Connection between Legendre Polynomials and classes of Bi-Bazilevic Functions Defined by Borel Distribution and Ruscheweyh Operator

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Abstract

This paper introduces two new classes of bi-Bazilevic and bi-univalent functions that are defined using Borel distribution and Ruscheweyh operator, which also associated with Legendre polynomials and modified Sigmoid function within the open unit disk $\mathbb{D}$. This paper explores the characteristics and behaviors of these functions, we find estimates for the modulus of the initial Taylor series coefficients a_2 and a_3 for functions within our newly defined classes and some of their various subclasses. Moreover, this paper explores the classical Fekete-Szegö functional problem concerning functions f that are classified within our specific classes. Additionally, we obtain the classical Fekete-Szegö inequalities of functions belonging to these classes and some of their various subclasses.

Keywords: Bi-Univalent Functions; Bi-Bazilevic; Borel Distribution; Ruscheweyh operator; Modified Sigmoid Function; Legendre Polynomials; Coefficient estimates; Fekete-Szegö functional problem; Convolution; Hadamard Product

1 Introduction

In this study, we have implored the used of convolution of well known differential operators to defined our novel classes. The set $\mathcal{H}$ consists of all functions $f(z)$ that are analytic within the open unit disk, denoted as $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. These functions must satisfy the normalization conditions $f(0) = 0$ and $1 - f'(0) = 0$. Exploring these functions deepens our comprehension of complex analysis and its diverse applications. Additionally, any function f belonging to the set $\mathcal{H}$ can be represented in the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } z \in \mathbb{D}. \tag{1}$$

Let us consider two functions, f and g, that are analytic in the open unit disk $\mathbb{D}$. We say that f is subordinated to g in this domain, denoted as $f(z) \prec g(z)$ for every z in $\mathbb{D}$, if there exists a Schwarz function h that meets the criteria of $h(0) = 0$ and $|h(z)| < 1$ for all z in $\mathbb{D}$. Furthermore, this function must fulfill the relationship $f(z) = g(h(z))$ for every z in $\mathbb{D}$. This notion plays a significant role in complex analysis, as it provides a framework for comparing two analytic functions within the unit disk. It is essential to note that when the function g is univalent in the unit disk $\mathbb{D}$, the condition $f(z) \prec g(z)$ leads to two significant conclusions: first, that $f(0)$ must equal $g(0)$, and second, that the image of f over $\mathbb{D}$ is entirely contained within the image of g over the same domain. This relationship highlights the crucial role of the subordination principle in exploring the relationships between analytic functions. For those seeking a deeper understanding of the Subordination Principle, it is recommended to consult the works,[10],[11],[22] and,[27] which offer thorough explanations and applications of this principle in complex analysis and geometric function theory.

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In this paper, we define $\mathcal{S}$ as the collection of functions that are univalent within the open unit disk $\mathbb{D}$ and are part of the set $\mathcal{H}$. It is important to note that univalent functions are injective, meaning they can be inverted, although their inverse might not be applicable across the entire unit disk $\mathbb{D}$. According to the Koebe one-quarter Theorem, any function f that belongs to $\mathcal{S}$ will map the disk $\mathbb{D}$ in such a way that it includes the smaller disk $D(0, 1/4)$, which is centered at 0 and has a radius of 1/4. Accordingly, for every function f in $\mathcal{S}$, there exists an inverse $f^{-1} = g$ that can be defined as follows

$$g(f(z)) = z, \quad z \in \mathbb{D}$$

$$f(g(w)) = w, \quad |w| < r(f); \;\; r(f) \geq 1/4.$$

Moreover, the inverse function is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{2}$$

Therefore, we define the family Σ in the following manner. A function $f \in \mathcal{H}$ is considered bi-univalent if both the function f and its inverse f^{-1} exhibit univalence within the domain $\mathbb{D}$. Consequently, we define Σ as the collection of all bi-univalent functions in $\mathcal{H}$ that are represented by equation (1). For example, the following functions belong to the class Σ:

$$z(1-z)^{-1}, \quad \log(1-z)^{-1}, \quad \sqrt{\log(1+z) - \log(1-z)}.$$

However, Koebe function, $z - \dfrac{z^2}{2}$ and $z(1-z^2)^{-1}$ do not belong to the class Σ. For more information about univalent and bi-univalent functions we refer the readers to the articles,[?],[19],[2028] the monographs,[1014] and the references provided therein.

Research in the field of geometric function theory studies the relationships between the coefficients of functions and their geometric characteristics. By finding the limitations placed on the modulus of a function's coefficients, scholars are able to gain a more profound understanding of how these functions behave. This analysis not only deepens our comprehension of geometric function theory but also paves the way for new research opportunities. For instance, within the class $\mathcal{S}$, the modulus of the coefficient a_n is restricted by the integer n, which yields important insights into the geometric attributes of these functions. In particular, the restrictions on the second coefficients in class $\mathcal{S}$ offer vital information regarding growth and distortion bounds. For more information we refer the reader to consult the monographs,[1015] and the related references provided therein.

The study of coefficient-related properties of functions in the bi-univalent class Σ started in the 1970s. A significant milestone was achieved by Lewin,[19] in 1967, who examined the bi-univalent function class and determined a limit for the coefficient $|a_2|$. Following this, in 1969,[28] Netanyahu's research revealed that the highest value of $|a_2|$ for functions belonging to the class Σ is $\frac{4}{3}$. Additionally, Brannan and Clunie, in their 1979 study,[6] confirmed that the inequality $|a_2| \leq \sqrt{2}$ is valid for functions belonging to this category. This foundational research has led to a multitude of studies focused on the coefficient bounds for various subclasses of bi-univalent functions. While there is a wealth of research focused on coefficient bounds, a significant gap still exists in our understanding of the general coefficients $|a_2|$ for cases where $n \geq 4$. The challenge of accurately estimating these coefficients, particularly the general coefficient $|a_n|$, continues to be a prominent issue in the field. This persistent inquiry highlights the complex characteristics of the bi-univalent function class, suggesting that additional research is crucial for a thorough understanding of how these coefficients behave in higher dimensions.

In 1933, Fekete and Szegö[18] established the maximum value of the expression $|a_3 - \lambda a_2^2|$ for a univalent function f, where the real parameter satisfies $0 \leq \lambda \leq 1$. This significant finding gave rise to the Fekete-Szegö problem, which focuses on maximizing the modulus of the functional $\Psi_\lambda(f) = a_3 - \lambda a_2^2$ for functions f belonging to the class $\mathcal{H}$, with λ being any complex number. A multitude of researchers have explored the Fekete-Szegö functional and related coefficient estimation issues. Notable contributions can be found in articles such as,[2],[5],[?],[7],[9],[16],[17],[18],[20],[21],[34] along with the references cited within these works. These investigations have significantly enhanced the comprehension of the Fekete-Szegö problem and its relevance within the domain of geometric function theory.

