^m$ $-$CS.

ii. Each FN$-\alpha^g$ $-$CS is FN-P^*CS.

Proof:

i. Assume that φ is FN-P^*CS. Consequently, $\mathrm{FNInt}\,(\omega_{\mathcal{H}_1}) = 0_{\mathcal{H}}$. The formula is , $\mathrm{FNCl}\,(\varphi) = 1_{\mathcal{H}}$.The circumstance for FN$-\alpha^m$ $-$CS is now consequently met because the interior represents empty and the closing is the entire space: $\mathrm{FNInt}(\mathrm{F\,NCl}(\omega_{\mathcal{H}_1})) = \mathrm{FNIint}(1_{\mathcal{H}}) = 0_{\mathcal{H}} \subseteq \mathrm{G}_{\mathcal{H}}$, where $\mathrm{G}_{\mathcal{H}}$ represents a FN-α-OS, as well as the circumstance is trivially.

ii. Assume that φ is a FN$-\alpha^g$ $-$CS. By definition, we have the following for any

FN$-\alpha^g$ $-$OS $\mathrm{G}_{\mathcal{H}}$ containing φ such that, $\mathrm{FNCl}(\varphi) \subseteq \mathrm{G}_{\mathcal{H}}$. This implies that each FN$-\alpha^g$ $-$OS $\mathrm{G}_{\mathcal{H}}$ contains the closure of φ, $\mathrm{FNCl}(\varphi)$. In order to demonstrate that φ is FN-P^*CS, we must first demonstrate that $\mathrm{FNInt}\,(\mathrm{FNCl}\,(\varphi)) = 0_{\mathcal{H}}$.Because $\mathrm{FNCl}(\varphi)$ is contained within any $\mathrm{FN} - \alpha - \mathrm{OS}$ that contains φ, no point in $\mathrm{FNCl}(\varphi)$ may be in the interior of a $\mathrm{FN} - \alpha - \mathrm{OS}$ without passing through its border. The interior of $\mathrm{FNCl}(\varphi)$ must thus be empty. Consequently, φ is FN-P^*CS.

Remark 3.10. It is not true that proposition 3.8 has been converted. This is demonstrated in an instance that follows.

Example 3.11

1. Consider $\mathcal{F}_{\mathcal{H}} = \{\mathcal{J}_n\}$, The following is how to establish the FNSs $\varphi_{1\mathcal{H}}, \varphi_{2\mathcal{H}}, \varphi_{3\mathcal{H}}$ in $\mathcal{F}_{\mathcal{H}}$:

- $\varphi_{1\mathcal{H}} = \{\mathcal{J}_n(0.3,0.5,0.7): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\varphi_{2\mathcal{H}} = \{\mathcal{J}_n(0.6,0.7,0.8): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\varphi_{3\mathcal{H}} = \{\mathcal{J}_n(0.4,0.6,0.9): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$

And the family $\tau_{\mathcal{H}} = \{0_{\mathcal{H}}, 1_{\mathcal{H}}, \omega_{1\mathcal{H}}, \omega_{2\mathcal{H}}, \omega_{3\mathcal{H}}\}$ be FNTS.

Assuming $\varphi = \varphi_{2\mathcal{H}}$, thus $\mathrm{FNCl}(\varphi_{2\mathcal{H}}) = 1_{\mathcal{H}}$, the entire space serves as the closing.

Hence, since, $\mathrm{FNInt}(\mathrm{FNCl}(\varphi_{2\mathcal{H}})) \subseteq \mathrm{G}_{\mathcal{H}}$, where $\mathrm{G}_{\mathcal{H}}$ is a FN-α-OS, the FN$-\alpha^m$ $-$CS requirement is met. But $\omega_{\mathcal{H}_1}$not satisfied $\mathrm{FN} - P^*\mathrm{CS}$ because $\mathrm{FNInt}(\varphi_{2\mathcal{H}}) \neq \emptyset$. Therefore, φ is FN$-\alpha^m$ $-$CS but not $\mathrm{FN} - P^*\mathrm{CS}$.

