Application of Neutrosophic Stratified Ranked Set Sampling: An Efficient Sampling Technique in the Estimation of Average Relative Humidity in USA

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Abstract

The study examined the shortcomings of conventional statistical techniques in managing unclear or ambiguous data and emphasized the necessity of implementing neutrosophic statistical techniques as a more enhanced remedy. Advanced techniques like neutrosophic statistics (NS) were developed since traditional statistical methods are unable to handle the uncertainty present in ambiguous data. In order to tackle this problem, the study suggested an innovative and novel sampling method called "neutrosophic stratified ranked set sampling (NSRSS)" in addition to specialized neutrosophic estimators for precisely predicting the population mean in the proximity of uncertainty. This novel strategy adjusted ranked set sampling (RSS) techniques to allow the special features of neutrosophic data. Furthermore, the study improved the precision of estimating the population mean in uncertain situations by introducing neutrosophic estimators that use subsidiary information inside the structure of stratified ranked set sampling (SRSS). The work provided theoretical insights into the performance of these estimators by presenting comprehensive formulations of bias and mean squared error (MSE). To illustrate the efficacy of the suggested techniques, the study includes simulation studies, numerical examples conducted using the computer language R. Evaluations utilizing MSE, and percentage relative efficiency (PRE) demonstrated the higher accuracy of the suggested estimators over conventional alternatives. The findings demonstrated the NSRSS's applicability, particularly for predicting population means in situations where heterogeneity and uncertainty are prevalent. Furthermore, it was demonstrated that the estimators and technique produced interval-based findings, which provided a more accurate depiction of the uncertainty related to population parameters. The reliability of the estimators in estimating population means was greatly improved by this interval estimation in combination with a lower MSE. A significant vacuum in the field of statistical research is filled by the study's introduction of estimators and a customized sampling approach made especially for neutrosophic data. This research significantly advances statistical theory and practice by extending traditional statistical approaches to efficiently handle ambiguous data, especially for applications where exact data is few, heterogeneous, or uncertain. The empirical validation through numerical illustrations and simulations conducted in R further solidifies the practicality and robustness of the proposed techniques, reinforcing their applicability to real-world scenarios.

Keywords: Mean squared error; Bias; Percentage relative efficiency; Neutrosophic statistics; Ranked Set Sampling; Study variable; Monte Carlo Simulation; Auxiliary variable

1. Introduction

Sampling plays a vital role in research due to its ability to address constraints related to cost, time, and feasibility. Improving the accuracy of estimators for population parameters while minimizing sampling errors is the main

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objective of sampling theory. A well-designed sampling approach is essential for obtaining meaningful and reliable insights into a broader population. When a sample is carefully selected, it effectively reflects the characteristics and diversity of the entire population, enabling researchers to make generalizable inferences. To increase estimators' efficiency, auxiliary information is frequently utilized, which can be implemented at different stages of the estimation process. For example, if we consider the primary variable as Y and an auxiliary variable as X, then cancer mortality could be the primary variable while dietary fat serves as the auxiliary.

Auxiliary data is typically available for every unit and is often pre-determined. When strongly correlated auxiliary data is not directly accessible, it can be derived from earlier research. Cochran[4] significantly contributed to modern sampling theory by proposing the use of subsidiary data, leading to the development of ratio estimators and product estimators by Murthy[11] that incorporate auxiliary variables. Many researchers have further enhanced sample surveys by utilizing auxiliary information. Sisodia and Dwivedi[17] modified ratio estimator, Upadhyaya and Singh[25] used transformations of auxiliary variables for population mean estimation, and further developments by Kadilar and Cingi[8], Singh et al.[15] are a few examples.

Stratified simple random sampling (SSRS) is used to improve accuracy for heterogeneous populations. For instance, stratified sampling, as opposed to simple random selection, which may misrepresent some segments, guarantees proportionate representation across income groups when examining the spending patterns of a city's citizens divided by income levels. The development of estimators for stratified sampling using subsidiary variables has been the subject of numerous studies. For example, estimators based on a single auxiliary variable were created by Kadilar and Cingi[7], Shabbir and Gupta[13], and Haq and Shabbir[6]. Muneer et al.[10] introduced a family of chain exponential estimators in the context of the SSRS framework in recent years.

Here, the focus is on RSS, a technique that seeks to improve productivity, cut expenses, simplify procedures, and save sample time. In several fields, such as healthcare, agriculture, geology, statistics, and mathematics, RSS has outperformed simple random sampling (SRS), especially when gathering data requires a large investment of time or money. McIntyre[9] first developed the RSS technique for population mean estimate, while Takahasi and Wakimoto[24] further its mathematical development. Dell and Clutter[5] demonstrated that RSS provides unbiased mean estimates under both ideal and non-ideal ranking scenarios.

RSS encompasses several variations, including double RSS, median RSS, and quartile RSS. Perfect ranking occurs when observations align with the actual values of the auxiliary variable, while imperfect ranking indicates some discrepancy. Many scholars have enhanced RSS using diverse approaches, including contributions by Stokes[22], Samawi and Muttlak[12], Bouza[1], Chen et al.[3], and several others. Their work has contributed to a wide range of estimation techniques applied across fields.

Samawi[13] introduced SRSS to improve the effectiveness of estimators. Subsequent advancements were made by Samawi and Siam[15], who developed combined and separate ratio estimators for SRSS.

Classical RSS assumes data precision without uncertainties, although real-world data often contain uncertainties or intervals. Fuzzy logic helps manage imprecise data, leading to fuzzy statistics for handling vague or uncertain characteristics. Neutrosophic (Nc) statistics, an extension of fuzzy logic, address both determined and indeterminate aspects, beneficial when data lacks precision or is unclear. Neutrosophic data examples include regional water level measurements, machinery size variations due to measurement errors, and interval-based temperature readings. Neutrosophy was introduced by Smarandache[19,20,21] and has since expanded in applications across sampling theory. Tahir et al.[23] contributed by presenting estimators of the neutrosophic ratio type. Despite its potential, the application of neutrosophic RSS (NRSS) to population parameter estimation has received little attention. The idea of NRSS along with generalized estimator for determining the population mean were presented by Vishwakarma and Singh[26].

The work done under NSRS and NRSS primarily addresses homogeneous neutrosophic data. However, we focus on extending SRSS to accommodate neutrosophic data for heterogeneous populations, proposing an enhanced NSRSS framework. In order to improve population mean estimation, our work involves creating new NSRSS estimators with a focus on lowering MSE and raising precision. This is especially pertinent for data characterized by uncertainty, vague intervals, or indeterminacy, common in real-world scenarios. Our research addresses the need for a stratified RSS framework when dealing with non-homogeneous imprecise data types.

Our study is organised as follows: An introduction is given in section 1, followed by discussions of the modified NSRSS method, needs, and research gaps in section 2 and 3. Section 4 provides the existing NSRSS estimators, section 5 details suggested enhanced NSRSS estimators, section 6 includes an empirical study with real data, section 7

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provides a simulation study with artificial data, and sections 8 and 9 cover discussions, conclusions, and future research directives.

2. Motivation, research gaps, and need

The primary aim of this article is to present an innovative method called "neutrosophic stratified ranked set sampling" (NSRSS) to address heterogeneous interval-type or Nc data. This research concentrates on sampling survey and represents the initial proposal of a stratified ranked set sampling approach specifically designed for neutrosophic data. Our aim also includes the development of NSRSS estimators for estimating population means. This is a pioneering effort in sampling theory, as we introduce SRSS for neutrosophic data, thereby laying the groundwork for further studies. Initially, we present NSRSS usual unbiased, ratio, product, exponential, and regression estimators, followed by some enhanced estimators for population mean estimation, which could encourage further research in sampling theory. By contrasting these estimators with well-known Nc techniques like ratio, product, exponential and regression estimators, this study makes a significant contribution to the expansion of sampling theory. SRSS is recognized as a preferable choice over SRS and RSS in cases where data are heterogeneous, making NSRSS a promising area for further research.

Our motivation for exploring NSRSS and its related estimators is driven by the need to introduce SRSS in a neutrosophic framework. Most prior work in survey sampling has dealt with precise, well-defined data, with classical methods producing accurate results, though sometimes with risks of over or under estimation. However, these traditional approaches are inadequate for handling indeterminate, vague, or interval-based data that represents Nc data, that is more common in real-life situations than clear-cut data. Consequently, there is a rising need for advanced NS. Conventional statistical techniques are often unsuitable for estimating unknown parameters in the presence of uncertain or set-type data. NS provide a suitable alternative in these situations. In cases where the population is heterogeneous, stratified sampling is preferable to SRS, as it allows for the division of the population into uniform groups or homogeneous strata, enhancing the precision of estimators used in parameter estimation. When a population has distinct subgroups with varied characteristics, stratified sampling enables a more precise representation of these differences. These considerations prompted us to investigate NSRSS and its estimators for accurate estimation of population parameters.

This study introduces the NSRSS technique and its enhanced estimators for estimation of population mean estimation to bridge the void between classical and NS. Despite extensive research in this area, we identified a gap in survey sampling literature addressing population mean estimation with auxiliary variables in a neutrosophic setting using SRSS. Our research seeks to fill this gap, contributing to the development of neutrosophic statistics. It is widely acknowledged by various authors that RSS is preferable to SRS while dealing with laborious, costly, or time-intensive measurements, and SRSS is even more effective than RSS when the data lacks homogeneity. Measurement challenges are further compounded in a neutrosophic context. Thus, our study introduces the NSRSS method to enhance the precision of population mean estimators in this complex setting.

3. Neutrosophic Stratified Ranked Set Sampling Methodology

There are various techniques available to represent neutrosophic observations, where the neutrosophic numbers might consist of an unspecified range denoted by [a, b]. Here, we define neutrosophic values as $Z_N = Z_L + Z_U I_N$ with $I_N \in [I_L, I_U]$, where N and I_N represents Nc number and the degree of indeterminacy respectively. As a result, our Nc data points will be confined within a specific range $Z_N \in [a, b]$, in this context, a and b represent the minimum and maximum bounds of the Nc data range.