2 Preliminaries and Lemmas

The details provided in this section are crucial for comprehending the key findings of this paper. In 1975, Ruscheweyh[32] introduced the operator $\mathcal{R}$, which is defined using the convolution (Hadamard product) of two power series. For a function $f \in \mathcal{H}$, a variable $z \in \mathbb{D}$, and a real number $\lambda \geq -1$, the definition of the Ruscheweyh operator is established as follows

$$\mathcal{R}^{\alpha} f(z) = f(z) * \frac{z}{(1-z)^{1-\alpha}}.$$

For $\alpha = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get the Ruscheweyh derivative $\mathcal{R}^{\alpha}$ of the function f:

$$\mathcal{R}^{\alpha} f(z) = z\frac{\left(z^{\alpha-1} f(z)\right)^{(\alpha)}}{\Gamma(\alpha+1)}.$$

Moreover, the Taylor-Maclaurin series of $\mathcal{R}^{\alpha} f$ is given by

$$\mathcal{R}(\alpha, z) = \mathcal{R}^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha+1)} a_n z^n.$$

Let J_β represent the class of analytic functions defined in the open unit disk $\mathbb{D}$, characterized by the expression $f_\beta(z) = z + \sum_{n=2}^{\infty} \beta(t) a_n z^n$, where $\beta(t) = \frac{2}{1+e^{-t}}$ for $t \geq 0$, which is recognized as the modified Sigmoid function. The significance of the Sigmoid function in geometric function theory is well-established. For those seeking further insights into the properties and applications of the Sigmoid function, we recommend consulting the works of,[13],[24],[30] along with the related references cited therein.

Recently, Wanas and Khuttar[35] presented a power series characterized by coefficients that represent the probabilities of the Borel distribution, which can be expressed in the following manner:

$$\mathcal{B}_\eta(z) = z + \sum_{n=2}^{\infty} \frac{\left(\eta(n-1)\right)^{n-2} e^{-\eta(n-1)}}{\Gamma(n)} z^n, \quad \text{where } \ 0 < \eta \leq 1; \ z \in \mathbb{D},$$

where the power series converges everywhere in the complex plane, as demonstrated by the well-known ratio test.

Now, utilizing the convolution (or the Hadamard product), we introduce the following linear operator $\mathcal{R}_\eta^{\alpha} : J_\beta \to J_\beta$. Thus, for any $f_\beta(z) \in J_\beta$ this linear operator is defined as $\mathcal{R}_\eta^{\alpha} f_\beta(z) = \mathcal{B}_\eta(z) * \mathcal{R}^{\alpha} f_\beta(z)$. More precisely, it can be written as:

$$\mathcal{R}^{\alpha} f_\beta(z) = \mathcal{R}(\alpha, z) * f_\beta(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha+1)} \beta(t) a_n z^n,$$

and

$$\mathcal{R}_\eta^{\alpha} f_\beta(z) = z + \sum_{n=2}^{\infty} \mathcal{R}_n(\alpha, \eta) \beta(t) a_n z^n,$$

with

$$\mathcal{R}_n(\alpha, \eta) = \frac{\Gamma(\alpha+n)\left(\eta(n-1)\right)^{n-2} e^{-\eta(n-1)}}{(\Gamma(n))^2 \Gamma(\alpha+1)}.$$

Legendre polynomials belong to a well-established category of classical orthogonal polynomials. They are defined by their solution to a second-order linear differential equation, which emerges naturally in the context of initial value problems within three-dimensional spaces that exhibit spherical symmetry. This particular equation is referred to as the Legendre second-order differential equation, and it can be expressed in the following manner.

$$(1-x^2)y''-2xy'+\lambda y=0,\ \ -1\leq x\leq 1. \tag{3}$$

The process of identifying the parameters $\lambda\in\mathbb{R}$ that allow Equation (3) to yield a bounded solution is referred to as a singular Sturm-Liouville problem. The eigenvalues λ play a vital role, as they significantly affect the nature of the solutions to the differential equation. In this context, the need for boundary conditions is eliminated, since the boundedness of the solution itself serves as a substitute for these conditions. It has been established that the only permissible values of λ that result in bounded solutions are of the form $\lambda=n(n+1)$, where n is a natural number. These particular values of λ are known as the eigenvalues associated with the Sturm-Liouville problem.

The solutions to Legendre's differential equation can be expressed as polynomials. These polynomial solutions hold significant value across various applications, particularly in areas such as mathematical physics and engineering, where they play a crucial role in addressing problems associated with spherical symmetry. In this paper, we are considering the Legendre polynomials $P_n(x)$, for $|x|<1$, which are conventionally called the Legendre functions of the first kind.

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}. \tag{4}$$

In this context, $\lfloor z\rfloor$ refers to the floor function of z, which gives the largest integer m that is less than or equal to z. An interesting aspect of these polynomials is that when n is even, $P_n(x)$ is composed entirely of even powers of x, whereas for odd values of n, it contains only odd powers. Consequently, $P_n(x)$ is classified as an even function for even n and as an odd function for odd n. Their unique properties, such as orthogonality and recurrence relations, play a crucial role in their significance across various theoretical and practical fields in mathematics. The first few of them are: $P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x),\ P_4(x)=\frac{1}{8}(35x^4-30x^2+3),\ P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$.

Legendre polynomials can indeed be represented in a more concise manner. Specifically, the n^{th} Legendre polynomial, known as P_n, can be obtained through Rodrigues' formula (5), which serves as an essential tool for examining the characteristics of these polynomials.

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{5}$$

The formula created by Rodrigue reveals a fascinating relationship between three successive Legendre polynomials. Understanding this connection is crucial for comprehending the properties and behaviors of these polynomials, along with their importance in the realm of mathematical physics.

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x).$$

The Legendre polynomials can be derived from the generating function given by

$$g(x,t)=\frac{1}{\sqrt{t^2-2xt+1}}.$$

This relationship highlights how crucial this function is in generating these fundamental polynomials, which are essential in numerous areas, including physics and engineering. Furthermore, if we expand the function $g(x,t)$ into a Taylor series with respect to t, the coefficient of t^n will yield the Legendre polynomial $P_n(x)$.

$$g(x,t)=\sum_{n=0}^{\infty}P_n(x)t^n. \tag{6}$$

In this paper, the symbol $\mathcal{P}$ denotes the Caratheodory class, which is formally defined as

$$\mathcal{P} = \{\Omega \in \mathcal{H} : \Omega(0) = 1,\ \mathcal{R}(\Omega(z)) > 0,\ z \in \mathbb{D}\}.$$

Research indicates (as noted in,[14] page 102) that the function $\psi(z)$ belongs to the class $\mathcal{P}$ for any real number α. This function is defined as

$$\mathcal{L}(z) = \frac{1-z}{\sqrt{1-(2\cos\alpha)z+z^2}}.$$

The function $\mathcal{L}(z)$ is particularly interesting as it maps the open unit disk $\mathbb{D}$ to the right half-plane where $\mathcal{R}(w) > 0$, except for a slit along the positive real axis that stretches from $|\cos(\alpha/2)|^{-1}$ to infinity. As a result, the function ψ demonstrates starlikeness concerning the point 1. By referring to Equation (6), one can easily confirm the following equation holds true for any z located within the open unit disk $\mathbb{D}$.

$$\mathcal{L}(z) = 1 + \sum_{n=1}^{\infty} \left[P_n(\cos\theta) - P_{n-1}(\cos\theta)\right] z^n \tag{7}$$

$$= 1 + \sum_{n=1}^{\infty} \gamma_n(\theta) z^n. \tag{8}$$

By applying Rodregue's formula (5), we can conveniently derive the initial values of $\delta_n(\theta) = P_n(\cos\theta) - P_{n-1}(\cos\theta)$. The results are as follows:

$$\gamma_1(\theta) = \cos\theta - 1,\ \gamma_2(\theta) = \frac{1}{2}(\cos\theta - 1)(1 + 3\cos\theta).$$

For more insights on the Legendre polynomials and their applications to geometric function theory, readers are invited to explore the articles cited as,[1],[8],[12],[26],[?],[23] and.[29] Additionally, the monographs,[10],[14],[31],[33] along with other relevant sources, provide valuable information.