2. Consider $\mathcal{F}_{\mathcal{H}} = \{\mathcal{J}_n\}$, The following is how to establish the FNSs $\varphi_{1\mathcal{H}}, \varphi_{2\mathcal{H}}, \varphi_{3\mathcal{H}}$ in $\mathcal{F}_{\mathcal{H}}$:

- $\varphi_{1\mathcal{H}} = \{\mathcal{J}_n(0.3,0.5,0.7): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\varphi_{2\mathcal{H}} = \{\mathcal{J}_n(0.6,0.7,0.8): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\varphi_{3\mathcal{H}} = \{\mathcal{J}_n(0.4,0.6,0.9): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\varphi_{3\mathcal{H}} = \{\mathcal{J}_n(0.2,0.6,0.7): \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$

And the family $\tau_{\mathcal{H}} = \{0_{\mathcal{H}}, 1_{\mathcal{H}}, \omega_{1\mathcal{H}}, \omega_{2\mathcal{H}}, \omega_{3\mathcal{H}}, \omega_{4\mathcal{H}}\}$ be FNTS.

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Assuming $\varphi = \varphi_{3\mathcal{H}}$, thus $\mathrm{FNCl}(\varphi_{3\mathcal{H}}) = 1_{\mathcal{H}}$, and $\mathrm{FNInt}(\varphi_{3\mathcal{H}}) = 0_{\mathcal{H}}$.

Since, $\mathrm{FNCl}(\varphi_{3\mathcal{H}}) = 1_{\mathcal{H}}$ and $\mathrm{FNInt}(\varphi_{3\mathcal{H}}) = 0_{\mathcal{H}}$, the FN-$P^{*}$CS condition is met. Nevertheless, even if $\mathrm{FNCl}(\varphi_{3\mathcal{H}}) = 1_{\mathcal{H}}$ is FN−α −OS, it does not always lie within every FN−α −OS like $G_{\mathcal{H}}$, thus the FN$-\alpha^{g}$ −CS requirement is not met.

Proposition 3.12. Suppose that $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ is a FNTS. If every $G_{\mathcal{H}}$ represents an FN $- P^{*}$CS that includes $G_{\mathcal{H}} \subseteq \varphi$ and $G_{\mathcal{H}} \subseteq \mathrm{FNInt}(\varphi)$, then φ is a FN $- P^{*}$OS in $\mathcal{F}_{\mathcal{H}}$.

Proof: Consider φ to be FN $- P^{*}$OS . For every FN − P*CS $G_{\mathcal{H}}$, through definition 3.1, such that $G_{\mathcal{H}} \subseteq \varphi$ and $G_{\mathcal{H}} \subseteq FNInt(\varphi)$. We must demonstrate that: $\mathrm{FNCl}(G_{\mathcal{H}}) \subseteq FN - cl(\varphi)$. The complement of φ within $\mathcal{F}_{\mathcal{H}}$ is $1_{\mathcal{H}-\varphi} \subseteq \mathrm{FNInt}(\varphi)$. Select $G_{\mathcal{H}} = 1_{\mathcal{H}-\varphi}$. Since $1_{\mathcal{H}-\varphi}$ is FN $- P^{*}$CS, by the closure properties of FN $- P^{*}$CS, we have: $\mathrm{FNCl}(1_{\mathcal{H}-\varphi}) \subseteq FNCl(\varphi)$. Given that $\mathrm{FNCl}(1_{\mathcal{H}-\varphi}) \subseteq \mathrm{FNCl}(\varphi)$ demonstrates that for each FN $- P^{*}$CS $G_{\mathcal{H}}$ fulfilling $G_{\mathcal{H}} \subseteq \varphi$ and $G_{\mathcal{H}} \subseteq \mathrm{FNInt}(\varphi)$, we include the following: $\mathrm{FNCl}(G_{\mathcal{H}}) \subseteq \mathrm{FNCl}(\varphi)$. Therefore, we have demonstrated the first way that the closure requirement occurs if φ is FN $- P^{*}$OS .

Conversely, assuming now that the following is true for any FN $- P^{*}$CS $G_{\mathcal{H}}$ that includes $G_{\mathcal{H}} \subseteq \varphi$ and $U \subseteq \mathrm{FNInt}(\varphi)$. We possess $\mathrm{FNCl}(G_{\mathcal{H}}) \subseteq \mathrm{FNCl}(\varphi)$.Now, φ must be shown to be FN − P^{*}OS.It must be demonstrated that for each FN $- P^{*}$CS $G_{\mathcal{H}}$ such that $G_{\mathcal{H}} \subseteq \varphi$ and $G_{\mathcal{H}} \subseteq \mathrm{FNInt}(\varphi)$, we include the following: $\mathrm{FNCl}(G_{\mathcal{H}}) \subseteq \mathrm{FNCl}(\varphi)$. Because we claimed that the closure requirement holds for all such sets $G_{\mathcal{H}}$, we infer that the closure requirement holds with all FN $- P^{*}$CS $G_{\mathcal{H}}$. Thus, φ represents FN $- P^{*}$OS.