The NRSS method involves choosing $m_N \in [m_L, m_U]$ bivariate random samples from a population of size N, where each sample has a size of $m_N \in [m_L, m_U]$. These samples are ranked based on the subsidiary variable $X_N \in [X_L, X_U]$, which is correlated with $Y_N \in [Y_L, Y_U]$. The ranking process is guided by the approach detailed in “Introduction to Neutrosophic Statistics” by Smarandache[20] “For ranking procedure, consider two sets: $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$. Their midpoints are computed as $X_{1midN} = [X_{1L} + X_{1U}]/2$ and $X_{2midN} = [X_{2L} + X_{2U}]/2$. The ranking is determined by comparing these midpoints: $X_{1N} \in [X_{1L}, X_{1U}]$ is ranked lower than $X_{2N} \in [X_{2L}, X_{2U}]$ if $X_{1midN} \leq X_{2midN}$. If the midpoints are equal $X_{1midN} = X_{2midN}$, the comparison shifts to their lower bounds $X_{1L} \leq X_{2L}$. If the lower bounds are also equal $X_{1L} = X_{2L}$, the upper bounds are compared $X_{1U} = X_{2U}$, leading to equality in the ranking $X_{1N} \in [X_{1L}, X_{1U}] = X_{2N} \in [X_{2L}, X_{2U}]$. Within the NRSS framework, the ranking proceeds by identifying the smallest unit from the first data set, $m_N \in [m_L, m_U]$, and retaining it as the first measurement, while discarding the remaining units. The second measurement is chosen as the second-smallest observation from the second data set, and the remaining observations are similarly eliminated. This process continues until $m_N \in [m_L, m_U]$ Nc bivariate units are

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selected up to the m^{th} term. After completing r cycles of this procedure, a total of $n_N = m_N r \in [n_L, n_U]$ bivariate NRSS units are obtained."

Although there is a $m_N^2 r$ total for $m_N \in [m_L, m_U]$ units in the NRSS data extraction, only $n_N = m_N r \in [n_L, n_U]$ for $m_N \in [m_L, m_U]$ units are actually counted for calculations. $\left(X_{j(i)N} \in \left[X_{j(i)L}, X_{j(i)U}\right], Y_{j[i]N} \in \left[Y_{j[i]L}, Y_{j[i]U}\right]; j = 1, 2, 3 \dots r; i = 1, 2, 3 \dots m\right)$ symbolise the paired Nc bivariate quantified collections of the i^{th} units in the j^{th} cycle. For example, overall bivariate NRSS units can be written as follows when neutrosophic sets are of size $m_N \in [m_L, m_U] = [3, 3]$ and replication r = 4.

$$\begin{bmatrix} (\mathbf{X_{1(1)N}}, \mathbf{Y_{1[1]N}}) & (X_{1(2)N}, Y_{1[2]N}) & (X_{1(3)N}, Y_{1[3]N}) \\ (X_{2(1)N}, Y_{2[1]N}) & (\mathbf{X_{2(2)N}}, \mathbf{Y_{2[2]N}}) & (X_{2(3)N}, Y_{2[2]N}) \\ (X_{3(1)N}, Y_{3[1]N}) & (X_{3(2)N}, Y_{3[2]N}) & (\mathbf{X_{3(3)N}}, \mathbf{Y_{3[3]N}}) \end{bmatrix}$$

Each of the 3 groups in the bivariate NRSS structure above has 3 observations and r =1 replication. The actual measurement is based on the 3 bivariate observations that are highlighted. There are observations for the actual measurement as well as a total of observations with $r = 1$ the actual measurement consists of $n_N = m_N r = [3 * 1, 3 * 1] = [3, 3]$ observations while there are a total of $m_N^2 r = [3^2 * 1, 3^2 * 1] = [9, 9]$ observations. If r = 4, the total number of observations increases to $m_N^2 r = [3^2 * 4, 3^2 * 4] = [36, 36]$ observations and $n_N = m_N r = [3 * 4, 3 * 4] = [12, 12]$ observations for the actual measurement.

Under NSRSS as a classical stratified sampling method, this sampling method also consists of dividing the whole heterogeneous neutrosophic population into various homogeneous neutrosophic subgroups or subpopulations such that units inside every subgroup are homogeneous and between subgroups/subpopulations are heterogeneous concerning characteristics under study or study variables. Such subgroups/subpopulations are known as strata, and each subgroup is a stratum. Then, we apply the NRSS method to each stratum to obtain neutrosophic separate stratified ranked set samples. Let $N_N = N_L + N_U I_N, I_N \in [I_L, I_U]$ be the finite neutrosophic heterogeneous population and separated into uniform, non-overlapping Nc strata of each size of $N_{hN} = N_{hL} + N_{hU} I_{hN}, I_{hN} \in [I_{hL}, I_{hU}]$ such that $\sum_{h=1}^{L} N_{hN} = N_N$, $h = 1, 2, \dots, L.$ Next, we create distinct stratified ranked set samples for each stratum using the NRSS approach $n_{hN} = n_{hL} + n_{hU} I_{hN}, I_{hN} \in [I_{hL}, I_{hU}] = m_{hN} r$ and $m_{hN} = m_{hL} + m_{hU} I_{hN}$, $I_{hN} \in [I_{hL}, I_{hU}]$ from each stratum population of size N_{hN} such that we have total separate stratified ranked set sample $n_N = n_L + n_U I_N, I_N \in [I_L, I_U]$, $n_N = \sum_{h=1}^{L} n_{hN}$, $h = 1, 2, \dots, L,$.

Also, let Y_N be the neutrosophic study characteristics and Y_{hjN} be the population value of the study character Y_N and y_{hjN} be the sample value of the j^{th} unit $(j = 1, 2, \dots, N_h)$ in the h^{th} stratum. Then,

Population means of h^{th} stratum = $\bar{Y}_{hN} = \bar{Y}_{hL} + \bar{Y}_{hU} I_{hN} = \dfrac{1}{N_{hN}} \sum_{j=1}^{N_{hN}} Y_{hjN}$, $I_{hN} \in [I_{hL}, I_{hU}]$.

Population mean = $\bar{Y}_N = \bar{Y}_N + \bar{Y}_N I_N = \dfrac{1}{N_{hN}} \sum_{h=1}^{L} \sum_{j=1}^{N_{hN}} Y_{hjN}, I_N \in [I_L, I_U]$

$$= \sum_{h=1}^{L} w_{hN} \bar{Y}_{hN}, \; w_{hN} = \frac{N_{hN}}{N_N} = w_{hL} + w_{hU} I_{hN}.$$

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Population mean square of h^{th} stratum = $S_{hN}^2 = S_{hL}^2 + S_{hU}^2 I_{hN} = \frac{1}{N_{hN}-1}\sum_{j=1}^{N_{hN}}\left(Y_{hjN} - \bar{Y}_{hN}\right)^2.$

Population mean square = $S_N^2 = S_L^2 + S_U^2 I_N = \frac{1}{N_N - 1}\sum_{h=1}^{L}\sum_{j=1}^{N_N}\left(Y_{hjN} - \bar{Y}_N\right)^2, I_N \in [I_L, I_U].$

Sample mean of h^{th} stratum = $\bar{y}_{[n]hN} = \bar{y}_{[n]hL} + \bar{y}_{[n]hU} I_{hN} = \frac{1}{n_{hN}}\sum_{j=1}^{n_{hN}} y_{hjN} = \frac{1}{rm_{hN}}\sum_{j=1}^{n_{hN}} y_{hjN}$, $I_{hN} \in [I_{hL}, I_{hU}]$

Sample mean = $\bar{y}_{[n]N} = \bar{y}_{[n]L} + \bar{y}_{[n]U} I_{hN} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}$, $I_N \in [I_L, I_U]$

Where $w_{hN} = \frac{N_{hN}}{N_N}$

Like in classical SRSS, neutrosophic unbiased estimator of population mean and its variance is represented as:

$$\bar{y}_{[n]N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}, w_{hN} = w_{hL} + w_{hU} I_{hN},\ I_{hN} \in [I_{hL}, I_{hU}] \qquad (1)$$

$$V(\bar{y}_{[n]N}) = \sum_{h=1}^{L} w_{hN}^2\, V(\bar{y}_{[n]hN}) = \sum_{h=1}^{L} w_{hN}^2 \left(\eta_{hN} C_{yhN}^2 - D_{yhN}^2\right) \qquad (2)$$

Let $\bar{y}_{[n]N} \in [\bar{y}_{[n]L}, \bar{y}_{[n]U}]$ and $\bar{x}_{(n)N} \in \left[\bar{x}_{(n)L}, \bar{y}_{(n)U}\right]$ considered as the sample means of the Nc study and subsidiary variables respectively. Let $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$ be the population means of the neutrosophic study and subsidiary variables, respectively. Let $\bar{y}_{[n]hN} \in [\bar{y}_{[n]hL}, \bar{y}_{[n]hU}]$ and $\bar{x}_{(n)hN} \in \left[\bar{x}_{(n)hL}, \bar{y}_{(n)hU}\right]$ be the stratum sample means also, $\bar{Y}_{hN} \in [\bar{Y}_{hL}, \bar{Y}_{hU}]$ and $\bar{X}_{hN} \in [\bar{X}_{hL}, \bar{X}_{hU}]$ be the stratum population means. The correlation coefficient within each stratum, linking the neutrosophic study variable and the subsidiary variable, is represented as $\rho_{yxhN} \in \left[\rho_{yxhL}, \rho_{yxhU}\right]$, $C_{xhN} \in \left[C_{xhL}, C_{xhU}\right]$ and $C_{yhN} \in \left[C_{yhL}, C_{yhU}\right]$ be the coefficient of variation (CV) within each stratum of neutrosophic study and subsidiary variables Y_N and X_N.