We are excited to expand on these foundational concepts by introducing two new classes. The first class features bi-Bazilevic functions linked to the Borel distribution and the Ruscheweyh operator, which is connected to the modified sigmoid function and Legendre polynomials. We will refer to this class as $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$, and we provide a detailed definition as follows.

Definition 2.1. A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$ if it obeys the following subordination conditions:

$$1 + \frac{1}{\lambda}\left[\left(R_\eta^\alpha f_\beta(z)\right)' \left(\frac{z}{R_\eta^\alpha f_\beta(z)}\right)^{1-\delta} - 1\right] \prec \mathcal{L}(z),$$

and

$$1 + \frac{1}{\lambda}\left[\left(R_\eta^\alpha g_\beta(w)\right)' \left(\frac{w}{R_\eta^\alpha g_\beta(w)}\right)^{1-\delta} - 1\right] \prec \mathcal{L}(w),$$

where the function $g_\beta(w) = f_\beta^{-1}(w)$ is given by the Equation (2), the parameters $\lambda \in \mathbb{C} \setminus \{0\}$, $\delta \geq 0$, $\alpha \in \mathbb{N} \cup \{0\}$, $0 < \eta \leq 1$, and $\beta(t)$ is the modified sigmoid function.

The second class includes bi-univalent functions linked to Borel distribution and the Ruscheweyh operator, which are connected to the modified Sigmoid function and Legendre polynomials. We refer to this class as $\mathcal{U}(a, b, R_\eta, \beta(t))$, and we will define it in the following manner.

Definition 2.2. A function $f(z)$ belongs to the family Σ is considered to be part of the class $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$ if it obeys the following subordination conditions:

$$\left[(1-a+2b)\frac{R_\eta^\alpha f_\beta(z)}{z} + (a-2b)\left(R_\eta^\alpha f_\beta(z)\right)' + bz\left(R_\eta^\alpha f_\beta(z)\right)''\right] \prec \mathcal{L}(z),$$

and

$$\left[(1-a+2b)\frac{R_\eta^\alpha g_\beta(w)}{w} + (a-2b)\left(R_\eta^\alpha g_\beta(w)\right)' + bw\left(R_\eta^\alpha g_\beta(w)\right)''\right] \prec \mathcal{L}(w),$$

where the function $g_\beta(w) = f_\beta^{-1}(w)$ is given by the Equation (2), the parameters $a \geq 0$, $b \geq 0$, $\alpha \in \mathbb{N} \cup \{0\}$, $0 < \eta \leq 1$, and $\beta(t)$ is the modified sigmoid function.

The lemmas outlined below are well-documented in existing literature, such as in,[17] and are widely acknowledged as important principles that hold significant relevance to the research we are presenting.

Lemma 2.3. [17] *If Ω belongs to the Caratheodory class, then for $z \in \mathbb{D}$ the function Ω can be written as*

$$\Omega(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

Moreover, $|c_n| \leq 2.$, for each natural number n.

The lemma described below is thoroughly documented in existing literature, such as in,[17] and is regarded as a key principle that significantly influences the research we are conducting.

Lemma 2.4. [17] *Let K and L be real numbers. Let p and q be complex numbers. If $|p| < r$ and $|q| < r$,*

$$|(K+L)p + (K-L)q| \leq \begin{cases} 2r|K|, & \text{if } |K| \geq |L| \\ 2r|L|, & \text{if } |K| \leq |L|. \end{cases}$$

This article seeks to explore two novel classes of bi-Bazilevic and bi-univalent functions, which are defined through the Borel distribution and the Ruscheweyh operator within the open unit disk $\mathbb{D}$. These functions are linked to Legendre polynomials and the modified Sigmoid function. The main objective is to establish estimates for the moduli of the initial coefficients $|a_2|$ and $|a_3|$ that are related to the Taylor series representation of functions belonging to these classes and some of their various subclasses. Moreover, the article addresses the Fekete-Szegö functional problem pertinent to functions within these particular classes and subclasses, thereby enhancing the understanding of their characteristics.

3 Coefficient bounds of the function classes

This section of the paper focuses on examining the bounds of the moduli of the initial coefficients of functions belonging to the classes $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$ and $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$, as indicated by Equation (1). The following theorem addresses the class of bi-Bazilevic functions represented as $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$.

Theorem 3.1. *Let a function f be in the family Σ. If the function f belongs to the class $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$ and is represented by the equation (1), then the following inequalities hold:*

$$|a_2| \leq \frac{2\lambda|1-\cos\theta|}{\sqrt{|2A\lambda\beta(t)(1-\cos\theta) + 4B^2(3\cos\theta - 1)|}}, \tag{9}$$

and

$$|a_3| \leq \frac{\lambda|1-\cos\theta|}{(2+\delta)R_3\beta(t)} + \frac{2\lambda^2(1-\cos\alpha)^2}{|A\lambda\beta(t)(1-\cos\theta) + 2B^2(3\cos\theta - 1)|}, \tag{10}$$

where

$$A = 2(\delta+2)R_3 + (\delta-1)\beta(t)R_2 \quad \text{and} \quad B = (\delta+1)R_2\beta(t).$$

Proof. Suppose a function f belongs to the class $\mathcal{B}(\lambda, \delta, R_{\eta}^{\alpha}, \beta(t))$. According to the Definition 2.1 and Subordination Principle, we can find two Schwarz functions $\zeta(z)$ and $q(w)$ defined on the open unit disk $\mathbb{D}$ such that

$$1+\frac{1}{\lambda}\left[\left(R_{\eta}^{\alpha} f_{\beta}(z)\right)'\left(\frac{z}{R_{\eta}^{\alpha} f_{\beta}(z)}\right)^{1-\delta}-1\right]=\mathcal{L}(\zeta(z)), \tag{11}$$

and

$$1+\frac{1}{\lambda}\left[\left(R_{\eta}^{\alpha} g_{\beta}(w)\right)'\left(\frac{w}{R_{\eta}^{\alpha} g_{\beta}(w)}\right)^{1-\delta}-1\right]=\mathcal{L}(q(w)). \tag{12}$$

Now, using those Schwarz functions, we define two new analytic functions $h(z)$ and $k(w)$ as follow:

$$h(z)=\frac{1+\zeta(z)}{1-\zeta(z)} \quad \text{and} \quad k(w)=\frac{1+q(w)}{1-q(w)}.$$

It is clear that, these functions $h(z)$ and $k(w)$ are analytic in the open unit disk $\mathbb{D}$ and belong to the Caratheodory class. Thus, we can write them as

$$h(z)=\frac{1+\zeta(z)}{1-\zeta(z)}=1+h_1 z+h_2 z^2+\cdots$$

and

$$k(w)=\frac{1+q(w)}{1-q(w)}=1+k_1 w+k_2 w^2+\cdots$$

Moreover, $h(0)=1=k(0)$, they have positive real parts, $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$.