Theorem 3.13. Each FN − Q*CS is FN − P*CS .

Proof: Suppose φ is FN − Q*CS in line with Definition 2.8. This implies $\mathrm{FNInt}(\varphi) = 0_{\mathcal{H}}$ (φ has an empty interior.), The fact that φ is FNCS indicates that $\mathrm{FNCl}(\varphi) = 1_{\mathcal{H}}$ (The whole space is the closure of φ). Thus, $FNInt(\varphi) = FNCl(\varphi) \setminus FN - bd(\varphi) = 0_{\mathcal{H}}$, also $FNCl(\varphi) = 1_{\mathcal{H}} = 1_{\mathcal{H}} \cup 1_{\mathcal{H}} = FNCl(\mathcal{M}) \cup FNCl(\mathcal{N})$.Therefore, φ is FN − P*CS.

Remark 3.14. The reverse above not true as shown illustration.

4. Fuzzy Neutrosophic P^{*}-generalized closed sets

This section will examine fuzzy neutrosophic P^{*}-generalized closed sets, a novel class of sets.

Definition 4.1. Assume that $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ is a FNTS. For FN $- P^{*}\mathrm{Cl}(\varphi)$ wherever, φ is FNOS, then φ is defined as fuzzy neutrosophic - P^{*}-generalized-closed set (FN- P*GCS). It is believed that φ is a fuzzy neutrosophic-P^{*}open set (FN − P*GOS), if the complement $1_{\mathcal{H}} - \varphi$ is FN-P^{*}GCS.

Theorem 4.2. Consider the FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$. A FNS φ is FN $- P^{*}$GCS iff, for each set $G_{\mathcal{H}} \subseteq \varphi$,

where $G_{\mathcal{H}}$ is FN $- P^{*}$GOS, 1_{φ} (the complement of φ) is FN-P^{*}GCS in $\mathcal{F}_{\mathcal{H}}$.

Proof: In $\mathcal{F}_{\mathcal{H}}$, consider φ be FN $- P^{*}$GOS, and allow $G_{\mathcal{H}}$ be any FN $- P^{*}$GCS such that $G_{\mathcal{H}} \subseteq \varphi$.

Because, φ is FN $- P^{*}$GOS, thus, $\mathrm{FNInt}(\varphi) = 0_{\mathcal{H}}$ and $\mathrm{FNCl}(\varphi) = 1_{\mathcal{H}}$. The complement for φ, 1_{φ}, must be demonstrated to be FN $- P^{*}$GCS. According to FN $- P^{*}$GCS characteristics, 1_{φ} must meet. The interior of 1_{φ} is empty, as indicated by the equation $\mathrm{FNInt}(1_{\varphi}) = 0_{\mathcal{H}}$.

$\mathrm{FNCl}(1_{\varphi}) = 1_{\mathcal{H}}$, indicating that the whole space is the closure of 1_{φ}.Because φ takes up the full space, the complement $1_{\mathcal{H}}$ has no interior points since $\mathrm{FNCl}(1_{\varphi}) = 1_{\mathcal{H}}$. Consequently,

$\mathrm{FNInt}(1_{\varphi}) = 0_{\mathcal{H}}$. Consequently, 1_{φ} meets both requirements to be FN $- P^{*}$GCS. $\mathrm{FNInt}(1_{\varphi}) = 0_{\mathcal{H}}$ and $\mathrm{FNCl}(1_{\varphi}) = 1_{\mathcal{H}}$. Therefore, φ is FN-$P^{*}$CS iff the complement 1_{φ} is FN-P^{*}Gfor FN $- P^{*}$GOS for every $G_{\mathcal{H}} \subseteq \varphi$.

Definition 4.3. Assume $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$ is FNTS and $\omega_{\mathcal{H}_1} = \langle \mathcal{J}_n, \mu_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), \sigma_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n), v_{\omega_{\mathcal{H}_1}}(\mathcal{J}_n) \rangle$ is FNS in $\mathcal{F}_{\mathcal{H}}$.