Let the error terms of the Nc stratum are $\epsilon_{h0N} \in [\epsilon_{h0L}, \epsilon_{h0U}]$, $\epsilon_{h1N} \in [\epsilon_{h1L}, \epsilon_{h1U}]$. The estimators' bias and MSE are calculated by expressing

$$\bar{y}_{[n]hN} = \bar{Y}_{hN}(1 + \epsilon_{h0N}), \bar{x}_{[n]hN} = \bar{X}_{hN}(1 + \epsilon_{h1N})$$

such that,

$E(\epsilon_{h0N}) = E(\epsilon_{h1N}) = 0,$

$E(\epsilon_{h0N}^2) = \eta_{hN} C_{yhN}^2 - D_{yhN}^2 = V_{yhN},$

$E(\epsilon_{h1N}^2) = \eta_{hN} C_{xhN}^2 - D_{xhN}^2 = V_{xhN},$

$E(\epsilon_{h0N}\epsilon_{h1N}) = \eta_{hN} C_{yxhN} - D_{yxhN} = V_{xyhN}.$

where,

$D_{yhN}^2 = \frac{1}{m_{hN}^2 r \bar{Y}_{hN}^2}\sum_{i=1}^{k}(\mu_{[iyhN]} - \bar{Y}_{hN})^2,$

$D_{xhN}^2 = \frac{1}{m_{hN}^2 r \bar{X}_{hN}^2}\sum_{i=1}^{k}(\mu_{(ixhN)} - \bar{X}_{hN})^2,$

$D_{yxhN} = \frac{1}{m_{hN}^2 r \bar{Y}_{hN}\bar{X}_{hN}}\sum_{i=1}^{k}(\mu_{[iyhN]} - \bar{Y}_{hN})(\mu_{(ixhN)} - \bar{X}_{hN}),$

$\eta_{hN} = \frac{1}{m_{hN} r}.$

here $\mu_{[iyhN]}$ and $\mu_{(ixhN)}$ represents the means of the i_h^{th} ranked set and defined as:

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$\mu_{[iyhN]} = \frac{1}{r}\sum_{j=1}^{r} y_{[i]hj}, \mu_{(ixhN)} = \frac{1}{r}\sum_{j=1}^{r} x_{(i)hj}.$

$$\eta_{hN} \in [\eta_{hL}, \eta_{hU}];\ \epsilon_{h0N}^2 \in [\epsilon_{h0L}^2, \epsilon_{h0U}^2];\ \epsilon_{h1N}^2 \in [\epsilon_{h1L}^2, \epsilon_{h1U}^2];\ C_{yhN}^2 \in [C_{yhL}^2, C_{yhU}^2];$$

$$C_{xhN}^2 \in [C_{xhL}^2, C_{xhU}^2];\ C_{yxhN} \in [C_{yxhL}, C_{yxhU}];\ D_{yhN}^2 \in [D_{yhL}^2, D_{yhU}^2];$$

$$D_{xhN}^2 \in [D_{xhL}^2, D_{xhU}^2];\ D_{yxhN} \in [D_{yxhL}, D_{yxhU}];\ V_{yhN} \in [V_{yhL}, V_{yhU}];\ V_{xhN} \in [V_{xhL}, V_{xhU}];\ V_{xyhN} \in [V_{xyhL}, V_{xyhU}].$$

4. Existing Neutrosophic Stratified Ranked Set Estimators

To compute population means within the framework of indeterminacy using SRSS, we have adapted various established ratio and product-type estimators into their NSRSS counterparts. Utilizing neutrosophic data, an NSRSS unbiased estimator with its corresponding MSE is provided as follows.

$$\tau_{[u]hN} = \bar{y}_{[n]hN} = \frac{1}{n_{hN}}\sum_{i=1}^{n_{hN}} y_{[i]hN} \quad (1)$$

$$\tau_{[u]N} = \bar{y}_{[n]N} = \sum_{h=1}^{L} w_{hN}\bar{y}_{[n]hN} \quad (2)$$

The variance of the estimator $\tau_{[u]N}$ is expressed as

$$Var\left(\tau_{[u]N}\right) = \sum_{h=1}^{L} w_{hN}^2 \bar{Y}_{hN}^2\, V_{yhN} \quad (3)$$

Building on the neutrosophic auxiliary information and drawing inspiration from the works of Samawi and Muttlak[12] and Bouza[2] we have introduced NSRSS ratio and product estimators for estimating the population mean, which are formulated as follows.

$$\tau_{[R]N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}\left(\frac{\bar{X}_{hN}}{\bar{x}_{(n)hN}}\right) \quad (4)$$

$$\tau_{[P]N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}\left(\frac{\bar{x}_{(n)hN}}{\bar{X}_{hN}}\right) \quad (5)$$

The ratio and product estimators' bias and MSE expressions under NSRSS up to first order approximation are provided by

$$Bias\left(\tau_{[R]N}\right) = \sum_{h=1}^{L} w_{hN}\bar{Y}_{hN}\left(V_{xhN} - V_{xyhN}\right) \quad (6)$$

$$Bias\left(\tau_{[P]N}\right) = \sum_{h=1}^{L} w_{hN}\bar{Y}_{hN}\left(V_{xyhN}\right) \quad (7)$$

$$MSE\left(\tau_{[R]N}\right) = \sum_{h=1}^{L} w_{hN}^2\, \bar{Y}_{hN}^2\left(V_{yhN} + V_{xhN} - 2V_{xyhN}\right) \quad (8)$$

$$MSE\left(\tau_{[P]N}\right) = \sum_{h=1}^{L} w_{hN}^2\, \bar{Y}_{hN}^2\left(V_{yhN} + V_{xhN} + 2V_{xyhN}\right) \quad (9)$$

The separate regression estimator employing NSRSS for the population mean $\bar{Y}_N$ is given as

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$$\tau_{[reg]N} = \sum_{h=1}^{L} w_{hN}(\bar{y}_{[n]hN} + b_{hN}(\bar{X}_{hN} - \bar{x}_{(n)hN})) \quad (10)$$

The estimator $\tau_{[reg]N}$'s bias and MSE expressions under NSRSS up to first order approximation are provided by

$$Bias(\tau_{[reg]N}) = 0 \quad (11)$$

$$MSE(\tau_{[reg]N}) = \sum_{h=1}^{L} w_{hN}^2(\bar{Y}_{hN}^2 V_{yhN} + b_{hN}^2 \bar{X}_{hN}^2 V_{xhN} - 2b_{hN}\bar{Y}_{hN}\bar{X}_{hN}V_{xyhN}) \quad (12)$$

$$Min\ MSE(\tau_{[reg]N}) = \sum_{h=1}^{L} w_{hN}^2\, \bar{Y}_{hN}^2 \left(V_{yhN} - \frac{{V_{xyhN}}^2}{V_{xhN}}\right) \quad (13)$$

where,

$$b_{hN} = \frac{\bar{Y}_{hN}}{\bar{X}_{hN}}\left(\frac{V_{xyhN}}{V_{xhN}}\right)$$

The separate ratio exponential estimator employing NSRSS for the population mean $\bar{Y}_N$ is given as

$$\tau_{[expR]N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}\, exp\left(\frac{\bar{X}_{hN} - \bar{x}_{(n)hN}}{\bar{X}_{hN} + \bar{x}_{(n)hN}}\right) \quad (14)$$

The estimator $\tau_{[expR]N}$'s bias and MSE expressions under NSRSS up to first order approximation are provided by

$$Bias(\tau_{[expR]N}) = \sum_{h=1}^{L} w_{hN}\bar{Y}_{hN}\left(\frac{3}{8}V_{xhN} - \frac{1}{2}V_{xyhN}\right) \quad (15)$$

$$MSE(\tau_{[expR]N}) = \sum_{h=1}^{L} w_{hN}^2\, \bar{Y}_{hN}^2 \left(V_{yhN} + \frac{V_{xhN}}{4} - V_{xyhN}\right) \quad (16)$$

The separate product exponential estimator employing NSRSS for the population mean $\bar{Y}_N$ is represented as

$$\tau_{[expP]N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{[n]hN}\, exp\left(\frac{\bar{x}_{(n)hN} - \bar{X}_{hN}}{\bar{x}_{(n)hN} + \bar{X}_{hN}}\right) \quad (17)$$

The estimator $\tau_{[expP]N}$'s bias and MSE expressions under NSRSS up to first order approximation are provided by

$$Bias(\tau_{[expP]N}) = \sum_{h=1}^{L} w_{hN}\bar{Y}_{hN}\left(\frac{1}{2}V_{xyhN} - \frac{1}{8}V_{xhN}\right) \quad (18)$$

$$MSE(\tau_{[expP]N}) = \sum_{h=1}^{L} w_{hN}^2\, \bar{Y}_{hN}^2 \left(V_{yhN} + \frac{V_{xhN}}{4} + V_{xyhN}\right) \quad (19)$$

where $\tau_{[u]hN} \in [\tau_{[u]hL}, \tau_{[u]hU}], \tau_{[u]N} \in [\tau_{[u]L}, \tau_{[u]U}], \tau_{[R]N} \in [\tau_{[R]L}, \tau_{[R]U}], \tau_{[P]N} \in [\tau_{[P]L}, \tau_{[P]U}],$

$$\tau_{[reg]N} \in [\tau_{[reg]L}, \tau_{[reg]U}], \tau_{[expR]N} \in [\tau_{[expR]L}, \tau_{[expR]U}], \tau_{[expP]N} \in [\tau_{[expP]L}, \tau_{[expP]U}],$$

$$Bias\ (\tau_{[u]N}) \in [Bias\ (\tau_{[u]L}), Bias\ (\tau_{[u]N})], Bias\ (\tau_{[R]N}) \in [Bias\ (\tau_{[R]L}), Bias\ (\tau_{[R]U})],$$

$$Bias\ (\tau_{[P]N}) \in [Bias\ (\tau_{[P]L}), Bias\ (\tau_{[P]U})], Bias\ (\tau_{[reg]N}) \in [Bias\ (\tau_{[reg]L}), Bias\ (\tau_{[reg]U})],$$

$$Bias\ (\tau_{[expR]N}) \in [Bias\ (\tau_{[expR]L}), Bias\ (\tau_{[expR]U})], Bias\ (\tau_{[expP]N}) \in [Bias\ (\tau_{[expP]L}), Bias\ (\tau_{[expP]U})]$$

$$MSE\ (\tau_{[u]N}) \in [MSE\ (\tau_{[u]L}), MSE\ (\tau_{[u]U})], MSE\ (\tau_{[R]N}) \in [MSE\ (\tau_{[R]L}), MSE\ (\tau_{[R]U})],$$

$$MSE\ (\tau_{[P]N}) \in [MSE\ (\tau_{[P]L}), MSE\ (\tau_{[P]U})], MSE\ (\tau_{[reg]N}) \in [MSE\ (\tau_{[reg]L}), MSE\ (\tau_{[reg]U})],$$

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$$MSE\left(\tau_{[expR]N}\right) \in \left[MSE\left(\tau_{[expR]L}\right), MSE\left(\tau_{[expR]U}\right)\right], MSE\left(\tau_{[expP]N}\right) \in \left[MSE\left(\tau_{[expP]L}\right), MSE\left(\tau_{[expP]U}\right)\right]$$

5. Proposed Neutrosophic Stratified Ranked Set Estimators

There is no estimator that performs optimally in every scenario. Therefore, the focus is on developing estimators that ensure least MSE and enhanced accuracy. The goal of this part is to create estimators that can produce accurate findings under a range of circumstances. To do this, we used auxiliary information to provide two novel estimators specifically designed for finite population mean estimation under NSRSS, while also adopting one estimator within the framework of NSRSS.

$$1).\ P_{1N} = \sum_{h=1}^{L} w_{hN}\left(\bar{y}_{[n]hN}\left(g_{1hN}+1\right) + g_{2hN}\log\left(\frac{\bar{x}_{(n)hN}}{\bar{X}_{hN}}\right)\right) \tag{20}$$

where, g_{1hN} and g_{2hN} are constants so that the estimators' MSE is as low as possible.