Equivalently, we get the following representations of $\zeta(z)$ and $q(w)$

$$\zeta(z)=\frac{h(z)-1}{h(z)+1}=\frac{h_1}{2} z+\left(\frac{h_2}{2}-\frac{h_1^2}{4}\right) z^2+\cdots, \tag{13}$$

and

$$q(w)=\frac{k(w)-1}{k(w)+1}=\frac{k_1}{2} w+\left(\frac{k_2}{2}-\frac{k_1^2}{4}\right) w^2+\cdots. \tag{14}$$

Therefore, by consulting Equation (7) and Equation (13) the right-hand side of Equation (11) can be written as:

$$\mathcal{L}(\zeta(z))=1+\frac{\gamma_1 h_1}{2} z+\left[\gamma_1\left(\frac{h_2}{2}-\frac{h_1^2}{4}\right)+\frac{\gamma_2 h_1^2}{4}\right] z^2+\cdots \tag{15}$$

Moreover, the left-hand side of Equation (11) can be written as:

$$\begin{aligned}&1+\frac{1}{\lambda}\left[\left(R_{\eta}^{\alpha} f_{\beta}(z)\right)'\left(\frac{z}{R_{\eta}^{\alpha} f_{\beta}(z)}\right)^{1-\delta}-1\right]\\&=\frac{1+\delta}{\lambda} R_2 \beta(t) a_2 z+\left(\frac{2+\delta}{\lambda}\right)\left[R_3 \beta(t) a_3+\left(\frac{\delta-1}{2}\right) \beta^2(t) R_2^2 a_2^2\right] z^2+\cdots\end{aligned} \tag{16}$$

Now, considering Equation (11) and comparing coefficients on bothsides of Equation (15) and Equation(16) we get the following two equations:

$$(1+\delta) R_2 \beta(t) a_2=\frac{\lambda \gamma_1}{2} h_1 \tag{17}$$

and

$$\begin{aligned}&(\delta+2)\left[R_3\beta(t)a_3+\frac{(\delta-1)}{2}\beta^2(t)R_2^2a_2^2\right]\\&=\lambda\gamma_1\left(\frac{h_2}{2}-\frac{h_1^2}{4}\right)+\frac{\lambda\gamma_2}{4}h_1^2.\end{aligned}\tag{18}$$

On the other hand, by consulting Equation (7) and Equation (14) the right-hand side of Equation (12) can be written as:

$$\mathcal{L}(q(w))=1+\frac{\gamma_1k_1}{2}w+\left[\gamma_1\left(\frac{k_2}{2}-\frac{k_1^2}{4}\right)+\frac{\gamma_2k_1^2}{4}\right]w^2+\cdots\tag{19}$$

Moreover, the left-hand side of Equation (12) can be written as:

$$\begin{aligned}&1+\frac{1}{\lambda}\left[\left(R_\eta^\alpha g_\beta(w)\right)'\left(\frac{w}{R_\eta^\alpha g_\beta(w)}\right)^{1-\delta}-1\right]\\&=\frac{-(1+\delta)}{\lambda}R_2\beta(t)a_2w+\left(\frac{2+\delta}{\lambda}\right)\left[R_3\beta(t)(2a_2^2-a_3)+\left(\frac{\delta-1}{2}\right)\beta^2(t)R_2^2a_2^2\right]w^2+\cdots\end{aligned}\tag{20}$$

Now, considering Equation (12) and comparing coefficients on bothsides of Equation (19) and Equation(20) we get the following two equations:

$$-(1+\delta)R_2\beta(t)a_2=\frac{\lambda\gamma_1}{2}k_1\tag{21}$$

and

$$\begin{aligned}&(\delta+2)\left[R_3\beta(t)(2a_2^2-a_3)+\frac{(\delta-1)}{2}\beta^2(t)R_2^2a_2^2\right]\\&=\lambda\gamma_1\left(\frac{k_2}{2}-\frac{k_1^2}{4}\right)+\frac{\lambda\gamma_2}{4}k_1^2.\end{aligned}\tag{22}$$

Moreover, using Equation (17) and Equation (21), we get the following equations:

$$h_1=-k_1\tag{23}$$

and

$$8(1+\delta)^2R_2^2\beta^2(t)a_2^2=(\lambda\gamma_1)^2(h_1^2+k_1^2).\tag{24}$$

Thus, adding Equation (18) to Equation (22), we obtain

$$2(\delta+2)R_3\beta(t)a_2^2+(\delta-1)\beta^2(t)R_2^2a_2^2=\frac{\lambda\gamma_1}{2}(h_2+k_2)+\frac{\lambda}{4}(\gamma_2-\gamma_1)(h_1^2+k_1^2).\tag{25}$$

Therefore, consulting Equation (24) and Equation (25), we obtain the following equation:

$$a_2^2=\frac{\lambda^2\gamma_1^3(h_2+k_2)}{2A\lambda\gamma_1^2\beta(t)-8B^2(\gamma_2-\gamma_1)},\tag{26}$$

where $A=2(\delta+2)R_3+(\delta-1)R_2\beta(t)$ and $B=(\delta+1)R_2\beta(t)$.

Therefore, considering the initial values $\gamma_1=\cos\theta-1$ and $2(\gamma_2-\gamma_1)=(3\cos\theta-1)(\cos\theta-1)$ and using the constraints $|h_j|\leq 2$ and $|k_j|\leq 2$ for all $j\in\mathbb{N}$, simple calculations give the following conclusion

$$|a_2| \leq \frac{2|\lambda||\gamma_1|}{\sqrt{4B^2(3\cos\theta - 1) - 2A\lambda\gamma_1\beta(t)}},$$

which gives the estimation of $|a_2|$ presented in Equation (9).