Then a FN-$P^{*}GCl(\omega_{\mathcal{H}_1})$ and FN-$P^{*}GInt(\omega_{\mathcal{H}_1})$ is expressed as:

i. FN $- P^{*}\mathrm{GCl}(\omega_{\mathcal{H}_1}) = \cap \{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}$ is FN $- P^{*}GCS$ in $\mathcal{F}_{\mathcal{H}}$ and $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}\}$,

ii. FN$-P^{*}\mathrm{GInt}(\omega_{\mathcal{H}_1}) = U\{\varphi_{\mathcal{H}_1} : \varphi_{\mathcal{H}_1}$ is FN- $P^{*}G$OS in $\mathcal{F}_{\mathcal{H}}$ and $\varphi_{\mathcal{H}_1} \subseteq \omega_{\mathcal{H}_1}\}$.

Theorem 4.4. Consider a FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$, and let $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$ the operators FN-$P^{*}GCl$ fulfills the following assertion for every $\omega_{\mathcal{H}_1}$ and $\varphi_{\mathcal{H}_1}$:

1. $FN - P^{*}\mathrm{GCl}(0_{\mathcal{H}}) = 0_{\mathcal{H}}$ and $FN - P^{*}\mathrm{GCl}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$.
2. For $\omega_{\mathcal{H}_1} \subseteq \varphi_{\mathcal{H}_1}$, subsequently FN$-P^{*}\mathrm{GCl}(\omega_{\mathcal{H}_1}) \subseteq FN - P^{*}\mathrm{GCl}(\varphi_{\mathcal{H}_1})$.

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3. Whenever $\omega_{\mathcal{H}_1}$ is FN-P^*GCS in $\mathcal{F}_{\mathcal{H}}$ then FN$-P^*GCl\left(\omega_{\mathcal{H}_1}\right) = \omega_{\mathcal{H}_1}$.

4. FN- $P^*\mathrm{GCl}(\omega_{\mathcal{H}_1}) = \mathrm{FN} - \mathrm{Cl}\left(\mathrm{FN} - P^*\mathrm{GCl}(\omega_{\mathcal{H}_1})\right)$.

5. $\mathrm{FN} - P^*\mathrm{GCl}(\omega_{\mathcal{H}_1}) \subseteq FN - P^*\mathrm{Cl}(\omega_{\mathcal{H}_1}) \subseteq \mathrm{Cl}(\omega_{\mathcal{H}_1})$.

Proof: Obvious.

Theorem 4.5. Assuming that a FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$, the operator $\mathrm{FN} - \mathrm{P^*GInt}$ satisfies the following assertion for each $\omega_{\mathcal{H}_1}, \varphi_{\mathcal{H}_1} \in I^X$:

(i) $\mathrm{FN} - \mathrm{P^*GInt}(0_{\mathcal{H}}) = 0_{\mathcal{H}}, \mathrm{FN} - \mathrm{P^*GInt}(1_{\mathcal{H}}) = 1_{\mathcal{H}}$,

(ii) $\mathrm{FN} - \mathrm{P^*GInt}(\omega_{\mathcal{H}_1}) \subseteq \omega_{\mathcal{H}_1}$,

(iii) $\mathrm{FN} - \mathrm{P^*GInt}(\omega_{\mathcal{H}_1} \cap \varphi_{\mathcal{H}_1}) = \mathrm{FN} - \mathrm{P^*GInt}(\omega_{\mathcal{H}_1}) \cap \mathrm{FN} - \mathrm{P^*GInt}(\varphi_{\mathcal{H}_1})$,

(iv) FN- $\mathrm{P^*GInt}(\omega_{\mathcal{H}_1}) = \mathrm{FN} - \mathrm{P^*GInt}\left(\mathrm{FN} - \mathrm{P^*GInt}(\omega_{\mathcal{H}_1})\right)$.

Proof: Obvious.

Theorem 4.6. Consider a FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$. When we have $\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) = 1_{\mathcal{H}}$ for each FN-P*GCS $\mathrm{G}_{\mathcal{H}}$ which includes $\mathrm{G}_{\mathcal{H}} \subseteq \varphi$, then a fuzzy neutrosophic set φ represents $\mathrm{FN} - \mathrm{P^*GOS}$.

Proof: Suppose φ is $\mathrm{FN} - \mathrm{P^*GOS}$. So, via the definition 4.1 of $\mathrm{FN} - \mathrm{P^*GOS}$, so that:

$\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) = 1_{\mathcal{H}}$ and $\mathrm{FNInt}(\mathrm{G}_{\mathcal{H}}) = 0_{\mathcal{H}}$. Let $\mathrm{G}_{\mathcal{H}}$ be any GCS that is FN-P*GCS such that $\mathrm{G}_{\mathcal{H}} \subseteq \varphi$. We must demonstrate that $\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) = 1_{\mathcal{H}}$. The closure for $\mathrm{G}_{\mathcal{H}}$ must equal the closure of φ, resulting in the entire space $1_{\mathcal{H}}$, as $\mathrm{G}_{\mathcal{H}} \subseteq \varphi$ and φ is FN-P*GOS. Thus, $\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) \subseteq \mathrm{FNCl}(\varphi) = 1_{\mathcal{H}}$. Consequently, the proof is completed in this direction: $\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) = 1_{\mathcal{H}}$.