Rewriting the estimator P_{1N} as defined in equation (20) in terms of ϵ's

$$P_{1N} = \sum_{h=1}^{L} w_{hN}\left(\bar{Y}_{hN}\left(1+\epsilon_{h0N}\right)\left(g_{1hN}+1\right) + g_{2hN}\log\left(\frac{\bar{X}_{hN}\left(1+\epsilon_{h1N}\right)}{\bar{X}_{hN}}\right)\right) \tag{21}$$

The bias of the estimator P_{1N} is determined by

$$Bias\left(P_{1N}\right) = \sum_{h=1}^{L} w_{hN}\left(\bar{Y}_{hN} g_{1hN} - \frac{g_{2hN}}{2} V_{xhN}\right) \tag{22}$$

We derive MSE, by considering expectations up to the first-order approximation

$$MSE\left(P_{1N}\right) = \sum_{h=1}^{L} w_{hN}^2\left(\bar{Y}_{hN}^2 V_{yhN} + g_{1hN}^2 A_{1hN} + g_{2hN}^2 B_{1hN} + 2g_{1hN}C_{1hN} + 2g_{2hN}D_{1hN} + 2g_{1hN}g_{2hN}E_{1hN}\right) \tag{23}$$

where,

$$A_{1hN} = \bar{Y}_{hN}^2\left(1+V_{yhN}\right)$$

$$B_{1hN} = V_{xhN}$$

$$C_{1hN} = \bar{Y}_{hN}^2 V_{yhN}$$

$$D_{1hN} = \bar{Y}_{hN} V_{yxhN}$$

$$E_{1hN} = \bar{Y}_{hN}\left(V_{yxhN} - \frac{1}{2}V_{xhN}\right)$$

To determine out the minimum MSE for P_{1N}, we take the partial derivatives of equation (23) w.r.t g_{1hN} & g_{2hN} and set them equal to zero and solve accordingly.

$$g_{1hN}{}^* = \frac{B_{1hN}C_{1hN} - D_{1hN}E_{1hN}}{E_{1hN}^2 - A_{1hN}B_{1hN}} \tag{24}$$

$$g_{2hN}{}^* = \frac{A_{1hN}D_{1hN} - C_{1hN}E_{1hN}}{E_{1hN}^2 - A_{1hN}B_{1hN}} \tag{25}$$

Substituting the optimal values of g_{1hN} & g_{2hN} into equation (23), the optimal value of MSE for P_{1N} is derived as follows.

$$Min\ MSE = \sum_{h=1}^{L} w_{hN}^2\left(C_{1hN} + \frac{B_{1hN}C_{1hN}^2 + A_{1hN}D_{1hN}^2 - 2C_{1hN}D_{1hN}E_{1hN}}{E_{1hN}^2 - A_{1hN}B_{1hN}}\right) \tag{26}$$

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$$2).\ P_{2N} = \sum_{h=1}^{L} w_{hN}\left(g_{3hN}\ \bar{y}_{[n]hN}\ +\ g_{4hN}\ exp\left(\frac{\bar{X}_{hN} - \bar{x}_{(n)hN}}{\bar{X}_{hN} + \bar{x}_{(n)hN}}\right)\left(1 + log\left(\frac{\bar{x}_{(n)hN}}{\bar{X}_{hN}}\right)\right)\right) \quad (27)$$

When P_2 from equation (27) is expressed in terms of ϵ's, we obtain

$$P_{2N} = \sum_{h=1}^{L} w_{hN}\left(g_{3hN}\bar{Y}_{hN}(1+\epsilon_{h0N}) + g_{4hN}\ exp\left(\frac{-\epsilon_{h1N}}{2+\epsilon_{h1N}}\right)(1 + log(1+\epsilon_{h1N})\right) \quad (28)$$

$$P_{2N} - \bar{Y}_N = \sum_{h=1}^{L} w_{hN}\left((g_{3hN}-1)\bar{Y}_{hN} + g_{3hN}\bar{Y}_{hN}\epsilon_{h0N} + g_{4hN}\left(1 + \frac{\epsilon_{h1N}}{2} - \frac{5{\epsilon_{h1N}}^2}{8}\right)\right) \quad (29)$$

$$Bias(P_{2N}) = \sum_{h=1}^{L} w_{hN}\left(\bar{Y}_{hN}(g_{3hN}-1) - g_{4hN}\left[1 - \frac{5}{8}V_{xhN}\right]\right) \quad (30)$$

Case 1: Weight's sum is unity ($g_{3hN} + g_{4hN} = 1$).

The value of MSE of P_2 is given by

$$MSE(P_{2N}) = \sum_{h=1}^{L} w_{hN}^2\bar{Y}_{hN}^2(V_{yhN} + {g_{4hN}}^2V_{xhN} - 2g_{4hN}V_{xyhN}) \quad (31)$$

To determine out the minimum MSE for P_{2N}, we take the partial derivatives of equation (31) w.r.t g_{4hN} and set them equal to zero and solve accordingly.

$${g_{4hN}}^* = \frac{V_{xyhN}}{V_{xhN}} \quad (32)$$

Substituting the optimal values of g_{4hN} into equation (31), the optimal value of MSE for P_{2N} is derived as follows.

$$Min\ MSE = \sum_{h=1}^{L} w_{hN}^2\bar{Y}_{hN}^2\left(V_{yhN} - \frac{{V_{xyhN}}^2}{V_{xhN}}\right) \quad (33)$$

Case 2: Weight's sum is flexible ($g_{3hN} + g_{4hN} \neq 1$).

$$P_{2N} - \bar{Y}_N = \sum_{h=1}^{L} w_{hN}\left((g_{3hN}-1)\bar{Y}_{hN} + g_{3hN}\bar{Y}_{hN}\epsilon_{h0N} + g_{4hN}\left(1 + \frac{\epsilon_{h1N}}{2} - \frac{5{\epsilon_{h1N}}^2}{8}\right)\right) \quad (34)$$

When we square both sides, we get

$$(P_{2N} - \bar{Y}_N)^2 = \sum_{h=1}^{L} w_{hN}^2(\bar{Y}_{hN}^2 + \bar{Y}_{hN}^2\ g_{3hN}^2(1+\epsilon_{h0N}^2) + g_{4hN}^2(1-\epsilon_{h1N}^2) - 2g_{3hN}\bar{Y}_{hN}^2 - 2g_{4hN}\bar{Y}_{hN}\left(1 - \frac{5{\epsilon_{h1N}}^2}{8}\right) + 2g_{3hN}g_{4hN}\left(1 - \frac{5{\epsilon_{h1N}}^2}{8} + \frac{\epsilon_{h0N}\epsilon_{h1N}}{2}\right) \quad (35)$$

We derive MSE, by considering expectations up to the first-order approximation

$$MSE(P_{2N}) = \sum_{h=1}^{L} w_{hN}^2(\bar{Y}_{hN}^2\ V_{yhN} + g_{3hN}^2A_{2hN} + g_{4hN}^2B_{2hN} - 2g_{3hN}C_{2hN} - 2g_{4hN}D_{2hN} + 2g_{3hN}g_{4hN}E_{2hN}) \quad (36)$$

where,

$$A_{2hN} = \bar{Y}_{hN}^2\ (1 + V_{yhN})$$

$$B_{2hN} = 1 - V_{xhN}$$

$$C_{2hN} = \bar{Y}_{hN}^2$$

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$$D_{2hN} = \bar{Y}_{hN}\left(1-\frac{5}{8}V_{xhN}\right)$$

$$E_{2hN} = \bar{Y}_{hN}\left(1-\frac{5}{8}V_{xhN}+\frac{1}{2}V_{xyhN}\right)$$

To determine out the minimum MSE for P_{2N}, we take the partial derivatives of equation (36) w.r.t g_{3hN} & g_{4hN} and set them equal to zero and solve accordingly.

$$g_{3hN}{}^{*} = \frac{B_{2hN}C_{2hN} - D_{2hN}E_{2hN}}{A_{2hN}B_{2hN} - E_{2hN}^2} \quad (37)$$

$$g_{4hN}{}^{*} = \frac{A_{2hN}D_{2hN} - C_{2hN}E_{2hN}}{A_{2hN}B_{2hN} - E_{2hN}^2} \quad (38)$$

Substituting the optimal values of g_{3hN} & g_{4hN} into equation (36), the optimal values MSE for P_{2N} is derived as follows.

$$Min\ MSE = \sum_{h=1}^{L} w_{hN}^2\left(C_{2hN} + \frac{B_{2hN}C_{2hN}^2 + A_{2hN}D_{2hN}^2 - 2C_{2hN}D_{2hN}E_{2hN}}{E_{2hN}^2 - A_{2hN}B_{2hN}}\right) \quad (39)$$

3). $$P_{3N} = \sum_{h=1}^{L} w_{hN}\left(g_{5hN}\,\bar{y}_{[n]hN} + g_{6hN}\left(\frac{\bar{X}_{hN}}{\bar{x}_{(n)hN}}\right)exp\left(\frac{\bar{X}_{hN} - \bar{x}_{(n)hN}}{\bar{X}_{hN} + \bar{x}_{(n)hN}}\right)\right) \quad (40)$$

When we translate P_{3N} from equation (40) into ϵ's, we obtain

$$P_{3N} = \sum_{h=1}^{L} w_{hN}\left(g_{5hN}\bar{Y}_{hN}(1+\epsilon_{h0N}) + g_{6hN}(1+\epsilon_{h1N})^{-1}\,exp\left(\frac{-\epsilon_{h1N}}{2+\epsilon_{h1N}}\right)\right) \quad (41)$$

$$P_{3N} - \bar{Y}_N = \sum_{h=1}^{L} w_{hN}\left((g_{5hN}-1)\bar{Y}_{hN} + g_{6hN}\bar{Y}_{hN}\epsilon_{h0N} + g_{6hN}\left(1 - \frac{3\epsilon_{h1N}}{2} + \frac{15\epsilon_{h1N}{}^2}{8}\right)\right) \quad (42)$$

$$Bias(P_{3N}) = \sum_{h=1}^{L} w_{hN}\left(\bar{Y}_{hN}(g_{5hN}-1) - g_{6hN}\left[1+\frac{15}{8}V_{xhN}\right]\right) \quad (43)$$