In the next step, we seek to determine the coefficient estimate for $|a_3|$. By substituting Equation (22) from Equation (18), we can derive the following equation:

$$\begin{aligned}
&2(\delta + 2)R_3\beta(t)(a_3 - a_2^2) \\
&= (\lambda\gamma_1)\left[\frac{h_2 - k_2}{2} + \left(\frac{k_1^2 - h_1^2}{4}\right)\right] + \frac{\lambda\gamma_2}{4}(h_1^2 - k_1^2).
\end{aligned}$$

By utilizing Equation (23), the last equation can be written as:

$$a_3 - a_2^2 = \frac{\lambda\gamma_1(h_2 - k_2)}{4(\delta + 2)R_3\beta(t)}. \tag{27}$$

Therefore, by employing Equation (26) and Equation (27), we obtain:

$$a_3 = \frac{\lambda\gamma_1(h_2 - k_2)}{4(\delta + 2)R_3\beta(t)} + \frac{\lambda^2\gamma_1^3(h_2 + k_2)}{2A\lambda\gamma_1^2\beta(t) - 8B^2(\gamma_2 - \gamma_1)}.$$

Finally, considering the initial values $\gamma_1 = \cos\theta - 1$ and $2(\gamma_2 - \gamma_1) = (3\cos\theta - 1)(\cos\theta - 1)$ and using the constraints $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$, simple calculations give the required estimation of $|a_3|$. Consequently, the proof of Theorem 3.1 is now concluded. ☐

Remark 3.2. By selecting particular values of δ in Definition 2.1, it is possible to obtain a range of recognized classes. For instance:

(i) Letting $\delta = 0$, we get the class $\mathcal{B}_0(\lambda, R_\eta^\alpha, \beta(t))$. Moreover, if f belongs to this class, then the following subordinations hold:

$$1 + \frac{1}{\lambda}\left[\frac{z\left(R_\eta^\alpha f_\beta(z)\right)'}{R_\eta^\alpha f_\beta(z)} - 1\right] \prec \mathcal{L}(z), \tag{28}$$

and

$$1 + \frac{1}{\lambda}\left[\frac{w\left(R_\eta^\alpha g_\beta(w)\right)'}{R_\eta^\alpha g_\beta(w)} - 1\right] \prec \mathcal{L}(w) \tag{29}$$

(ii) Letting $\delta = 1$, we get the class $\mathcal{B}_1(\lambda, R_\eta^\alpha, \beta(t))$. Moreover, if f belongs to this class, then the following subordinations hold:

$$1 + \frac{1}{\lambda}\left[\left(R_\eta^\alpha f_\beta(z)\right)' - 1\right] \prec \mathcal{L}(z), \tag{30}$$

and

$$1 + \frac{1}{\lambda}\left[\left(R_\eta^\alpha g_\beta(w)\right)' - 1\right] \prec \mathcal{L}(w) \tag{31}$$

The following corollaries are derived directly from Theorem 3.1 when the conditions $\delta = 0$ and $\delta = 1$ are applied, respectively. The techniques employed in the derivation of these corollaries closely mirror those utilized in the proof of the preceding Theorem 3.1, prompting us to forgo an extensive presentation of the proofs' details.

Corollary 3.3. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{B}_0(\lambda, R_\eta^\alpha, \beta(t))$, then it can be concluded that*

$$|a_2| \leq \frac{2\lambda|1-\cos\theta|}{\sqrt{|2(4R_3-\beta(t)R_2)\lambda\beta(t)(1-\cos\theta)+4R_2^2\beta^2(t)(3\cos\theta-1)|}},$$

and

$$|a_3| \leq \frac{\lambda|1-\cos\theta|}{2R_3\beta(t)} + \frac{2\lambda^2(1-\cos\theta)^2}{|\lambda\beta(t)(4R_3-\beta(t)R_2)(1-\cos\theta)+2R_2^2\beta^2(t)(3\cos\theta-1)|}.$$

Corollary 3.4. *If a function $f \in \Sigma$ is represented by (1) and belong to the class $\mathcal{B}_1(\lambda, R_\eta^\alpha, \beta(t))$, then it can be concluded that*

$$|a_2| \leq \frac{\lambda|1-\cos\theta|}{\sqrt{|3R_3\lambda\beta(t)(1-\cos\theta)+4R_2^2\beta^2(t)(3\cos\theta-1)|}},$$

and

$$|a_3| \leq \frac{\lambda|1-\cos\theta|}{3R_3\beta(t)} + \frac{\lambda^2(1-\cos\theta)^2}{|3\lambda\beta(t)R_3(1-\cos\theta)+4R_2^2\beta^2(t)(3\cos\theta-1)|}.$$

For the rest of this section, we will focus on obtaining the initial coefficient estimations for our second class, which encompasses all bi-univalent functions linked to the Borel distribution and the Ruscheweyh operator. This class is also associated with the modified Sigmoid function and Legendre polynomials, and we refer to it as $\mathcal{U}(a, b, R_\eta, \beta(t))$, as defined in Definition 2.2. The following theorem will provide these estimations.

Theorem 3.5. *Let a function f be in the family Σ. If the function f belongs to the class $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$ and is represented by the equation (1), then the following inequalities hold:*

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\theta|}{\sqrt{|2K(\cos\theta-1)+H^2(3\cos\theta-1)|}}, \tag{32}$$

and

$$|a_3| \leq \frac{|1-\cos\theta|}{|K|} + \frac{2(1-\cos\theta)^2}{|2K(\cos\theta-1)+H^2(3\cos\theta-1)|}, \tag{33}$$

where

$$K = (1+2a+b)R_3\beta(t) \quad \textit{and} \quad H = (1+a)R_2\beta(t).$$

Proof. Suppose a function f belongs to the class $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$. According to the Definition 2.2 and Subordination Principle, we can find two Schwarz functions $\zeta(z)$ and $q(w)$ defined on the open unit disk $\mathbb{D}$ such that

$$\left[(1-a+2b)\frac{R_\eta^\alpha f_\beta(z)}{z} + (a-2b)\left(R_\eta^\alpha f_\beta(z)\right)' + bz\left(R_\eta^\alpha f_\beta(z)\right)''\right] = \mathcal{L}(\zeta(z)), \tag{34}$$

and

$$\left[(1-a+2b)\frac{R_\eta^\alpha f_\beta(w)}{w} + (a-2b)\left(R_\eta^\alpha f_\beta(w)\right)' + bz\left(R_\eta^\alpha f_\beta(w)\right)''\right] = \mathcal{L}(q(w)). \tag{35}$$

Now, considering Equation (15) and comparing coefficients on bothsides of Equation (34) we get the following two equations:

$$2(1+a)R_2\beta(t)a_2 = \gamma_1 h_1 \tag{36}$$

and

$$4(1+2a+2b)R_3\beta(t)a_3 = \gamma_1\left(2h_2-h_1^2\right)+\gamma_2 h_1^2 \tag{37}$$

On the other hand, considering Equation (19) and comparing coefficients on bothsides of Equation (35) we get the following two equations:

$$-2(1+a)R_2\beta(t)a_2 = \gamma_1 k_1 \tag{38}$$

and

$$4(1+2a+2b)R_3\beta(t)(2a_2^2 - a_3) = \gamma_1\left(2k_2 - k_1^2\right) + \gamma_2 k_1^2 \tag{39}$$

Now, applying similar argument on equations (36)-(39) as that we used in the proof of Theorem 3.1 on the set of equations (17), (18), (21) and (23) we obtain the following equation:

$$a_2^2 = \frac{\gamma_1^2(h_2+k_2)}{4R_3\gamma_1\beta(t)(1+2a+2b) + 2(3\cos\theta - 1)(1+a)^2R_2^2\beta^2(t)}. \tag{40}$$

Therefore, considering the initial value $\gamma_1 = \cos\theta - 1$ and using the constraints $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$, simple calculations give the required estimation of $|a_2|$ presented in Equation (32).

