W- Hausdorff Separation Axiom in Second Order Interval Valued Fuzzy Topological Spaces

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Abstract

We studied and introduced a concept SIVFT then present the concept of SIVF subspace and SIVF product topology in SIVF topological spaces. W-Hausdorff Separation Axiom in SIVF topological spaces and its basics are studied.

Keywords: IVF set (IVFs); IVF topology (IVFT); Second order fuzzy set (SFs); Second order fuzzy topology (SFT); Second order IVF set (SIVFs); Second order IVF topology (SIVFT); SIVF subspace; SIVFW-Hausdorff space

1. Introduction

Decision-making is always plagued by an element of uncertainty and it is an unavoidable aspect in every occupation. To overcome the problems of uncertainty Zadeh [29] framed the fuzzy set with broad application in abundant fields and is used to express imprecise and vague concepts in natural language. Zadeh [30] initiated IVFs in which membership are intervals of numbers. IVFs furnish a sufficient description of uncertainty than conventional Fs's. Chang [5] introduced fuzzy topological spaces based on Fs's of Zadeh [29]. Lowen [17] modified the definition of Chang [5] in order to study the structures of fuzzy topological spaces in detail. Further, the author introduced two functors ω and i to investigate the connections between fuzzy topological spaces and topological spaces. Mondal and Samanta [20] introduced the concept of IVF topology by following the definition of Chang and Lowen's FT. The continuous function from IVFs to IVFs is also defined. Liang [16] proposed SIVFs, which is isolated case of type 2 Fs's. Using SFs's, Kalaichelvi [12] extended FT of Chang and Lowen to SFT. Several versions of Hausdorff separation axiom have been defined ([2],[8],[12],[19],[22],[28]). Hausdorff separation axioms in fuzzy topological spaces introduced by Gantner et al. [9] were extended to second order fuzzy topological spaces by Kalaichelvi [12] and were denoted as W-Hausdorff axioms respectively. Using the concept of IVFs, we [3] introduced the notion of the IVF W-Hausdorff Space are defined by extending the definition of fuzzy Hausdorff space of Gantner and some basic properties regarding this concept are proved. Using SIVFs's, we extend IVFT of Mondal and Samantha's to SIVFT. The aim of this work is to develop SIVFW- Hausdorff space as an extension of IVF W- Hausdorff and some of its basic properties are studied.

2. Preliminaries

Definition: 2.1 [29]

Let $\mho$ be an arbitrary nonempty set. Let I = [0, 1]. A Fs in $\mho$ is a mapping from $\mho$ into I that is a Fs is an element of $I^{\mho}$.

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Definition: 2.2 [11]

$\mathcal{A}_{IV}$: Ʊ → [I] said IVFs in Ʊ, [I] be closed subintervals of [0, 1]. All IVFs's on Ʊ is denoted by IVF (Ʊ).

For every $\mathcal{A}_{IV}$ ∈ IVF(Ʊ) and $\mathscr{v}$ ∈ Ʊ, $\mathcal{A}_{IV}$ ($\mathscr{v}$) = [$\mathcal{A}^{LF}$ ($\mathscr{v}$), $\mathcal{A}^{UF}$ ($\mathscr{v}$)] called membership for element $\mathscr{v}$ to $\mathcal{A}_{IV}$, where $\mathcal{A}^{LF}$: Ʊ → I and $\mathcal{A}^{UF}$: Ʊ → I called lower Fs and upper Fs in Ʊ. $\mathcal{A}_{IV}$ can also be represented as $\mathcal{A}_{IV}$ = [$\mathcal{A}^{LF}$, $\mathcal{A}^{UF}$]

Definition: 2.3[20]

$\mathfrak{Y}_{IV}$ be family of IVFs on Ʊ. The family $\mathfrak{Y}_{IV}$ $\widehat{\subset}$ IVF (Ʊ) called IVFT iff $\mathfrak{Y}_{IV}$ satisfies

(i) 0_{IV}, 1_{IV} ∈ $\mathfrak{Y}_{IV}$

(ii) $\mathcal{A}_{IV}$, $\mathcal{B}_{IV}$ ∈ $\mathfrak{Y}_{IV}$ implies $\mathcal{A}_{IV} \widehat{\cap} \mathcal{B}_{IV}$ ∈ $\mathfrak{Y}_{IV}$

(iii) $\mathcal{A}_{IV_j}$ ∈ $\mathfrak{Y}_{IV}$ for each j ∈ J implies $(\bigcup_{j\in J} (\mathcal{A}_{IV_j}) \in \mathfrak{Y}_{IV}$

(Ʊ, $\mathfrak{Y}_{IV}$) is IVF topological space.