Case 1: Weights' sum is unity ($g_{5hN} + g_{6hN} = 1$)

The expression of MSE of P_{3N} is given by

$$MSE(P_{3N}) = \sum_{h=1}^{L} w_{hN}^2\bar{Y}_{hN}^2(V_{yhN} + g_{4hN}{}^2V_{xhN} - 2g_{4hN}V_{xyhN}) \quad (44)$$

To determine out the minimum MSE for P_{3N}, we take the partial derivatives of equation (44) w.r.t g_{6hN} and set them equal to zero and solve accordingly.

$$g_{6hN}{}^{*} = \frac{V_{xyhN}}{V_{xhN}} \quad (45)$$

Substituting the optimal values of g_{6hN} into equation (44), the optimal value of MSE for P_{3N} is derived as follows.

$$Min\ MSE = \sum_{h=1}^{L} w_{hN}^2\bar{Y}_{hN}^2\left(V_{yhN} - \frac{V_{xyhN}{}^2}{V_{xhN}}\right) \quad (46)$$

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Case 2: Weights' sum is unity ($g_{5hN} + g_{6hN} \neq 1$)

$$P_{3N} - \bar{Y}_N = \sum_{h=1}^{L} w_{hN} \left((g_{5hN} - 1)\bar{Y}_{hN} + g_{5hN}\bar{Y}_{hN}\epsilon_{h0N} + g_{6hN}\left(1 - \frac{3\epsilon_{h1N}}{2} + \frac{15{\epsilon_{h1N}}^2}{8}\right)\right) \quad (47)$$

When we square both sides, we get

$$(P_{3N} - \bar{Y}_N)^2 = \sum_{h=1}^{L} w_{hN}^2 (\bar{Y}_{hN}^2 + \bar{Y}_{hN}^2\, g_{5hN}^2(1+\epsilon_{h0N}^2) + g_{6hN}^2(1+6\epsilon_{h1N}^2) - 2g_{3hN}\bar{Y}_{hN}^2 - 2g_{4hN}\bar{Y}_{hN}\left(1 + \frac{15{\epsilon_{h1N}}^2}{8}\right) + 2g_{3hN}g_{4hN}\left(1 - \frac{3\epsilon_{h0N}\epsilon_{h1N}}{2} + \frac{15{\epsilon_{h1N}}^2}{8}\right) \quad (48)$$

We derive MSE, by considering expectations up to the first-order approximation

$$MSE(P_{3N}) = \sum_{h=1}^{L} w_{hN}^2 (\bar{Y}_{hN}^2\, V_{yhN} + g_{5hN}^2 A_{3hN} + g_{6hN}^2 B_{3hN} - 2g_{5hN}C_{3hN} - 2g_{6hN}D_{3hN} + 2g_{5hN}g_{6hN}E_{3hN}) \quad (49)$$

where,

$$A_{3hN} = \bar{Y}_{hN}^2\,(1 + V_{yhN})$$

$$B_{3hN} = 1 + 6V_{xhN}$$

$$C_{3hN} = \bar{Y}_{hN}^2$$

$$D_{3hN} = \bar{Y}_{hN}\left(1 + \frac{15}{8}V_{xhN}\right)$$

$$E_{3hN} = \bar{Y}_{hN}\left(1 + \frac{15}{8}V_{xhN} - \frac{3}{2}V_{xyhN}\right)$$

To determine out the minimum MSE for P_{3N}, we take the partial derivatives of equation (49) w.r.t g_{5hN} & g_{6hN} and set them equal to zero and solve accordingly.

$${g_{5hN}}^* = \frac{B_{3hN}C_{3hN} - D_{3hN}E_{3hN}}{A_{3hN}B_{3hN} - E_{3hN}^2} \quad (50)$$

$${g_{6hN}}^* = \frac{A_{3hN}D_{3hN} - C_{3hN}E_{3hN}}{A_{3hN}B_{3hN} - E_{3hN}^2} \quad (51)$$

Substituting the optimal values of g_{5hN} & g_{6hN} into equation (49), the optimal value of MSE for P_{3N} is derived as follows.

$$Min\,MSE = \sum_{h=1}^{L} w_{hN}^2 \left(C_{3hN} + \frac{B_{3hN}C_{3hN}^2 + A_{3hN}D_{3hN}^2 - 2C_{3hN}D_{3hN}E_{3hN}}{E_{3hN}^2 - A_{3hN}B_{3hN}} \right) \quad (52)$$

Where $P_{iN} \in [P_{iL}, P_{iU}], Bias\ (P_{iN}) \in [Bias\ (P_{iL}), Bias\ (P_{iU})]$,

$$MSE\ (P_{iN}) \in [MSE\ (P_{iL}), MSE\ (P_{iU})], A_{iN} \in [A_{iL}, A_{iU}], B_{iN} \in [B_{iL}, B_{iU}], C_{iN} \in [C_{iL}, C_{iU}],$$

$$D_{iN} \in [D_{iL}, D_{iU}], E_{iN} \in [E_{iL}, E_{iU}],\ g_{jN} \in [\, g_{jN}\,,\ g_{jN}\,], i = 1,2,3\ and\ j = 1,2,3,4,5,6.$$

6. Numerical Illustration

This section examines how well the recommended estimators perform in comparison to the other estimators that are currently in use and taken into consideration in this work. To numerically elucidate the characteristics of Nc stratified

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estimators, we have compiled real-life indeterminate climate data of the USA state. We have taken two states, Alabama and Georgia, as strata, and then November month is taken for the data from both states. There are many variables, but we are considering Dew Point Temperature vs Relative Humidity variables only here. The Dew Point Temperature variable is taken as the subsidiary variable $X_{hN} \in [X_{hL}, X_{hU}]$, and the Relative Humidity as the study variable $Y_{hN} \in [Y_{hL}, Y_{hU}]$. The parameter descriptions are given in Singh et al.[17]. In addition, one can visit for the data on this link: https://mrcc.purdue.edu.

Table 1: Description of real data parameters for estimating means in the context of NSRSS.

Parameters	Value	Parameters	Value
1st Stratum			
N_{1N}	[19, 19]	S_{y1N}	[13.55, 23.30]
n_{1N}	[9, 9]	C_{x1N}	[0.2295, 0.5549]
$\bar{X}_{1N}$	[19.58, 61.95]	C_{y1N}	[0.1405, 0.8261]
$\bar{Y}_{1N}$	[28.21, 96.47]	$\beta_{2(x)1N}$	[6.4058, 6.0892]
S_{x1N}	[10.86, 14.22]	ρ_{yx1N}	[0.946, 0.941]
2nd Stratum			
N_{2N}	[22, 22]	S_{y2N}	[11.78, 21.88]
n_{2N}	[12, 12]	C_{x2N}	[0.1393, 0.5794]
$\bar{X}_{2N}$	[22.55, 62.23]	C_{y2N}	[0.1255, 0.6887]
$\bar{Y}_{2N}$	[31.77, 93.86]	$\beta_{2(x)2N}$	[2.5435, 8.5923]
S_{x2N}	[8.67, 13.06]	ρ_{yx2N}	[0.8854, 0.9481]

Additionally, $n_{1N} = 9$ samples were taken from the first strata of size 19, and $n_{2N} = 12$ samples are taken from the second strata of size 22. The NSRSS sampling method, as described in Section 3, is applied concurrently to sample data for both the study and subsidiary variables. The formula for calculating the PRE is defined as follows:

$$PRE(Estimators) = \frac{MSE(\tau_{[u]N})}{MSE(estimator)} \times 100 \qquad (53)$$

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Table 2: The MSE and PRE of both suggested and existing estimators

Estimators	MSE		PRE	
$\tau_{[u]N}$	[7.259338,	17.83989]	[100,	100]
$\tau_{[R]N}$	[2.260564,	4.565784]	[321.1295,	390.7301]
$\tau_{[P]N}$	[41.95204,	50.70008]	[17.3039,	35.18711]
$\tau_{[reg]N}$	[0.487483,	2.999074]	[1489.148,	594.8467]
$\tau_{[expR]N}$	[1.100254,	8.702535]	[659.7873,	204.9965]
$\tau_{[expP]N}$	[21.44598,	31.26969]	[33.84941,	57.0517]
$\boldsymbol{P_{1N}}$	**[0.479819,**	**2.830461]**	**[1512.934,**	**630.2822]**
$\boldsymbol{P_{2N}}$	**[0.218061,**	**0.631283]**	**[3329.042,**	**2825.975]**
$\boldsymbol{P_{3N}}$	**[0.227592,**	**0.647094]**	**[3189.634,**	**2756.925]**

7. Simulation Studies

The proposed estimator's performance is validated by conducting simulation studies, comparing it against other established estimators, such as traditional estimators, ratio estimators, and regression estimators. This simulation was specifically conducted within the framework of neutrosophic analysis. Our population is separately stratified into two distinct categories, denoted as stratum 1 and stratum 2. The steps below are used to do this.

1. The distribution of Nc random variables will be Nc normal,

i.e. $(X_N, Y_N) \sim NN[(\mu_{xN}, \sigma^2_{xN}), (\mu_{yN}, \sigma^2_{yN})], X_N \in [X_L, X_U], Y_N \in [Y_L, Y_U], \mu_{xN} \in [\mu_{xL}, \mu_{xU}], \mu_{yN} \in [\mu_{yL}, \mu_{yU}], \sigma^2_{xN} \in [\sigma^2_{xL}, \sigma^2_{xU}], \sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$. We created 4-variate random datasets with N=1000 from a 4-variate normal distribution distribution with mean $(\mu_{xL}, \mu_{yL}, \mu_{xU}, \mu_{yU})$ and covariance matrix

$$\begin{bmatrix} \sigma^2_{xL} & \rho_{xyL}\sigma_{xL}\sigma_{yL} & 0 & 0 \\ \rho_{xyL}\sigma_{xL}\sigma_{yL} & \sigma^2_{yU} & 0 & 0 \\ 0 & 0 & \sigma^2_{xU} & \rho_{xyU}\sigma_{xU}\sigma_{yU} \\ 0 & 0 & \rho_{xyU}\sigma_{xU}\sigma_{yU} & \sigma^2_{yU} \end{bmatrix}.$$

2. The parameters have been computed for this artificially generated population of size N = 1000.

3. A sample of size n_N has been selected from this artificially generated population.

4. These sample datasets were used to calculate the MSE for all estimators being evaluated.

5. The simulation process was repeated 10,000 times, and the average of these 10,000 MSE values was computed as the final MSE for each estimator of the population mean.