Thus, $\mathrm{FNCl}(\mathrm{G}_{\mathcal{H}}) \subseteq \mathrm{FNCl}(\varphi) = 1_{\mathcal{H}}$, completing the proof in this direction. Let φ be FN-P*GOS. We must demonstrate that $\mathrm{FNGCl}(\varphi) = 1_{\mathcal{H}}$. We can infer from the premise that $\mathrm{FNGCl}(\varphi) = 1_{\mathcal{H}}$, for each $\mathrm{FN} - \mathrm{P^*GCS}$ U that is equal to $\mathrm{G}_{\mathcal{H}} \subseteq \varphi$.So, φ closure must be the full space $\mathcal{F}_{\mathcal{H}}$ since it represents the largest FN-P*GOS. Consequently, we deduce that φ is $\mathrm{FN} - \mathrm{P^*GOS}$, since $\mathrm{FNGCl}(\varphi) = 1_{\mathcal{H}}$.

Theorem 4.7. Each $\mathrm{FN} - \mathrm{P^*CS}$ is $\mathrm{FN} - \mathrm{P^*GCS}$.

Proof. Obvious.

Remark 4.8. The reverse above not true as shown illustration.

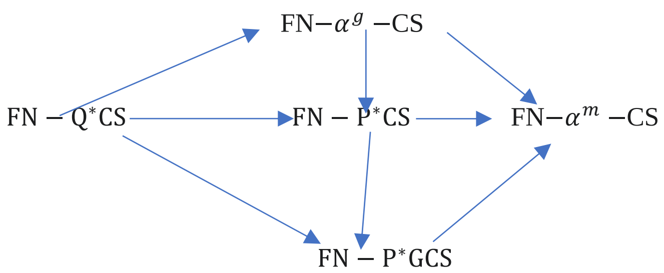
Example 4.9. Assume $\mathcal{F}_{\mathcal{H}} = \{\mathcal{J}_n\}$, put $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{F}_{\mathcal{H}}$ and construct FNSs as follows:

- $\mathcal{M} = \{\mathcal{J}_n(0.2,0.3,0.7) : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$
- $\mathcal{N} = \{\mathcal{J}_n(0.6,0.7,0.1) : \mathcal{J}_n \in \mathcal{F}_{\mathcal{H}}\}$

And the family $\tau_{\mathcal{H}} = \{0_{\mathcal{H}}, 1_{\mathcal{H}}, \mathcal{M}, \mathcal{N}\}$ be FNTS. Set $\varphi = \mathcal{M}$. Then $\mathrm{FN} - \mathrm{P^*GCS}$ demands that $\mathrm{FNGInt}(\mathrm{FNGCl}(\mathcal{M})) = \emptyset$. Because, the closure $\mathrm{FNGCl}(\mathcal{M})$ is $1_{\mathcal{H}}$. Because the interior of $1_{\mathcal{H}}$ is not empty, this condition fulfils, as $\mathrm{FNGInt}(\mathrm{FNGCl}(\mathcal{M})) = 0_{\mathcal{H}}$.Hence, $\varphi = \mathcal{M}$ *is* $\mathrm{FN} - \mathrm{P^*GCS}$.

Nevertheless, it cannot be included in any FNOS smaller than $1_{\mathcal{H}}$ since $\mathrm{FNCl}(\mathcal{M}) = 1_{\mathcal{H}}$. Therefore, the need to be included within each FNOS is not met by the closure for $\mathcal{M}$, resulting in the entire space $1_{\mathcal{H}}$. $\mathcal{M}$ is not $\mathrm{FN} - \mathrm{P^*CS}$ as a result.