Definition: 2.4 [3]

An IVF topological space (Ʊ, $\mathfrak{Y}_{IV}$) is said to be an IVFW-Hausdorff, denoted by (IVFW-H) or IVFW–T_2, if for all pair of disjoint points x, y ∈ Ʊ, there exists two IVF open sets $\mathcal{A}_{IV}$ = [$\mathcal{A}^{LF}$, $\mathcal{A}^{UF}$] ∈$\mathfrak{Y}_{IV}$ and $\mathcal{B}_{IV}$ = [$\mathcal{B}^{LF}$, $\mathcal{B}^{UF}$] ∈$\mathfrak{Y}_{IV}$ such that $\mathcal{A}^{LF}$ ($\mathscr{v}$) = 1, $\mathcal{A}^{UF}$ ($\mathscr{v}$) = 1, $\mathcal{B}^{LF}$ (y) = 1, $\mathcal{B}^{UF}$(y) = 1 and $\mathcal{A}_{IV} \widehat{\cap} \mathcal{B}_{IV} = 0_{IV}$. .

Definition: 2.5 [16]

A SIVFs in Ʊ is a map IVFs: Ʊ→ $[I]^I$. The family of SIVFs's is denoted by SIVF (Ʊ).

For every $\mathcal{A}_{SIV}$∈ SIVF(Ʊ), $\mathscr{v}$ ∈ Ʊ and α ∈ I. $\mathcal{A}_{SIV}$ ($\mathscr{v}$) (α) = [$\mathcal{A}^{LSF}$ ($\mathscr{v}$)(α), $\mathcal{A}^{USF}$ ($\mathscr{v}$)(α)] called membership element $\mathscr{v}$ to $\mathcal{A}_{SIV}$, where $\mathcal{A}^{LSF}$: Ʊ → I^I and $\mathcal{A}^{USF}$: Ʊ → I^I called lower SFs and upper SFs in Ʊ. Here $\mathcal{A}_{SIV}$ written [$\mathcal{A}^{LSF}$, $\mathcal{A}^{LSF}$].

Two SIVFs's $\mathcal{A}_{SIV}$, $\mathcal{B}_{SIV}$ in SIVF (Ʊ), ∀ $\mathscr{v}$ ∈ Ʊ and ∀ α ∈ I

(i) $\mathcal{A}_{SIV} \hat{\hat{\subseteq}} \mathcal{B}_{SIV}$ iff $\mathcal{A}^{LSF}$ ($\mathscr{v}$) ≤ $\mathcal{B}^{LSF}$ ($\mathscr{v}$) and $\mathcal{A}^{USF}$ ($\mathscr{v}$) ≤ $\mathcal{B}^{USF}$ ($\mathscr{v}$),

iff $\mathcal{A}^{LSF}$ ($\mathscr{v}$) (α) ≤ $\mathcal{B}^{LSF}$ ($\mathscr{v}$) (α) and $\mathcal{A}^{USF}$ ($\mathscr{v}$) (α) ≤ $\mathcal{B}^{USF}$ ($\mathscr{v}$) (α),

(ii) $\mathcal{A}_{SIV} = \mathcal{B}_{SIV}$ iff $\mathcal{A}_{SIV} \hat{\hat{\subseteq}} \mathcal{B}_{SIV}$ and $\mathcal{B}_{SIV} \hat{\hat{\subseteq}} \mathcal{A}_{SIV}$

(iii) The union $\mathcal{A}_{SIV} \hat{\hat{\cup}} \mathcal{B}_{SIV}$ and intersection $\mathcal{A}_{SIV} \hat{\hat{\cap}} \mathcal{B}_{SIV}$ are defined respectively, by

$$\mathcal{A}_{SIV} \hat{\hat{\cup}} \mathcal{B}_{SIV} = [\mathcal{A}^{LSF} \vee \mathcal{B}^{LSF}, \mathcal{A}^{USF} \vee \mathcal{B}^{USF}]$$

$$\mathcal{A}_{SIV} \hat{\hat{\cap}} \mathcal{B}_{SIV} = [\mathcal{A}^{LSF} \wedge \mathcal{B}^{LSF}, \mathcal{A}^{USF} \wedge \mathcal{B}^{USF}]$$

(iv) The constant SIVFs's 0_{SIV} and 1_{SIV} defined $0_{SIV} = [0_{SF}, 0_{SF}]$, $1_{SIV} = [1_{SF}, 1_{SF}]$

3. Second Order Interval Valued Fuzzy Topological Spaces

Definition: 3.1

$\mathfrak{Y}_{SIV}$ collection of SIVFs on Ʊ called SIVFT (Chang) iff $\mathfrak{Y}_{SIV}$ satisfies

(i) 0_{SIV}, 1_{SIV} ∈ $\mathfrak{Y}_{SIV}$

(ii) $\mathcal{A}_{SIV}$, $\mathcal{B}_{SIV}$ ∈ $\mathfrak{Y}_{SIV}$ implies $\mathcal{A}_{SIV} \cap \mathcal{B}_{SIV}$ ∈ $\mathfrak{Y}_{SIV}$

(iii) $\mathcal{A}_{SIV_j}$ ∈ $\mathfrak{Y}_{SIV}$ for each j ∈ J implies $(\bigcup_j \mathcal{A}_{SIV_j}) \in \mathfrak{Y}_{SIV}$

The pair (Ʊ, $\mathfrak{Y}_{SIV}$) is SIVF topological space (Chang).