6. The PRE of each of the estimator with respect to $\tau_{[u]N}$ has been determined by applying the formula.

7. This process can also be applied to different populations with distinct parameters.

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Table 3: Summary of simulated data parameters for estimating means in the context of NSRSS.

Parameters	Neutrosophic	Classical	Parameters	Neutrosophic	Classical
Population 1	1st Stratum		2nd Stratum		
N_{1N}	[200, 200]	200	N_{2N}	[200, 200]	200
$\bar{X}_{1N}$	[40, 50]	46	$\bar{X}_{2N}$	[120, 130]	123
$\bar{Y}_{1N}$	[50, 60]	54	$\bar{Y}_{2N}$	[130, 140]	137
S_{x1N}	[9, 12]	11	S_{x2N}	[12, 24]	15
S_{y1N}	[10, 13]	12	S_{y2N}	[22, 25]	23
Population 2					
N_{1N}	[200, 200]	200	N_{2N}	[200, 200]	200
$\bar{X}_{1N}$	[70, 80]	75	$\bar{X}_{2N}$	[150, 160]	155
$\bar{Y}_{1N}$	[90, 100]	96	$\bar{Y}_{2N}$	[180, 190]	184
S_{x1N}	[12, 19]	16	S_{x2N}	[19, 25]	21
S_{y1N}	[13, 20]	18	S_{y2N}	[21, 27]	25

Table 4: The PREs and MSEs of both suggested and existing estimators within the NSRSS framework for Population 1

n_N = 24	ρ_{xyN} = [0.9, 0.9]				ρ_{xyN} = [0.8, 0.8]			
Estimators	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[5.99674,	14.5457]	[100,	100]	[6.70853,	14.40247]	[100,	100]
$\tau_{[R]N}$	[2.05206,	3.02691]	[292,	481]	[3.98848,	5.9795]	[168,	241]
$\tau_{[P]N}$	[21.53535,	59.66501]	[24,	28]	[21.1488,	56.71128]	[25,	32]
$\tau_{[reg]N}$	[1.72288,	2.2787]	[348,	638]	[3.22543,	4.31126]	[208,	334]
$\tau_{[expR]N}$	[2.57516,	4.58624]	[233,	317]	[3.88348,	5.95525]	[173,	242]
$\tau_{[expP]N}$	[12.3168,	32.90529]	[44,	49]	[12.46364,	31.32115]	[46,	54]

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P_{1N}	[1.71895,	2.26005]	[349,	644]	[3.21884,	4.28749]	[208,	336]
P_{2N}	[0.96441,	2.09121]	[622,	696]	[1.21279,	3.09257]	[466,	553]
P_{3N}	[0.66257,	0.87245]	[905,	1667]	[1.29282,	1.76833]	[519,	814]
Estimators	ρ_{xyN} = [0.7, 0.7]				ρ_{xyN} = [0.6, 0.6]			
	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[7.33904,	14.76347]	[100,	100]	[8.16538,	14.709]	[100,	100]
$\tau_{[R]N}$	[5.79743,	8.80431]	[127,	168]	[7.62111,	11.93466]	[107,	123]
$\tau_{[P]N}$	[20.63853,	52.95935]	[28,	36]	[20.21633,	50.2402]	[29,	40]
$\tau_{[reg]N}$	[4.52665,	6.44316]	[162,	229]	[5.84707,	8.09475]	[140,	182]
$\tau_{[expR]N}$	[5.0985,	7.7543]	[144,	190]	[6.45491,	9.22722]	[126,	159]
$\tau_{[expP]N}$	[12.51905,	29.83182]	[49,	59]	[12.75252,	28.37999]	[52,	64]
P_{1N}	[4.51718,	6.41318]	[162,	230]	[5.83418,	8.05956]	[140,	183]
P_{2N}	[1.3007,	3.34004]	[442,	564]	[1.29779,	3.58643]	[410,	629]
P_{3N}	[1.87745,	2.68269]	[391,	550]	[2.46094,	3.63675]	[332,	404]
n_N = 36	ρ_{xyN} = [0.9, 0.9]				ρ_{xyN} = [0.8, 0.8]			
Estimators	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[4.14692,	10.38599]	[100,	100]	[4.68196,	10.2829]	[100,	100]
$\tau_{[R]N}$	[1.45859,	2.14749]	[284,	484]	[2.82771,	4.24873]	[166,	242]
$\tau_{[P]N}$	[14.75004,	42.52071]	[24,	28]	[14.53956,	40.38289]	[25,	32]
$\tau_{[reg]N}$	[1.27559,	1.70387]	[325,	610]	[2.38545,	3.22707]	[196,	319]
$\tau_{[expR]N}$	[1.8134,	3.27971]	[229,	317]	[2.75441,	4.25759]	[170,	242]
$\tau_{[expP]N}$	[8.45913,	23.46632]	[44,	49]	[8.61034,	22.32467]	[46,	54]
P_{1N}	[1.2737,	1.6947]	[326,	613]	[2.38221,	3.21515]	[197,	320]
P_{2N}	[0.6931,	1.56433]	[598,	664]	[0.86572,	2.3029]	[447,	541]
P_{3N}	[0.49081,	0.66019]	[845,	1573]	[0.95229,	1.33183]	[492,	772]
Estimators	ρ_{xyN} = [0.7, 0.7]				ρ_{xyN} = [0.6, 0.6]			

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	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[5.15938,	10.47569]	[100,	100]	[5.75434,	10.48278]	[100,	100]
$\tau_{[R]N}$	[4.10583,	6.24057]	[126,	168]	[5.37867,	8.45519]	[107,	124]
$\tau_{[P]N}$	[14.2385,	37.65536]	[28,	36]	[13.99259,	35.82328]	[29,	41]
$\tau_{[reg]N}$	[3.34456,	4.75838]	[154,	220]	[4.30892,	6.03096]	[134,	174]
$\tau_{[expR]N}$	[3.62941,	5.49006]	[142,	191]	[4.58368,	6.55487]	[126,	160]
$\tau_{[expP]N}$	[8.69574,	21.19745]	[49,	59]	[8.89064,	20.23892]	[52,	65]
P_{1N}	[3.33987,	4.74333]	[154,	221]	[4.30248,	6.01303]	[134,	174]
P_{2N}	[0.92359,	2.48823]	[421,	559]	[0.9228,	2.6805]	[391,	624]
P_{3N}	[1.38230,	2.00263]	[373,	523]	[1.80262,	2.72061]	[319,	385]

Table 5: The PREs and MSEs of both recommended and existing estimators within the NSRSS framework for Population 2

n_N = 24	ρ_{xyN} = [0.9, 0.9]				ρ_{xyN} = [0.8, 0.8]			
Estimators	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[6.33409,	20.82624]	[100,	100]	[6.96397,	21.64214]	[100,	100]
$\tau_{[R]N}$	[2.23469,	4.9437]	[283,	421]	[4.25967,	9.11246]	[163,	238]
$\tau_{[P]N}$	[23.85232,	90.98492]	[23,	27]	[23.33807,	85.75316]	[25,	32]
$\tau_{[reg]N}$	[1.79683,	3.29563]	[353,	632]	[3.29547,	6.64487]	[211,	326]
$\tau_{[expR]N}$	[2.60704,	6.10045]	[243,	341]	[3.90309,	8.92963]	[178,	242]
$\tau_{[expP]N}$	[13.41585,	49.12106]	[42,	47]	[13.44229,	47.24998]	[46,	52]
P_{1N}	[1.79457,	3.27307]	[353,	636]	[3.29186,	6.61524]	[212,	327]
P_{2N}	[1.14734,	3.64488]	[552,	571]	[1.43845,	4.76444]	[454,	484]
P_{3N}	[0.71274,	1.31424]	[889,	1585]	[1.37624,	2.72585]	[506,	794]
Estimators	ρ_{xyN} = [0.7, 0.7]				ρ_{xyN} = [0.6, 0.6]			
	MSE		PRE		MSE		PRE	
$\tau_{[u]N}$	[7.61888,	20.64343]	[100,	100]	[8.43717,	20.53845]	[100,	100]

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$\boldsymbol{\tau}_{[R]N}$	[6.20828,	13.84982]	[123,	149]	[8.18271,	18.317]	[103,	112]
$\boldsymbol{\tau}_{[P]N}$	[22.79331,	81.42014]	[25,	33]	[22.3359,	76.82708]	[27,	38]
$\boldsymbol{\tau}_{[reg]N}$	[4.64649,	8.96223]	[164,	230]	[5.98512,	11.26629]	[141,	182]
$\boldsymbol{\tau}_{[expR]N}$	[5.1931,	10.49874]	[147,	197]	[6.6044,	12.66933]	[128,	162]
$\boldsymbol{\tau}_{[expP]N}$	[13.48561,	44.28389]	[47,	56]	[13.681,	41.92437]	[49,	62]
$\boldsymbol{P}_{1N}$	[4.64136,	8.92852]	[164,	231]	[5.97828,	11.2277]	[141,	183]
$\boldsymbol{P}_{2N}$	[1.53563,	5.74515]	[359,	496]	[1.54129,	5.97125]	[344,	547]
$\boldsymbol{P}_{3N}$	[2.01247,	3.99382]	[379,	517]	[2.65872,	5.31573]	[317,	386]
$\boldsymbol{n}_N$ = 36	ρ_{xyN} = [0.9, 0.9]				ρ_{xyN} = [0.8, 0.8]			
Estimators	MSE		PRE		MSE		PRE	
$\boldsymbol{\tau}_{[u]N}$	[4.38596,	14.80505]	[100,	100]	[4.86685,	15.38448]	[100,	100]
$\boldsymbol{\tau}_{[R]N}$	[1.58153,	3.51391]	[277,	421]	[3.02058,	6.46028]	[161,	238]
$\boldsymbol{\tau}_{[P]N}$	[16.36886,	64.67041]	[23,	27]	[16.0519,	61.07339]	[25,	30]
$\boldsymbol{\tau}_{[reg]N}$	[1.32371,	2.45155]	[331,	604]	[2.43928,	4.90893]	[200,	313]
$\boldsymbol{\tau}_{[expR]N}$	[1.83644,	4.3377]	[239,	341]	[2.77636,	6.32679]	[175,	243]
$\boldsymbol{\tau}_{[expP]N}$	[9.2301,	34.91596]	[42,	48]	[9.29203,	33.63334]	[46,	52]
$\boldsymbol{P}_{1N}$	[1.32263,	2.44051]	[332,	607]	[2.4375,	4.89418]	[200,	314]
$\boldsymbol{P}_{2N}$	[0.82392,	2.74491]	[532,	539]	[1.02455,	3.55519]	[433,	475]
$\boldsymbol{P}_{3N}$	[0.52577,	0.99155]	[834,	1493]	[1.01568,	2.03624]	[479,	756]
Estimators	ρ_{xyN} = [0.7, 0.7]				ρ_{xyN} = [0.6, 0.6]			
	MSE		PRE		MSE		PRE	
$\boldsymbol{\tau}_{[u]N}$	[5.34889,	14.63937]	[100,	100]	[5.92875,	14.60706]	[100,	100]
$\boldsymbol{\tau}_{[R]N}$	[4.39296,	9.81505]	[122,	149]	[5.75661,	13.01336]	[103,	112]
$\boldsymbol{\tau}_{[P]N}$	[15.70954,	57.87504]	[25,	34]	[15.43541,	54.54652]	[27,	38]
$\boldsymbol{\tau}_{[reg]N}$	[3.42888,	6.61932]	[156,	221]	[4.39799,	8.38299]	[135,	174]
$\boldsymbol{\tau}_{[expR]N}$	[3.69534,	7.42579]	[145,	197]	[4.67586,	9.01699]	[127,	162]