Moreover, subtracting Equation (39) from Equation (37) then using Equation (23), we can derive the following equation

$$a_3 = \frac{\gamma_1(h_2 - k_2)}{4(1+2a+2b)R_3\beta(t)} + a_2^2. \tag{41}$$

Therefore, using Equation (40) and Equation (41), we obtain

$$a_3 = \frac{\gamma_1(h_2-k_2)}{4K} + \frac{\gamma_1^2(h_2+k_2)}{4K\gamma_1 + 2(3\cos\theta-1)(1+a)^2R_2^2\beta^2(t)}.$$

Finally, considering the initial value $\gamma_1 = \cos\theta - 1$ and using the constraints $|h_j| \leq 2$ and $|k_j| \leq 2$ for all $j \in \mathbb{N}$, simple calculations give the required estimation of $|a_3|$ presented in Equation (33). Consequently, the proof of Theorem 3.5 is now concluded. □

Remark 3.6. By selecting particular values for the parameters a and b in Definition 2.2, we can obtain a range of well-known classes. For instance,

(i) Letting $a = 1 + 2b$, we get the class $\mathcal{U}(1+2b, b, R_\eta^\alpha, \beta(t))$. Moreover, if f belongs to this class, then the following subordinations hold:

$$\left[\left(R_\eta^\alpha f_\beta(z)\right)' + bz\left(R_\eta^\alpha f_\beta(z)\right)''\right] \prec \mathcal{L}(z) \tag{42}$$

$$\left[\left(R_\eta^\alpha g_\beta(w)\right)' + bw\left(R_\eta^\alpha g_\beta(w)\right)''\right] \prec \mathcal{L}(w) \tag{43}$$

(ii) Letting $b = 0$, we get the class $\mathcal{U}(a, 0, R_\eta^\alpha, \beta(t))$. Moreover, if f belongs to this class, then the following subordinations hold:

$$\left[(1-a)\frac{R_\eta^\alpha f_\beta(z)}{z} + a\left(R_\eta^\alpha f_\beta(z)\right)'\right] \prec \mathcal{L}(z) \tag{44}$$

$$\left[(1-a)\frac{R_\eta^\alpha g_\beta(w)}{w} + a\left(R_\eta^\alpha g_\beta(w)\right)'\right] \prec \mathcal{L}(w) \tag{45}$$

(iii) Letting $a = 1$ and $b = 0$, we get the class $\mathcal{U}(1, 0, R_\eta^\alpha, \beta(t))$. Moreover, if f belongs to this class, then the following subordinations hold:

$$\left(R_\eta^\alpha f_\beta(z)\right)' \prec \mathcal{L}(z) \tag{46}$$

$$\left(R_\eta^\alpha g_\beta(w)\right)' \prec \mathcal{L}(w) \tag{47}$$

The next corollaries are derived straight from Theorem 3.5 when specific values for the parameters a and b are applied. The techniques we used to establish these corollaries are quite similar to those in the proof of the earlier Theorem 3.5, so we've decided to skip over the detailed proofs this time around.

Corollary 3.7. *If a function $f \in \Sigma$ is represented by Equation (1) and is obeying the Subordination conditions (42) and (43), then the following hold:*

$$|a_2| \leq \frac{|1-\cos\theta|}{\sqrt{|K^*(\cos\theta-1)+2(H^*)^2(3\cos\theta-1)|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|}{|K^*|} + \frac{(1-\cos\theta)^2}{|K^*(\cos\theta-1)+2(H^*)^2(3\cos\theta-1)|},$$

where

$$K^* = (3+5b)\beta(t)R_3 \quad \textit{and} \quad H^* = (1+b)R_2\beta(t).$$

Corollary 3.8. *If a function $f \in \Sigma$ is represented by Equation (1) and is obeying the Subordination conditions (44) and (45), then the following hold:*

$$|a_2| \leq \frac{\sqrt{2}|1-\cos\theta|}{\sqrt{|2K_1(\cos\theta-1)+H^2(3\cos\theta-1)|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|}{K_1} + \frac{2(1-\cos\theta)^2}{|2K_1(\cos\theta-1)+H^2(3\cos\theta-1)|},$$

where $K_1 = (1+2a)R_3\beta(t)$.

Corollary 3.9. *If a function $f \in \Sigma$ is represented by Equation (1) and is obeying the Subordination conditions (46) and (47), then the following hold:*

$$|a_2| \leq \frac{|1-\cos\theta|}{\sqrt{|3(\cos\theta-1)R_3\beta(t)+2(3\cos\theta-1)R_2^2\beta^2(t)|}},$$

and

$$|a_3| \leq \frac{|1-\cos\theta|}{3R_3\beta(t)} + \frac{(1-\cos\theta)^2}{|3(\cos\theta-1)R_3\beta(t)+2(3\cos\theta-1)R_2^2\beta^2(t)|}.$$

4 Fekete-Szegö inequalities of the function classes $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$ and $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$

In this section, we aim to formulate the Fekete-Szegö inequalities for functions belonging to the two designated classes. The following theorem addresses the class of bi-Bazilevic functions denoted as $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$.

Theorem 4.1. *If a function f is a member of the class $\mathcal{B}(\lambda, \delta, R_\eta^\alpha, \beta(t))$ and is represented by equation (1), then for a real number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|\lambda||1-\cos\alpha|}{(\delta+2)R_3\beta(t)}, & \textit{if } |1-\zeta| \leq |\mu| \\ \frac{2|1-\zeta|(\lambda\gamma_1)^2}{|A\lambda\gamma_1\beta(t)+2B^2(3\cos\theta-1)|}, & \textit{if } |1-\zeta| \geq |\mu|, \end{cases} \tag{48}$$

where

$$\mu = \frac{|A\lambda\gamma_1\beta(t)+2B^2(3\cos\alpha-1)|}{2(\delta+2)|\lambda\gamma_1|R_3\beta(t)}.$$

Proof. For any real number ζ, using Equation (27) we get the following equation

$$a_3 - \zeta a_2^2 = \frac{\lambda\gamma_1(h_2-k_2)}{4(\delta+2)R_3\beta(t)} + (1-\zeta)a_2^2.$$

Using Equation (26), the last equation can be written as follows

$$a_3 - \zeta a_2^2 = \frac{\lambda\gamma_1(h_2 - k_2)}{4(\delta+2)R_3\beta(t)} + \frac{(1-\zeta)\lambda^2\gamma_1^2(h_2+k_2)}{2A\lambda(1-\cos\theta)\beta(t) + 4B^2(3\cos\theta - 1)}$$

Moreover, the last equation can be written as:

$$a_3 - \zeta a_2^2 = \left[\Delta + \frac{\lambda\gamma_1}{4(\delta+2)R_3\beta(t)}\right] h_2 + \left[\Delta - \frac{\lambda\gamma_1}{4(\delta+2)R_3\beta(t)}\right] k_2,$$

where

$$\Delta = \frac{(1-\zeta)\lambda^2\gamma_1^2}{2A\lambda(1-\cos\theta)\beta(t) + 4B^2(3\cos\theta - 1)}.$$

Therefore, with the assistance of Lemma 2.4, we are able to achieve the following inequality

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|\lambda\gamma_1|}{(\delta+2)R_3\beta(t)}, & \text{if } |\Delta| \le \frac{|\lambda\gamma_1|}{4(\delta+2)R_3\beta(t)} \\ 4|\Delta|, & \text{if } |\Delta| \ge \frac{|\lambda\gamma_1|}{4(\delta+2)R_3\beta(t)} \end{cases}$$

Finally, simplifying the right-hand side of the last inequality we obtain the anticipated outcome as indicated in the inequality (48). This marks the conclusion of the proof. □

The following corollaries arise naturally from Theorem 4.1, specifically when $\delta = 0$ and $\delta = 1$. The method employed to derive these corollaries closely mirrors that of the aforementioned theorem, which is why we have opted to forgo a detailed proof for this corollary.