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Example: 3.2

A nonnull set $\mho$ and +ve integer n. $\mathfrak{Y}_{SIV} = \{1_{SIV}\} \cup \{\mathcal{A}_{SIV} \in$ SIVF($\mho$) / $\mathcal{A}^{LSF}(\mathscr{v})(\alpha) = 0$, $\mathcal{A}^{USF}(\mathscr{v})(\alpha) = 0$, for every $\mathscr{v} \in \mho$ and for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . . n}. Then $\mathfrak{Y}_{SIV}$ is a SIVFT on $\mho$.

Proof:

(i) Since $0_{SF}(\mathscr{v})(\alpha) = 0$, $0_{SF}(\mathscr{v})(\alpha) = 0$, for every $\mathscr{v} \in \mho$ and for every $\alpha \in$ I

Therefore $0_{SIV} \in \mathfrak{Y}_{SIV}$

(ii) $\mathcal{A}_{SIV_j} \in \mathfrak{Y}_{SIV}$ for j ∈ J

implies $\mathcal{A}^{LSF}{}_j(\mathscr{v})(\alpha) = 0$, $\mathcal{A}^{USF}{}_j(\mathscr{v})(\alpha) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, .. n, for every $\mathscr{v} \in \mho$ and for every j ∈ J.

Therefore $\bigvee_{j \in J}(\mathcal{A}^{LSF}{}_j(\mathscr{v})(\alpha)) = 0$, $\bigvee_{j \in J}(\mathcal{A}^{USF}{}_j(\mathscr{v})(\alpha)) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . . n, for every $\mathscr{v} \in \mho$

implies $(\bigcup_j \mathcal{A}_{SIV_j}) \in \mathfrak{Y}_{SIV}$

(iii) $\mathcal{A}_{SIV_\mu} \in \mathfrak{Y}_{SIV}$ for μ = 1 to m

implies $\mathcal{A}^{LSF}{}_\mu(\mathscr{v})(\alpha) = 0$, $\mathcal{A}^{USF}{}_\mu(\mathscr{v})(\alpha) = 0$, for $\alpha \neq \frac{r}{n}$, r = 0, 1, . . n, for every $\mathscr{v} \in \mho$ and for μ = 1 to m

Therefore $\bigwedge_{\mu=1}^{m}(\mathcal{A}^{LSF}{}_\mu(\mathscr{v})(\alpha)) = 0$, $\bigwedge_{\mu=1}^{m}(\mathcal{A}^{USF}{}_\mu(\mathscr{v})(\alpha)) = 0$, $\alpha \neq \frac{r}{n}$, r = 0,….. . n,

for every $\mathscr{v} \in \mho$

implies $(\bigcap_{\mu=1}^{m} \mathcal{A}_{SIV_i}) \in \mathfrak{Y}_{SIV}$

Therefore $\mathfrak{Y}_{SIV}$ is a SIVFT on $\mho$.

4. W-Hausdorff Separation Axiom in Second Order Interval Valued Fuzzy Topological Spaces

Definition: 4.1

SIVF topological space $(\mho, \mathfrak{Y}_{SIV})$ said (SIVFW-H)$_1$ if ∀ $\mathscr{v}$, y ∈ $\mho$, $\mathscr{v} \neq$ y, there exists two IVF open sets $\mathcal{A}_{SIV} = [\mathcal{A}^{LSF}, \mathcal{A}^{USF}]$, $\mathcal{B}_{SIV} = [\mathcal{B}^{LSF}, \mathcal{B}^{USF}] \in \mathfrak{Y}_{SIV}$, such that $\mathcal{A}^{LSF}(\mathscr{v}) = 1$, $\mathcal{A}^{USF}(\mathscr{v}) = 1$, $\mathcal{B}^{LSF}(y) = 1$, $\mathcal{B}^{USF}(y) = 1$ and $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$.

Definition: 4.2

SIVF topological space $(\mho, \mathfrak{Y}_{SIV})$ said to SIVFW-Hausdorff of type 2, denoted (SIVFW-H)$_2$, by replacing $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$ in above definition by $\mathcal{A}_{SIV} \cap_2 \mathcal{B}_{SIV} = 0_{SIV}$.

Note: 4.3

$(\mho, \mathfrak{Y}_{SIV})$ is (SIVFW-H)$_1$ implies $(\mho, \mathfrak{Y}_{SIV})$ is (SIVFW-H)$_2$ since $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$ implies $\mathcal{A}_{SIV} \cap_2 \mathcal{B}_{SIV} = 0_{SIV}$. The converse of the above implication is not true. Since $\mathcal{A}_{SIV} \cap_2 \mathcal{B}_{SIV} = 0_{SIV}$ need not imply that $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$.