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$\boldsymbol{\tau}_{[expP]N}$	[9.35363, 31.45579]	[47, 57]	[9.51527, 29.78357]	[49, 62]
$\boldsymbol{P}_{1N}$	[3.42635, 6.60245]	[156, 222]	[4.3946, 8.36349]	[135, 175]
$\boldsymbol{P}_{2N}$	[1.09014, 4.2836]	[342, 491]	[1.09685, 4.44168]	[329, 541]
$\boldsymbol{P}_{3N}$	[1.48147, 2.97639]	[361, 492]	[1.94138, 3.97453]	[305, 368]

Table 6: PREs of the NSRSS estimators compared to Population 1 estimators under NSSRS

$\boldsymbol{n}_N$ =24	$\boldsymbol{\rho}_{xyN}$ = [0.9, 0.9]	$\boldsymbol{\rho}_{xyN}$ = [0.8, 0.8]	$\boldsymbol{\rho}_{xyN}$ = [0.7, 0.7]	$\boldsymbol{\rho}_{xyN}$ = [0.6, 0.6]
	PRE	PRE	PRE	PRE
$\boldsymbol{\tau}_{[u]N}$	[119, 209]	[122, 187]	[118, 160]	[119, 152]
$\boldsymbol{\tau}_{[R]N}$	[118, 118]	[122, 130]	[117, 125]	[123, 140]
$\boldsymbol{\tau}_{[P]N}$	[121, 250]	[120, 233]	[121, 226]	[120, 210]
$\boldsymbol{\tau}_{[reg]N}$	[116, 119]	[125, 126]	[117, 118]	[123, 127]
$\boldsymbol{\tau}_{[expR]N}$	[117, 151]	[124, 139]	[116, 119]	[120, 126]
$\boldsymbol{\tau}_{[expP]N}$	[120, 237]	[121, 218]	[120, 201]	[120, 186]
$\boldsymbol{P}_{1N}$	[115, 119]	[125, 125]	[117, 118]	[123, 127]
$\boldsymbol{P}_{2N}$	[124, 203]	[120, 230]	[124, 264]	[125, 256]
$\boldsymbol{P}_{3N}$	[119, 120]	[124, 132]	[118, 130]	[124, 144]
$\boldsymbol{n}_N$ =36	ρ_{xyN} = [0.9, 0.9]	ρ_{xyN} = [0.8, 0.8]	ρ_{xyN} = [0.7, 0.7]	ρ_{xyN} = [0.6, 0.6]
	PRE	PRE	PRE	PRE
$\boldsymbol{\tau}_{[u]N}$	[122, 244]	[125, 205]	[121, 171]	[121, 162]
$\boldsymbol{\tau}_{[R]N}$	[122, 122]	[125, 135]	[121, 132]	[126, 148]
$\boldsymbol{\tau}_{[P]N}$	[124, 300]	[123, 277]	[124, 260]	[123, 242]
$\boldsymbol{\tau}_{[reg]N}$	[119, 122]	[128, 129]	[120, 121]	[126, 130]
$\boldsymbol{\tau}_{[expR]N}$	[119, 162]	[128, 143]	[118, 124]	[123, 130]
$\boldsymbol{\tau}_{[expP]N}$	[123, 231]	[124, 251]	[123, 224]	[207, 123]
$\boldsymbol{P}_{1N}$	[119, 122]	[127, 129]	[120, 121]	[126, 130]

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$\boldsymbol{P_{2N}}$	[128,	240]	[123,	283]	[128,	228]	[129,	229]
$\boldsymbol{P_{3N}}$	[123,	125]	[128,	139]	[121,	141]	[128,	156]

Table 7: PREs of the NSRSS estimators compared to Population 2 estimators under NSSRS

$\boldsymbol{n_N}$=24	$\boldsymbol{\rho_{xyN}}$ = [0.9, 0.9]		$\boldsymbol{\rho_{xyN}}$ = [0.8, 0.8]		$\boldsymbol{\rho_{xyN}}$ = [0.7, 0.7]		$\boldsymbol{\rho_{xyN}}$ = [0.6, 0.6]	
	PRE		PRE		PRE		PRE	
$\boldsymbol{\tau_{[u]N}}$	[109,	114]	[102,	113]	[107,	110]	[107,	109]
$\boldsymbol{\tau_{[R]N}}$	[107,	108]	[102,	104]	[108,	110]	[103,	107]
$\boldsymbol{\tau_{[P]N}}$	[101,	121]	[101,	119]	[101,	120]	[101,	114]
$\boldsymbol{\tau_{[reg]N}}$	[106,	107]	[103,	105]	[106,	108]	[101,	106]
$\boldsymbol{\tau_{[expR]N}}$	[102,	107]	[104,	106]	[106,	109]	[101,	104]
$\boldsymbol{\tau_{[expP]N}}$	[102,	120]	[101,	117]	[109,	115]	[101,	101]
$\boldsymbol{P_{1N}}$	[106,	107]	[103,	105]	[106,	108]	[101,	106]
$\boldsymbol{P_{2N}}$	[102,	121]	[109,	119]	[102,	128]	[103,	123]
$\boldsymbol{P_{3N}}$	[108,	109]	[102,	106]	[102,	107]	[103,	113]
$\boldsymbol{n_N}$=36	ρ_{xyN} = [0.9, 0.9]		ρ_{xyN} = [0.8, 0.8]		ρ_{xyN} = [0.7, 0.7]		ρ_{xyN} = [0.6, 0.6]	
	PRE		PRE		PRE		PRE	
$\boldsymbol{\tau_{[u]N}}$	[109,	113]	[101,	121]	[098,	105]	[107,	109]
$\boldsymbol{\tau_{[R]N}}$	[106,	109]	[102,	104]	[098,	108]	[103,	107]
$\boldsymbol{\tau_{[P]N}}$	[101,	123]	[101,	119]	[100,	119]	[101,	114]
$\boldsymbol{\tau_{[reg]N}}$	[105,	108]	[103,	104]	[096,	108]	[101,	106]
$\boldsymbol{\tau_{[expR]N}}$	[101,	107]	[103,	106]	[096,	108]	[101,	105]
$\boldsymbol{\tau_{[expP]N}}$	[109,	120]	[102,	117]	[099,	113]	[101,	111]
$\boldsymbol{P_{1N}}$	[105,	108]	[103,	104]	[096,	108]	[101,	106]
$\boldsymbol{P_{2N}}$	[102,	122]	[110,	119]	[102,	129]	[102,	122]
$\boldsymbol{P_{3N}}$	[109,	111]	[102,	105]	[102,	107]	[103,	110]

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Table 8: The MSEs and PREs of the proposed and existing estimators, comparing neutrosophic and classical approaches for Population 2 for n_N =24.

ρ_{xyN} = [0.9, 0.9]	MSE			PRE		
Estimators	Neutrosophic		Classical	Neutrosophic		Classical
$\tau_{[u]N}$	[6.33409,	20.82624]	10.82059	[100,	100]	100
$\tau_{[R]N}$	[2.23469,	4.9437]	2.94943	[283,	421]	367
$\tau_{[P]N}$	[23.85232,	90.98492]	39.50349	[23,	27]	27
$\tau_{[reg]N}$	[1.79683,	3.29563]	2.44849	[353,	632]	442
$\tau_{[expR]N}$	[2.60704,	6.10045]	4.28355	[243,	341]	253
$\tau_{[expP]N}$	[13.41585,	49.12106]	22.56057	[42,	47]	48
P_{1N}	[1.79457,	3.27307]	2.44245	[353,	636]	443
P_{2N}	[1.14734,	3.64488]	1.53196	[552,	571]	706
P_{3N}	[0.71274,	1.31424]	0.91239	[889,	1585]	1186
ρ_{xyN} = [0.8, 0.8]	MSE			PRE		
Estimators	Neutrosophic		Classical	Neutrosophic		Classical
$\tau_{[u]N}$	[6.96397,	21.64214]	11.82107	[100,	100]	100
$\tau_{[R]N}$	[4.25967,	9.11246]	5.72606	[163,	238]	206
$\tau_{[P]N}$	[23.33807,	85.75316]	38.45086	[25,	30]	31
$\tau_{[reg]N}$	[3.29547,	6.64487]	4.71948	[211,	326]	250
$\tau_{[expR]N}$	[3.90309,	8.92963]	6.20672	[178,	242]	190
$\tau_{[expP]N}$	[13.44229,	47.24998]	22.56911	[46,	52]	52
P_{1N}	[3.29186,	6.61524]	4.71002	[212,	327]	251
P_{2N}	[1.43845,	4.76444]	1.99172	[454,	484]	594
P_{3N}	[1.37624,	2.72585]	1.79433	[506,	794]	659
ρ_{xyN} = [0.7, 0.7]	MSE			PRE		
Estimators	Neutrosophic		Classical	Neutrosophic		Classical
$\tau_{[u]N}$	[7.61888,	20.64343]	12.79321	[100,	100]	100