Remark 4.10. Figure 1 below illustrates the relationship among several classes in FNTS $(\mathcal{F}_{\mathcal{H}}, \tau_{\mathcal{H}})$:



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5. Conclusion

This work introduces a new and sophisticated idea in fuzzy neutrosophic set theory: the FN- $P^{*}CS$. This unique concept expands the previous structure by combining fuzziness and neutrosophic concepts into closed set theory, resulting in a larger and more flexible method to dealing with ambiguity and indeterminacy in computational modeling. Moreover, this work investigates and establishes essential features within the FN- $P^{*}CS$ offering light on its behavior and prospective applications in fields where ambiguity and indeterminacy are common. These qualities provide more insight into how this particular set interacts in the context for fuzzy neutrosophic structures, making it more useful in real-world applications. Furthermore, the study explores the connection among the FN- $P^{*}CS$- along with other existing set categories in fuzzy neutrosophic topology. These comparisons assist to highlight the importance of this novel model in the wider field for fuzzy neutrosophic principle, as well as how it links to or differs from existing set frameworks, so adding to the overall advancement for fuzzy neutrosophic topology. The interaction of these sets demonstrates the flexibility and adaptation found in the FN- $P^{*}CS$ in both mathematical and practical situations.

References

[1] C. L. Chang, "Fuzzy Topological Spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182-190, 1968.

[2] F. Smarandache, "Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set," *Journal of Defense Resources Management (JoDRM)*, vol. 1, no. 1, pp. 107-116, 2010.

[3] A. A. Salama and S. A. Alblowi, "Neutrosophic Set and Neutrosophic Topological Spaces," *Neutrosophic Sets and Systems*, vol. 3, no. 4, pp. 31-35, 2012.

[4] I. Arockiarani and J. M. Jency, "More on Fuzzy Neutrosophic Sets and Fuzzy Neutrosophic Topological Spaces," *International Journal of Innovative Research and Studies*, vol. 3, no. 5, pp. 643-652, 2014.

[5] F. M. Mohammed and S. F. Matar, "Fuzzy Neutrosophic α^m -Closed Sets in Fuzzy Neutrosophic Topological Spaces," *Neutrosophic Sets and Systems*, vol. 21, pp. 56-66, 2018.

[6] A. Al-Quran, F. Al-Sharqi, A. U. Rahman, and Z. M. Rodzi, "The q-rung orthopair fuzzy-valued neutrosophic sets: Axiomatic properties, aggregation operators and applications," *AIMS Mathematics*, vol. 9, no. 2, pp. 5038-5070, 2024.

[7] Y. Veereswari, "An Introduction to Fuzzy Neutrosophic Topological Spaces," *International Journal of Mathematical Analysis (IJMA)*, vol. 8, pp. 144-149, 2017.

[8] R. P. Agarwal, S. Milles, B. Ziane, A. Mennouni, and L. Zedam, "Ideals and filters on neutrosophic topologies generated by neutrosophic relations," *Axioms*, vol. 13, no. 5, p. 292, 2024.

[9] C. Maheswari and S. Chandrasekar, "Neutrosophic gb-Closed Sets and Neutrosophic gb-Continuity," *Infinite Study*, 2020.

[10] S. R. Alavi and M. A. Zadeh, "Fuzzy Topological Structures and Their Applications," *Journal of Mathematical Analysis and Applications*, vol. 498, no. 1, pp. 123-135, 2021.

[11] R. Thomas and S. Anila, "On Neutrosophic Semi-Preopen Sets and Semi-Preclosed Sets in a Neutrosophic Topological Space," *International Journal of Scientific Research in Mathematical and Statistical Sciences*, vol. 5, no. 5, pp. 138-143, 2018.

[12] S. Narmatha, E. Glory Bebina, and R. Vishnu Priyaa, "On πgβ–Closed Sets and Mappings in Neutrosophic Topological Spaces," *International Journal of Innovative Technology and Exploring Engineering (IJITEE)*, vol. 8, no. 125, pp. 505-510, 2019.

[13] W. Al-Omeri and S. Jafari, "On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces," *Mathematics*, pp. 1–12, 2019.

[14] H. Y. Mohammed, F. M. Mohammed, and G. Al-mahbashi, "On Q* -Closed Sets in Fuzzy Neutrosophic Topology: Principles, Proofs, and Examples," *Neutrosophic Systems with Applications*, vol. 20, pp. 67-74, 2024.

[15] J. N. Ismail et al., "Enhancing Decision Accuracy in DEMATEL using Bonferroni Mean Aggregation under Pythagorean Neutrosophic Environment," *Journal of Fuzzy Extension and Applications (JFEA)*, vol. 4, no. 4, pp. 281-298, 2023.

[16] C. Maheswari and S. Chandrasekar, "Neutrosophic gb-Closed Sets and Neutrosophic gb-Continuity," *Infinite Study*, 2020.

DOI: https://doi.org/10.54216/IJNS.260311