Definition: 4.4

Let (Ʊ, $\mathfrak{Y}_{SIV}$) be SIVF topological space, a nonempty $\mathfrak{M}$ is subset of Ʊ. Let $\mathcal{A}_{SIV} \in \mathfrak{Y}_{SIV}$. Define the restriction function $\mathcal{A}_{SIV}/\mathfrak{M} : \mathfrak{M} \rightarrow [I]^I$ as follows

$$(\mathcal{A}^{LSF} / \mathfrak{M})(\mathfrak{p}) = \mathcal{A}^{LSF}(\mathfrak{p}),\ (\mathcal{A}^{USF} / \mathfrak{M})(\mathfrak{p}) = \mathcal{A}^{USF}(\mathfrak{p}), \text{ if } \mathfrak{p} \in \mathfrak{M}.$$

Define $\mathfrak{Y}_{SIV}/\ \mathfrak{M}$ = {$\mathcal{A}_{SIV}/\mathfrak{M}$ such that $\mathcal{A}_{SIV} \in \mathfrak{Y}_{SIV}$}. Then $\mathfrak{Y}_{SIV}/\ \mathfrak{M}$ called SIVF subspace topology of $\mathfrak{M}$ and ($\mathfrak{M}$, $\mathfrak{Y}_{SIV}/\ \mathfrak{M}$) called SIVF subspace of (Ʊ, $\mathfrak{Y}_{SIV}$).

Theorem: 4.5

1. SIVF subspace of (SIVFW-H)$_1$ space is (SIVFW-H)$_1$
2. SIVF subspace of (SIVFW-H)$_2$ space is (SIVFW-H)$_2$

Proof:

(Ʊ, $\mathfrak{Y}_{SIV}$) is (SIVFW-H)$_1$ space and $\mathfrak{M}$ be nonempty subset of Ʊ.

Consider $y_1, y_2 \in \mathfrak{M}$ where $y_1 \neq y_2$. Then $y_1, y_2 \in$ Ʊ, there exists two SIVF open set $\mathcal{A}_{SIV} = [\mathcal{A}^{LSF}, \mathcal{A}^{USF}]$, $\mathcal{B}_{SIV} = [\mathcal{B}^{LSF}, \mathcal{B}^{USF}] \in \mathfrak{Y}_{SIV}$, such that $\mathcal{A}^{LSF}(y_1) = 1$, $\mathcal{A}^{USF}(y_1) = 1$, $\mathcal{B}^{LSF}(y_2) = 1$, $\mathcal{B}^{USF}(y_2) = 1$ and $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$

Let $\mathcal{A}_{SIV}/\mathfrak{M}$, $\mathcal{B}_{SIV}/\mathfrak{M} \in \mathfrak{Y}_{SIV}/\ \mathfrak{M}$, where

$$(\mathcal{A}_{SIV}/\mathfrak{M}) = [\mathcal{A}^{LSF}/\mathfrak{M}, \mathcal{A}^{USF}/\mathfrak{M}],\ (\mathcal{B}_{SIV}/\mathfrak{M}) = [\mathcal{B}^{LSF}/\mathfrak{M}, \mathcal{B}^{USF}/\mathfrak{M}]$$

Therefore

$(\mathcal{A}^{LSF} / \mathfrak{M})(y_1) = \mathcal{A}^{LSF}(y_1) = 1$

$(\mathcal{A}^{USF} / \mathfrak{M})(y_1) = \mathcal{A}^{USF}(y_1) = 1$

$(\mathcal{B}^{LSF} / \mathfrak{M})(y_2) = \mathcal{B}^{LSF}(y_2) = 1$

$(\mathcal{B}^{USF}/\mathfrak{M})(y_2) = \mathcal{B}^{USF}(y_2) = 1$

Also $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$ implies $\mathcal{A}^{LSF}(\mathscr{v}) = 0$, $\mathcal{A}^{USF}(\mathscr{v}) = 0$ or $\mathcal{B}^{LSF}(\mathscr{v}) = 0$ and

$\mathcal{B}^{USF}(\mathscr{v}) = 0$, $\forall\ \mathscr{v} \in \mathfrak{M} \subseteq$ Ʊ

implies $(\mathcal{A}^{LSF} / \mathfrak{M})(\mathscr{v}) = 0$, $(\mathcal{A}^{USF} / \mathfrak{M})(\mathscr{v}) = 0$ or

$(\mathcal{B}^{LSF} / \mathfrak{M})(\mathscr{v}) = 0$, $(\mathcal{B}^{USF} / \mathfrak{M})(\mathscr{v}) = 0$ $\quad \forall\ \mathscr{v} \in \mathfrak{M} \subseteq$ Ʊ

implies $\mathcal{A}_{SIV}/\mathfrak{M} \cap_1 \mathcal{B}_{SIV}/\mathfrak{M} = 0_{SIV}$