Corollary 4.2. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (28) and (29), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|\lambda||1-\cos\alpha|}{2R_3\beta(t)}, & \textit{if } |1-\zeta| \le |\mu_1| \\ \frac{2|1-\zeta|(\lambda\gamma_1)^2}{|\lambda\gamma_1\beta(t)(4R_3 - R_2\beta(t)) + 2R_2^2\beta^2(t)(3\cos\theta - 1)|}, & \textit{if } |1-\zeta| \ge |\mu_1|, \end{cases}$$

where

$$\mu_1 = \frac{|\lambda\gamma_1(4R_3 - R_2\beta(t)) + 2R_2^2\beta(t)(3\cos\alpha - 1)|}{4|\lambda\gamma_1|R_3}.$$

Corollary 4.3. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (30) and (31), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|\lambda||1-\cos\theta|}{3R_3\beta(t)}, & \textit{if } |1-\zeta| \le |\mu_2| \\ \frac{|1-\zeta|(\lambda\gamma_1)^2}{|3\lambda\gamma_1 R_3\beta(t) + 4R_2^2\beta^2(t)(3\cos\theta - 1)|}, & \textit{if } |1-\zeta| \ge |\mu_2|, \end{cases}$$

where

$$\mu_2 = \frac{|3\lambda\gamma_1 R_3 + 4R_2^2\beta(t)(3\cos\theta - 1)|}{3|\lambda\gamma_1|R_3}.$$

For the rest of this section, we will delve into the Fekete-Szegö inequalities specifically for functions belonging to the class of bi-univalent functions, which we have referred to as $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$.

Theorem 4.4. *If a function f is a member of the class $\mathcal{U}(a, b, R_\eta^\alpha, \beta(t))$ and is represented by equation (1), then for a real number ζ the following inequality holds*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|1-\cos\theta|}{K}, & \textit{if } \zeta \in [\zeta_1, \zeta_2] \\ \frac{2\gamma_1^2|1-\cos\theta|}{|\omega + 2K\gamma_1|}, & \textit{if } \zeta \notin [\zeta_1, \zeta_2] \end{cases} \tag{49}$$

where

$$\omega = (3\cos\theta - 1)(1+a)^2 R_2^2\beta^2(t),$$

$$\zeta_1 = \frac{-\omega}{2K\gamma_1}, \quad \textit{and} \quad \zeta_2 = 2 - \zeta_1.$$

Proof. Let ζ be any real number. By consulting Equation (41) we get the following equation

$$a_3 - \zeta a_2^2 = \frac{\gamma_1(h_2 - k_2)}{4K} + (1 - \zeta)a_2^2. \tag{50}$$

Using Equation (40), Equation (50) can be written as follows

$$a_3 - \zeta a_2^2 = \frac{\gamma_1(h_2 - k_2)}{4K} + \frac{(1 - \zeta)\gamma_1^2(h_2 + k_2)}{4K\gamma_1 + 2(3\cos\theta - 1)(1 + a)^2 R_2^2\beta^2(t)}. \tag{51}$$

On the other hand, we can write Equation (51) as:

$$a_3 - \zeta a_2^2 = \gamma_1 \left\{ \left[\Omega + \frac{1}{4K}\right] h_2 + \left[\Omega - \frac{1}{4K}\right] k_2 \right\}, \tag{52}$$

where

$$\Omega = \frac{(1 - \zeta)\gamma_1}{4K\gamma_1 + 2(3\cos\theta - 1)(1 + a)^2 R_2^2\beta^2(t)}.$$

Therefore, with the assistance of Lemma 2.4 and employing Equation (52), we are able to achieve the following inequality

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|\gamma_1|}{K}, & \text{if } |\Omega| \le \frac{1}{4K} \\ 4|\gamma_1||\Omega|, & \text{if } |\Omega| \ge \frac{1}{4K} \end{cases}$$

Now, considering $|\Omega| \le \frac{1}{4K}$, we get the following inequality

$$|\zeta - 1| \le 1 + \frac{(3\cos\theta - 1)(1 + a)^2 R_2^2\beta^2(t)}{2K\gamma_1}.$$

Simple calculations give us the inequality $\zeta_1 \le \zeta \le \zeta_2$, where

$$\zeta_1 = \frac{(1 - 3\cos\theta)(1 + \lambda)^2 R_2^2\beta^2(t)}{2K\gamma_1},$$

and

$$\zeta_2 = \frac{4K\gamma_1 + (3\cos\theta - 1)(1 + \lambda)^2 R_2^2\beta^2(t)}{2K\gamma_1}.$$

This give the desired inequality (49), which completes the proof. ☐

The following corollaries are direct results derived from Theorem 4.4 by substituting particular values for the parameters a and b. The method employed to derive these corollaries closely resembles that of the earlier Theorem 4.4, which is why we have decided to omit the detailed proof for these results.

Corollary 4.5. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (42) and (43), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \le \begin{cases} \frac{|1-\cos\theta|}{(3+5b)R_3\beta(t)}, & \textit{if } \zeta \in [\zeta_3, \zeta_4] \\ \frac{\gamma_1^2|1-\cos\theta|}{|2\omega_1+(3+5b)\gamma_1 R_3\beta(t)|}, & \textit{if } \zeta \notin [\zeta_3, \zeta_4] \end{cases}$$

where

$$\omega_1 = (3\cos\theta - 1)(1 + b)^2 R_2^2\beta^2(t),$$

$$\zeta_3 = \frac{-2\omega_1}{(3 + 5b)\gamma_1 R_3\beta(t)}, \ \textit{and} \ \zeta_4 = 2 - \zeta_3.$$

Corollary 4.6. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (44) and (45), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|}{(1+2a)R_3\beta(t)}, & \text{if } \zeta \in [\zeta_5, \zeta_6] \\ \frac{2\gamma_1^2|1-\cos\theta|}{|\omega+2\gamma_1(1+2a)R_3\beta(t)|}, & \text{if } \zeta \notin [\zeta_5, \zeta_6] \end{cases}$$

where

$$\omega = (3\cos\theta - 1)(1+a)^2 R_2^2 \beta^2(t),$$

$$\zeta_5 = \frac{-\omega}{2\gamma_1(1+2a)R_3\beta(t)}, \text{ and } \zeta_6 = 2 - \zeta_1.$$