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$\tau_{[R]N}$	[6.20828, 13.84982]	8.44015	[123, 149]	152
$\tau_{[P]N}$	[22.79331, 81.42014]	37.43026	[25, 33]	34
$\tau_{[reg]N}$	[4.64649, 8.96223]	6.83309	[164, 230]	187
$\tau_{[expR]N}$	[5.1931, 10.49874]	8.08118	[147, 197]	158
$\tau_{[expP]N}$	[13.48561, 44.28389]	22.57624	[47, 56]	57
P_{1N}	[4.64136, 8.92852]	6.81967	[164, 231]	188
P_{2N}	[1.53563, 5.74515]	2.17067	[359, 469]	589
P_{3N}	[2.01247, 3.99382]	2.64714	[379, 517]	483

ρ_{xyN} = [0.6, 0.6]	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\tau_{[u]N}$	[8.43717, 20.53845]	13.66185	[100, 100]	100
$\tau_{[R]N}$	[8.18271, 18.317]	11.04799	[103, 112]	124
$\tau_{[P]N}$	[22.3359, 76.82708]	36.29881	[27, 38]	38
$\tau_{[reg]N}$	[5.98512, 11.26629]	8.73903	[141, 182]	156
$\tau_{[expR]N}$	[6.6044, 12.66933]	9.85203	[128, 162]	139
$\tau_{[expP]N}$	[13.681, 41.92437]	22.47744	[49, 62]	61
P_{1N}	[5.97828, 11.2277]	8.72146	[141, 183]	157
P_{2N}	[1.54129, 5.97125]	2.23488	[344, 547]	611
P_{3N}	[2.65872, 5.31573]	3.46083	[317, 386]	395

Table 9: The MSEs and PREs of the proposed and existing estimators, comparing neutrosophic and classical approaches for Population 2 for n_N =36.

ρ_{xyN} = [0.9, 0.9]	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\tau_{[u]N}$	[4.38596, 14.80505]	7.46179	[100, 100]	100
$\tau_{[R]N}$	[1.58153, 3.51391]	2.08743	[277, 421]	357
$\tau_{[P]N}$	[16.36886, 64.67041]	27.02848	[23, 27]	28

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$\boldsymbol{\tau}_{[reg]N}$	[1.32371, 2.45155]	1.81122	[331, 604]	412
$\boldsymbol{\tau}_{[expR]N}$	[1.83644, 4.3377]	3.00057	[239, 341]	249
$\boldsymbol{\tau}_{[expP]N}$	[9.2301, 34.91596]	15.4711	[42, 48]	48
$\boldsymbol{P}_{1N}$	[1.32263, 2.44051]	1.80838	[332, 607]	413
$\boldsymbol{P}_{2N}$	[0.82392, 2.74491]	1.10305	[532, 539]	676
$\boldsymbol{P}_{3N}$	[0.52577, 0.99155]	0.67596	[834, 1493]	1104
$\boldsymbol{\rho}_{xyN}$ = [0.8, 0.8]	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\boldsymbol{\tau}_{[u]N}$	[4.86685, 15.38448]	8.20566	[100, 100]	100
$\boldsymbol{\tau}_{[R]N}$	[3.02058, 6.46028]	4.05929	[161, 238]	202
$\boldsymbol{\tau}_{[P]N}$	[16.0519, 61.07339]	26.35684	[25, 30]	31
$\boldsymbol{\tau}_{[reg]N}$	[2.43928, 4.90893]	3.49232	[200, 313]	235
$\boldsymbol{\tau}_{[expR]N}$	[2.77636, 6.32679]	4.38187	[175, 243]	187
$\boldsymbol{\tau}_{[expP]N}$	[9.29203, 33.63334]	15.53065	[46, 52]	53
$\boldsymbol{P}_{1N}$	[2.43750, 4.89418]	3.48774	[200, 314]	235
$\boldsymbol{P}_{2N}$	[1.02455, 3.55519]	1.42207	[433, 475]	577
$\boldsymbol{P}_{3N}$	[1.01568, 2.03624]	1.32854	[479, 756]	618
$\boldsymbol{\rho}_{xyN}$ = [0.7, 0.7]	MSE		PRE	
Estimators	Neutrosophic	Classical	Neutrosophic	Classical
$\boldsymbol{\tau}_{[u]N}$	[5.34889, 14.63937]	8.91654	[100, 100]	100
$\boldsymbol{\tau}_{[R]N}$	[4.39296, 9.81505]	5.95609	[122, 149]	150
$\boldsymbol{\tau}_{[P]N}$	[15.70954, 57.87504]	25.70691	[25, 35]	35
$\boldsymbol{\tau}_{[reg]N}$	[3.42888, 6.61932]	5.03206	[156, 221]	177
$\boldsymbol{\tau}_{[expR]N}$	[3.69534, 7.42579]	5.70757	[145, 197]	156
$\boldsymbol{\tau}_{[expP]N}$	[9.35363, 31.45579]	15.58299	[47, 57]	57
$\boldsymbol{P}_{1N}$	[3.42635, 6.60245]	5.02552	[156, 222]	177

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$\boldsymbol{P_{2N}}$	[1.09014,	4.2836]	1.54156	[342,	421]	578
$\boldsymbol{P_{3N}}$	[1.48147,	2.97639]	1.95129	[361,	492]	457
$\boldsymbol{\rho_{xyN}}$ = [0.6, 0.6]	MSE			PRE		
Estimators	Neutrosophic		Classical	Neutrosophic		Classical
$\boldsymbol{\tau_{[u]N}}$	[5.92875,	14.60706]	9.54324	[100,	100]	100
$\boldsymbol{\tau_{[R]N}}$	[5.75661,	13.01336]	7.77318	[103,	112]	123
$\boldsymbol{\tau_{[P]N}}$	[15.43541,	54.54652]	24.96728	[28,	37]	38
$\boldsymbol{\tau_{[reg]N}}$	[4.39799,	8.38299]	6.40867	[135,	174]	149
$\boldsymbol{\tau_{[expR]N}}$	[4.67586,	9.01699]	6.95146	[127,	162]	137
$\boldsymbol{\tau_{[expP]N}}$	[9.51527,	29.78357]	15.54851	[49,	62]	61
$\boldsymbol{P_{1N}}$	[4.3946,	8.36349]	6.4001	[135,	175]	149
$\boldsymbol{P_{2N}}$	[1.09685,	4.44168]	1.58358	[329,	541]	603
$\boldsymbol{P_{3N}}$	[1.94138,	3.97453]	2.54237	[305,	368]	375

8. Discussion

The study approximated the first NSRSS estimators to the first order using mathematical formulations. Simulation experiments and numerical examples were conducted to assess the properties of the suggested NSRSS estimators. Real-world climate data from a U.S. state was utilized for numerical illustrations, whereas the simulations used generated Nc datasets with different correlation coefficients and sample sizes. Tables 2, 4, 5, 6, 7, 8, and 9 summarize the findings, displaying MSEs and PREs for both suggested and existing NSRSS estimators. PRE values for the NSRSS estimators were compared to NSSRS estimators, with results detailed in Tables 6 and 7.

The excellence of the recommended estimators is demonstrated by the presentation of MSEs and PREs for both the suggested and current estimators in Table 2. Table 2's bold language highlights how well the suggested NSRSS estimators performed, consistently showing lower MSEs and higher PREs than the current approaches.

Likewise, simulation findings for suggested neutrosophic data under various correlation coefficients and sample sizes are shown in Tables 4 and 5. These tables confirm that the suggested estimators outperform the existing ones in terms of performance, obtaining higher PREs and lower MSEs. Furthermore, using the suggested estimator, MSE values fall but PRE values grow when sample sizes and correlation coefficients climb. This pattern shows that the suggested estimators continue to perform similarly to traditional SRSS under NSRSS.

In particular, the PRE values of the recommended NSRSS estimators and those of the NSSRS estimators are contrasted in Tables 6 and 7. Since RSS is generally acknowledged as an improvement over SRS, the data show that all PRE values are more than 100, highlighting the superiority of NSRSS estimators over their NSSRS counterparts.

Tables 8 and 9 offer a comparison between classical SRSS and NSRSS, using MSEs and PREs as benchmarks. These tables show that MSEs obtained through classical SRSS fall within the range of values derived from NSRSS, with NSRSS consistently delivering more outcomes that are effective. The analysis demonstrates that classical SRSS is inadequate for handling ambiguous or indeterminate data. NSRSS, however, excels in estimating uncertain or interval-based data, offering reliable results for Nc datasets compared to classical methods.

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9. Conclusion

This research addresses the difficulties challenged by unclear or uncertain data within the context of traditional statistical methods. The lack of prior studies tackling this specific issue highlights the novelty and importance of this work; it offers a substantial improvement in survey sampling methods, especially when it comes to managing ambiguous and diverse data.

In this manuscript, we introduce the first NSRSS method within the field of sampling theory. We propose enhanced NSRSS estimators aimed at estimating population means by incorporating subsidiary information. We used first-order approximations to determine the expression of bias and MSE for these recommended estimators in order to assess their efficacy. The performance of these proposed estimators was compared to existing alternatives through analyses using both simulated datasets using R and real-world data of climate from a U.S. state. Numerical examples and simulation analysis revealed that the recommended estimators consistently outperformed existing methods under the NSRSS framework. Among these, the estimator labeled P_{2N} demonstrated the best performance. Notably, the sensitivity examination of the suggested estimators under NSRSS aligns closely with the behavior observed in classical SRSS. As sample sizes and correlation coefficients increase, the MSE of the suggested estimators decreases, while the PRE improves.

Further comparisons between NSRSS estimators and their NSSRS counterparts confirm that NSRSS offers a more robust alternative, similar to the way classical SRSS is superior to SSRS. The proposed estimators' MSEs under conventional SRSS range from the lower to the higher MSE values found using NSRSS. The same trend applies to PREs, emphasizing the enhanced efficacy of NSRSS over traditional methods. This study highlights NSRSS's superior ability to manage neutrosophic data and produce more accurate population mean estimations than existing techniques.

The findings from numerical illustrations and simulations make a strong case for adopting the proposed NSRSS estimators in various practical applications. Given the limited research on Nc SRSS estimators, there are many opportunities for further development. Future work could explore variations such as unbalanced NRSS, extreme NRSS, and double NRSS, similar to extensions in classical RSS. Additionally, replacing the recommended estimators with alternative techniques or methodologies could be investigated. Beyond sampling theory, potential research directions include statistical domains such as control charts, reliability analysis, hypothesis testing, and non-parametric estimation, offering exciting avenues for advancing this field.

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