Hence ($\mathfrak{M}$, $\mathfrak{Y}_{SIV}/\mathfrak{M}$) is (SIVFW-H)$_1$

($\mathfrak{M}$, $\mathfrak{Y}_{SIV}/\mathfrak{M}$) is a (SIVFW-H)$_2$ as $(\frac{\mathcal{A}_{SIV}}{\mathfrak{M}} \cap_1 \frac{\mathcal{B}_{SIV}}{\mathfrak{M}}) = 0_{SIV}$

implies $(\mathcal{A}_{SIV}/\mathfrak{M} \cap_2 \mathcal{B}_{SIV}/\mathfrak{M}) = 0_{SIV}$.

Definition: 4.6

Let (Ʊ,$\mathfrak{Y}_{SIV_1}$), ($\mathfrak{M}$,$\mathfrak{Y}_{SIV_2}$) be two SIVF topological spaces. If $\mathcal{A}_{SIV} \in \mathfrak{Y}_{SIV_1}$ and $\mathcal{B}_{SIV} \in \mathfrak{Y}_{SIV_2}$, then the product $(\mathcal{A}_{SIV} * \mathcal{B}_{SIV})$ in Ʊ x $\mathfrak{M}$ is defined as follows :

$[\mathcal{A}_{SIV} * \mathcal{B}_{SIV}] = [\mathcal{A}^{LSF} * \mathcal{B}^{LSF}, \mathcal{A}^{USF} * \mathcal{B}^{USF}]$ where

$(\mathcal{A}^{LSF} * \mathcal{B}^{LSF})(\mathscr{v}, y)(\alpha) = \mathcal{A}^{LSF}(\mathscr{v})(\alpha) \wedge \mathcal{B}^{LSF}(y)(\alpha)$

$(\mathcal{A}^{USF} * \mathcal{B}^{USF})(\mathscr{v}, y)(\alpha) = \mathcal{A}^{USF}(\mathscr{v})(\alpha) \wedge \mathcal{B}^{USF}(y)(\alpha)$

$\forall\ (\mathscr{v}, y) \in$ Ʊ x $\mathfrak{M}$, $\alpha \in I$.

Product topology $\mathfrak{Y}_{SIV_1}$ x $\mathfrak{Y}_{SIV_2}$ on Ʊ x $\mathfrak{M}$ is SIVFT having the collection {$\mathcal{A}_{SIV} * \mathcal{B}_{SIV}$ / $\mathcal{A}_{SIV} \in \mathfrak{Y}_{SIV_1}$, $\mathcal{B}_{SIV} \in \mathfrak{Y}_{SIV_2}$} as basis.

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Theorem:4.7

1. Product of two $(\text{SIVFW-H})_1$ spaces is $(\text{SIVFW-H})_1$
2. Product of two $(\text{SIVFW-H})_2$ spaces is $(\text{SIVFW-H})_2$

Proof:

$(\mathfrak{U}, \mathfrak{Y}_{SIV_1})$ and $(\mathfrak{M}, \mathfrak{Y}_{SIV_2})$ be $(\text{SIVFW-H})_1$ spaces.

Consider distinct points $(\mathscr{v}_1, y_1), (\mathscr{v}_2, y_2) \in \mathfrak{U} \text{ x } \mathfrak{M}$

Either $\mathscr{v}_1 \neq \mathscr{v}_2$ or $y_1 \neq y_2$

take $\mathscr{v}_1 \neq \mathscr{v}_2$

Therefore there exists two SIVF open sets $\mathcal{A}_{SIV} = [\mathcal{A}^{LSF}, \mathcal{A}^{USF}]$, $\mathcal{B}_{SIV} = [\mathcal{B}^{LSF}, \mathcal{B}^{USF}] \in \mathfrak{Y}_{SIV_1}$ such that $\mathcal{A}^{LSF}(\mathscr{v}_1) = 1$, $\mathcal{A}^{USF}(\mathscr{v}_1) = 1$, $\mathcal{B}^{LSF}(\mathscr{v}_2) = 1$, $\mathcal{B}^{USF}(\mathscr{v}_2) = 1$ and

$\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$

$\mathcal{A}_{SIV}, \mathcal{B}_{SIV} \in \mathfrak{Y}_{SIV_1}$ implies $\mathcal{A}_{SIV} * 1_{SIV} \in \mathfrak{Y}_{SIV_1} \text{ x } \mathfrak{Y}_{SIV_2}$, $\mathcal{B}_{SIV} * 1_{SIV} \in \mathfrak{Y}_{SIV_1} \text{ x } \mathfrak{Y}_{SIV_2}$

where

$\mathcal{A}_{SIV} * 1_{SIV} = [\mathcal{A}^{LSF} * 1_{SF}, \mathcal{A}^{USF} * 1_{SF}]$ and $\mathcal{B}_{SIV} * 1_{SIV} = [\mathcal{B}^{LSF} * 1_{SF}, \mathcal{B}^{USF} * 1_{SF}]$