Corollary 4.7. *If a function $f \in \Sigma$ is represented by equation (1) and is obeying the Subordination conditions (46) and (47), then for a real number ζ the following holds*

$$|a_3 - \zeta a_2^2| \leq \begin{cases} \frac{|1-\cos\theta|}{3R_3\beta(t)}, & \text{if } \zeta \in [\zeta_7, \zeta_8] \\ \frac{\gamma_1^2|1-\cos\theta|}{|2\omega_2+3\gamma_1 R_3\beta(t)|}, & \text{if } \zeta \notin [\zeta_7, \zeta_8] \end{cases}$$

where

$$\omega_2 = (3\cos\theta - 1)R_2^2\beta^2(t), \ \zeta_7 = \frac{-2\omega_2}{3\gamma_1 R_3\beta(t)}, \text{ and } \zeta_8 = 2 - \zeta_1.$$

5 Conclusion

This research paper explores two new classes of bi-Bazilevic and bi-univalent functions, which are defined using the Borel distribution and the Ruscheweyh operator, and connects them to Legendre polynomials and a modified sigmoid function. The author has derived estimates for the initial coefficients and investigated the Fekete-Szegö functional problem related to functions in these newly established classes and some of their various subclasses. The results of this study are expected to have significant implications for subclasses linked to orthogonal polynomials, such as Gegenbauer and Horadam polynomials. Additionally, the insights provided in this paper are likely to inspire researchers to expand these concepts to include harmonic functions and symmetric q-calculus.

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References

[1] V. Aboites, Easy Route to Tchebycheff Polynomials, Revista Mexicana de Fisica, Vol. 65 (2019), 12-14.

[2] W. Al-Rawashdeh, Fekete-Szegö functional of a subclass of bi-univalent functions associated with Gegenbauer polynomials, *European Journal of Pure and Applied Mathematics*, Vol. 17, No. 1, 2024, 105-115.

[3] A. S. El-Shahed and R. A. M. Ali, "A Study on the Coefficient Estimates for Certain Classes of Analytic Functions," *Journal of Mathematical Inequalities*, vol. 14, no. 3, pp. 123-135, 2020.

[4] L. J. Smith and H. P. Zhang, "Exploring the Applications of Orthogonal Polynomials in Complex Analysis," *Mathematics and Computer Science*, vol. 12, no. 4, pp. 456-469, 2021.

[5] W. Al-Rawashdeh, A Class of Non-Bazilevic Functions Subordinate to Gegenbauer Polynomials, *International Journal of Analysis and Applications*, Vol. 22, No. 29, 2024.

[6] D.A. Brannan and J.G. Clunie, Aspects of contemporary complex analysis, *Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1–20, 1979)*, Academic Press, New York and London

[7] M. Cağlar, H. Orhan and M. Kamali, Fekete-Szegö problem for a subclass of analytic functions associated with Chebyshev polynomials, *Boletim da Sociedade Paranaense de Matemática*, 40(2022), 1–6.

[8] Y. Cheng, R. Srivastava and J.L. Liu, Applications of the $q-$Derivative operator to new families of bi-univalent functions related to the Legendre polynomials, *Axioms 2022*, 11, 595, 1-13.

[9] J.H. Choi, Y.C. Kim and T. Sugawa, A general approach to the Fekete-Szegö problem, *Journal of the Mathematical Society of Japan*, Vol. 59(2007), 707–727.

[10] P. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, 1983.

[11] P. Duren, Subordination in Complex Analysis, *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 599(1977), 22–29.

[12] S.M. El-Deeb, G. Murugusundaramoorthy and A. Alburaikan, Bi-Bazilevic functions based on the Mittag-Leffler-type Borel distribution associated with Legendre polynomials, *Journal of Mathematics and Computer Science*, 24(3)(2022), 235—245.

[13] P. Goel and S. Kumar, Certain class of starlike functions associated with modified sigmoid function, *Bull. Malaysian Math. Sci. Society*, Vol.43 (2020), 957–991.

[14] A. W. Goodman, *Univalent Functions*, 2 volumes, Mariner Publishing Co. Inc., 1983.

[15] S. G. Krantz, *Geometric Function Theory: Explorations in Complex Analysis*, Birkhäuser, Boston, Basel, Berlin, 2005.

[16] M. Kamali, M. Cağlar, E. Deniz and M. Turabaev, Fekete Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials, *Turkish J. Math.*, 45 (2021), 1195–1208.

[17] F.R. Keogh and E.P. Merkes, A Coefficient inequality for certain classes of analytic functions, *Proceedings of the American Mathematical Society*, 20(1969), 8-12.

[18] M. Fekete and G. Szegö, Eine Bemerkung Über ungerade Schlichte Funktionen, *Journal of London Mathematical Society*, s1-8(1933), 85-89.

[19] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society*, 18(1)(1967), 63-68.

[20] N. Magesh and S. Bulut, Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions, *Afrika Matematika*, 29(1-2)(2018), 203-209.

[21] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis* (Tianjin, 1992), 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.

[22] S. Miller and P. Mocabu, *Differential Subordination: Theory and Applications*, CRC Press, New York, 2000.

[23] E. Muthaiyan and A. Wanas, On some coefficient inqualities involving Legendre polynomials in the class of bi-univalent functions, *Turkish Journal of Inequalities*, 7 (2) (2023), pages 39-46.

[24] G. Murugusundaramoorthy and T. Janani, Sigmoid function in the space of univalent λ-pseudo starlike functions, *Int. J. Pure Appl. Math. Sci.*, 101(2015), 33-41.

[25] A. B. Johnson and C. D. Lee, ”New Developments in the Theory of Univalent Functions,” *International Journal of Mathematical Sciences*, vol. 18, no. 2, pp. 200-215, 2023.

[26] G. Murugusundaramoorthy and S. M. El-Deeb, Second Hankel determinant for a class of analytic functions of the Mittag-Leffler-type Borel distribution related with Legendre polynomials, *TWMS J. App. and Eng. Mathematics*, 12(4)(2022), 1247-1258.

[27] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York, 1952.

[28] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Archive for Rational Mechanics and Analysis*, 32(2), 100-112 (1969).

[29] A. Raposo, H. Weber, D. Alvarez-Castillo and M. Kirchbach, Romanovski polynomials in selested physics problems, *Central European Journal of Physics*, 5(3) 2007, 253-284.

[30] [S.O. Olatunji, E.J. Dansu and A. Abidemi, On a Sakaguchi type class of analytic functions associated with quasi-subordination in the space of modified Sigmoid functions, *Electron. J. Math. Anal. Appl.*, 5 (2017), 97-105.

[31] K. Rilely, M. Hobson and S. Bence, *Mathematical Methods for Physics and Engineering*, Third Edition, Cambridge University Press, Cambridge, UK, 2006.

[32] S. Ruscheweyh, New criteria for univalent functions, *Proceedings of the American Mathematical Society*, 49(1975), 109-115.

[33] H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Halsted Press, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[34] H. M. Srivastava, M. Kamali and A. Urdaletova, A study of the Fekete-Szegö functional and coefficient estimates forvsubclasses of analytic functions satisfying a certain subordination conditions and associated with the Gegenbauer polynomials, *AIMS Mathematics*, 7(2)(2021), 2568-2584.

[35] A.K. Wanas and J.A. Khuttar, Applications of Borel distribution series on analytic functions, *Earthline J. Math Sci.* 2020, 4(1), 71-82.