Consider

$(\mathcal{A}^{LSF} * 1_{SF})(\mathscr{v}_1, y_1) = \mathcal{A}^{LSF}(\mathscr{v}_1) \wedge 1_{SF}(y_1)$, for every $(\mathscr{v}_1, y_1) \in \mathfrak{U} \text{ x } \mathfrak{M}$

$= 1$

$(\mathcal{A}^{USF} * 1_{SF})(\mathscr{v}_1, y_1) = \mathcal{A}^{USF}(\mathscr{v}_1) \wedge 1_{SF}(y_1)$, for every $(\mathscr{v}_1, y_1) \in \mathfrak{U} \text{ x } \mathfrak{M}$

$= 1$

$(\mathcal{B}^{LSF} * 1_{SF})(\mathscr{v}_2, y_2) = \mathcal{B}^{LSF}(\mathscr{v}_2) \wedge 1_{SF}(y_2)$, for every $(\mathscr{v}_2, y_2) \in \mathfrak{U} \text{ x } \mathfrak{M}$

$= 1$

$(\mathcal{B}^{USF} * 1_{SF})(\mathscr{v}_2, y_2) = \mathcal{B}^{USF}(\mathscr{v}_2) \wedge 1_{SF}(y_2)$, for every $(\mathscr{v}_2, y_2) \in \mathfrak{U} \text{ x } \mathfrak{M}$

$= 1$

For $(\mathscr{v}, y) \in \mathfrak{U} \text{ x } \mathfrak{M}$

Consider

$(\mathcal{A}^{LSF} * 1_{SF})(\mathscr{v}, y) \neq 0$, $(\mathcal{A}^{USF} * 1_{SF})(\mathscr{v}, y) \neq 0$

implies $(\mathcal{A}^{LSF} * 1_{SF})(\mathscr{v}, y)(\alpha) \neq 0$, $(\mathcal{A}^{USF} * 1_{SF})(\mathscr{v}, y)(\alpha) \neq 0$, for some $\alpha \in I$

implies $\mathcal{A}^{LSF}(\mathscr{v})(\alpha) \wedge 1_{SF}(y)(\alpha) \neq 0$,

$\mathcal{A}^{USF}(\mathscr{v})(\alpha) \wedge 1_{SF}(y)(\alpha) \neq 0$, for some $\alpha \in I$

implies $\mathcal{A}^{LSF}(\mathscr{v})(\alpha) \neq 0$, $\mathcal{A}^{USF}(\mathscr{v})(\alpha) \neq 0$, $\forall \mathscr{v} \in \mathfrak{U}$, for some $\alpha \in I$

implies $\mathcal{A}^{LSF}(\mathscr{v}) \neq 0$, $\mathcal{A}^{USF}(\mathscr{v}) \neq 0$, $\forall \mathscr{v} \in \mathfrak{U}$

implies $\mathcal{B}^{LSF}(\mathscr{v}) = 0$, $\mathcal{B}^{USF}(\mathscr{v}) = 0$, $\forall \mathscr{v} \in \mathfrak{U}$ (since $\mathcal{A}_{SIV}, \cap_1 \mathcal{B}_{SIV}, = 0_{SIV}$)

implies $\mathcal{B}^{LSF}(\mathscr{v})(\alpha) = 0$, $\mathcal{B}^{USF}(\mathscr{v})(\alpha) = 0$, $\forall \mathscr{v} \in \mathfrak{U}$, $\forall \alpha \in I$

implies $\mathcal{B}^{LSF}(\mathscr{v})(\alpha) \wedge 1_{SF}(y)(\alpha) = 0$, $\mathcal{B}^{USF}(\mathscr{v})(\alpha) \wedge 1_{SF}(y)(\alpha) = 0$, $\forall \mathscr{v} \in \mathfrak{U}$,

$y \in \mathfrak{M}, \alpha \in I$

implies $(\mathcal{B}^{LSF} * 1_{SF})(\mathscr{v}, y)(\alpha) = 0$, $(\mathcal{B}^{USF} * 1_{SF})(\mathscr{v}, y)(\alpha) = 0$,

$\forall\ (\mathscr{v}, y) \in \mho \times \mathfrak{M}$ and $\alpha \in I$

implies $(\mathcal{B}^{LSF} * 1_{SF})\,(\mathscr{v}, y) = 0$, $(\mathcal{B}^{USF} * 1_{SF})\,(\mathscr{v}, y) = 0$

Therefore $(\mathcal{A}_{SIV} * 1_{SIV}) \cap_1 (\mathcal{B}_{SIV} * 1_{SIV}) = 0_{SIV}$

Similarly, $y_1 \neq y_2$

Therefore $(\mho \times \mathfrak{M}, \mathfrak{Y}_{SIV_1} \times \mathfrak{Y}_{SIV_2})$ is $(\text{SIVFW-H})_1$

$(\mho \times \mathfrak{M}, \mathfrak{Y}_{SIV_1} \times \mathfrak{Y}_{SIV_2})$ is $(\text{SIVFW-H})_2$ as $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$ implies $\mathcal{A}_{SIV} \cap_2 \mathcal{B}_{SIV} = 0_{SIV}$

Definition: 4.8

$\{(\mho_j, \mathfrak{Y}_{SIV_j}) \,/\, j \in \mathcal{T}\}$ Collection of SIVF topological spaces, $\mho = \prod_{j \in \mathcal{T}} \mho_j$. SIVF product topology on $\mho$ is SIVF open sets of the form $\mathcal{A}_{SIV} = [\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}{}_j, \prod_{j\in\mathcal{T}} \mathcal{A}^{USF}{}_j]$, where $\mathcal{A}_{SIV_j} \in \mathfrak{Y}_{SIV_j}$ and $\mathcal{A}_{SIV_j} = 1_{SIV}$ except j's. Here

$(\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}{}_j)\,(\mathscr{v}_j)\,(\alpha) = \wedge_{j \in \mathcal{T}} \mathcal{A}^{LSF}{}_j, (\mathscr{v}_j)\,(\alpha)$,

$(\prod_{j\in\mathcal{T}} \mathcal{A}^{USF}{}_j)\,(\mathscr{v}_j)\,(\alpha) = \wedge_{j \in \mathcal{T}} \mathcal{A}^{USF}{}_j\,(\mathscr{v}_j)\,(\alpha)$

$\forall\ (\mathscr{v}_j) \in \prod_{j\in\mathcal{T}} \mho_j$ and for some $\alpha \in I$.

Theorem: 4.9

1. Arbitrary product of $(\text{SIVFW-H})_1$ space is $(\text{SIVFW-H})_1$
2. Arbitrary product of $(\text{SIVFW-H})_2$ space is $(\text{SIVFW-H})_2$

Proof:

$\{(\mho_j, \mathfrak{Y}_{SIV_j}) \,/\, j \in \mathcal{T}\}$ be $(\text{SIVFW-H})_1$ spaces.

$\mho = \prod_{j\in\mathcal{T}} \mho_j$ and $\mathfrak{Y}_{SIV} = \prod_{j\in\mathcal{T}} \mathfrak{Y}_{SIV_j}$

Take distinct IVF points $(\mathscr{v}_j), (y_j) \in \prod_{j\in J} \mho_j$.

Therefore $\mathscr{v}_k \neq y_k$, for some $k \in \mathcal{T}$, there exists SIVF open sets $\mathcal{A}_{SIV} = [\mathcal{A}^{LSF}{}_k, \mathcal{A}^{USF}{}_k,]$,

$\mathcal{B}_{SIV} = [\mathcal{B}^{LSF}{}_k, \mathcal{B}^{USF}{}_k,] \in \mathfrak{Y}_{SIV_k}$, such that $\mathcal{A}^{LSF}{}_k, (\mathscr{v}_k) = 1$, $\mathcal{A}^{USF}{}_k\,(\mathscr{v}_k) = 1$, $\mathcal{B}^{LSF}{}_k\,(y_k) = 1$, $\mathcal{B}^{USF}{}_k\,(y_k) = 1$ and $\mathcal{A}_{SIV_k} \cap_1 \mathcal{B}_{SIV_k} = 0_{SIV}$

Let $\mathcal{A}_{SIV} = \prod_{j\in\mathcal{T}} \mathcal{A}_{SIV_j}$, where $\mathcal{A}_{SIV_j} = 1_{SIV}$, for $j \neq k$

$\mathcal{B}_{SIV} = \prod_{j\in\mathcal{T}} \mathcal{B}_{SIV_j}$, where $\mathcal{B}_{SIV_j} = 1_{SIV}$, for $j \neq k$

Then $\mathcal{A}_{SIV}, \mathcal{B}_{SIV} \in \prod_{j\in\mathcal{T}} \mathfrak{Y}_{SIV_j}$

$$\mathcal{A}_{SIV} = \prod_{j\in\mathcal{T}} \mathcal{A}_{SIV_j} = [\textstyle\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}{}_j, \prod_{j\in\mathcal{T}} \mathcal{A}^{USF}{}_j]$$

$$\begin{aligned}(\textstyle\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}{}_j)\,(\mathscr{v}_j) &= \wedge_{j \in \mathcal{T}} \mathcal{A}^{LSF}{}_j\,(\mathscr{v}_j)\\ &= \mathcal{A}^{LSF}{}_k, (\mathscr{v}_k), \text{ for some } k \in \mathcal{T}\\ &= 1\end{aligned}$$

$$\begin{aligned}(\textstyle\prod_{j\in\mathcal{T}} \mathcal{A}^{USF}{}_j)\,(\mathscr{v}_j) &= \wedge_{j \in \mathcal{T}} \mathcal{A}^{USF}{}_j\,(\mathscr{v}_j)\\ &= \mathcal{A}^{USF}{}_k\,(\mathscr{v}_k), \text{ for some } k \in \mathcal{T}\\ &= 1\end{aligned}$$

$$\mathcal{B}_{SIV} = \prod_{j\in\mathcal{T}} \mathcal{B}_{SIV_j} = [\textstyle\prod_{j\in\mathcal{T}} \mathcal{B}^{LSF}{}_j, \prod_{j\in\mathcal{T}} \mathcal{B}^{USF}{}_j]$$

$$(\textstyle\prod_{j\in\mathcal{T}} \mathcal{B}^{LSF}{}_j)\,(y_j) = \wedge_{j \in \mathcal{T}} \mathcal{B}^{LSF}{}_j\,(y_j)$$

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$= \mathcal{B}^{LSF}{}_{k}\,(y_k)$, for some $k \in \mathcal{T}$

$= \quad 1$

$(\prod_{j\in\mathcal{T}} \mathcal{B}^{USF}j)\ (y_j) \quad = \quad \wedge_{j\,\in\,\mathcal{T}} \mathcal{B}^{USF}{}_{j}\,(y_j)$

$= \mathcal{B}^{USF}{}_{k}\,(y_k)$, for some $k \in \mathcal{T}$

$= \quad 1$

Consider $\forall\, \mathcal{v} \in \mho$

$\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}j\ (\mathcal{v}) \neq 0$ and $\prod_{j\in\mathcal{T}} \mathcal{A}^{USF}j\ (\mathcal{v}) \neq 0$

$(\prod_{j\in\mathcal{T}} \mathcal{A}^{LSF}j\ (\mathcal{v}))\ (\alpha) \neq 0$ and $(\prod_{j\in\mathcal{T}} \mathcal{A}^{USF}j\ (\mathcal{v}))\ (\alpha) \neq 0$, for some $\alpha \in I$

$\wedge_{j\,\in\,\mathcal{T}}(\mathcal{A}^{LSF}{}_{j}\ (\mathcal{v})\ (\alpha)) \neq 0$ and $\wedge_{j\,\in\,\mathcal{T}}(\mathcal{A}^{USF}{}_{j}\ (\mathcal{v})\ (\alpha)) \neq 0$

$\mathcal{A}^{LSF}{}_{k}\ (\mathcal{v})\ (\alpha) \neq 0$ and $\mathcal{A}^{USF}{}_{k}\ (\mathcal{v})\ (\alpha) \neq 0$, for $k \neq j$

$\mathcal{B}^{LSF}{}_{k}\ (\mathcal{v})\ (\alpha) = 0$ and $\mathcal{B}^{USF}{}_{k}\ (\mathcal{v})\ (\alpha) = 0$, $\forall\ \alpha \in I$

$\wedge_{j\,\in\,\mathcal{T}}(\mathcal{B}^{LSF}{}_{k}\ (\mathcal{v})\ (\alpha) = 0$ and $\wedge_{j\,\in\,\mathcal{T}}(\mathcal{B}^{USF}{}_{k}\ (\mathcal{v})\ (\alpha) =$

$(\prod_{j\in\mathcal{T}} \mathcal{B}^{LSF}j)\ (\mathcal{v})\ (\alpha) = 0$ and $(\prod_{j\in\mathcal{T}} \mathcal{B}^{USF}j)\ (\mathcal{v})\ (\alpha) = 0$

$(\prod_{j\in\mathcal{T}} \mathcal{B}^{LSF}j)\ (\mathcal{v}) = 0$ and $(\prod_{j\in\mathcal{T}} \mathcal{B}^{USF}j)\ (\mathcal{v}) = 0$

implies $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$

Hence $(\mho, \mathfrak{Y}_{SIV})$ is $(SIVFW\text{-}H)_1$

$(\mho, \mathfrak{Y}_{SIV})$ is $(SIVFW\text{-}H)_2$ as $\mathcal{A}_{SIV} \cap_1 \mathcal{B}_{SIV} = 0_{SIV}$ implies $\mathcal{A}_{SIV} \cap_2 \mathcal{B}_{SIV} = 0_{SIV}$

5. Conclusion

The level of this research was to evolve W-Hausdorff Separation axioms in sequence to assemble a framework that will furnish method for object ranking. Thus, we initiated a new definition of Second order IVFT, second order IVF Subspace and second order IVF product topology. W- Hausdorff separation axiom in second order IVF topological spaces were presented, and furthermore, its basics are studied.

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