HyperWeighted Graph, SuperHyperWeighted Graph, and MultiWeighted Graph

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Abstract

A weighted graph is a graph in which each edge is assigned a numerical value (weight), typically representing cost, distance, or intensity. In this paper, we revisit and further explore three generalizations of weighted graphs: the *Hyperweighted Graph*, the *Superhyperweighted Graph*, and the *MultiWeighted Graph*. These advanced structures were initially introduced in.[10] Our objective is to enhance understanding and broaden awareness of their theoretical foundations and potential applications through renewed analysis and formal refinement.

Keywords: HyperWeighted Graph; SuperHyperWeighted Graph; Weighted Graph; Weighted Set

1 Preliminaries and Definitions

This section introduces the fundamental concepts and definitions that underpin the discussions in this paper. Throughout, all sets are assumed to be finite.

Definition 1.1 (Universal Set)**.** A *universal set*, denoted by U, is a set that contains all elements under consideration in a given context. In this paper, every set is assumed to be a subset of the universal set U.

Definition 1.2 (Base Set)**.** [9] A *base set* S is the underlying set from which more elaborate structures, such as powersets and hyperstructures, are constructed. It is defined as

$$S = \{x \mid x \text{ belongs to a specified domain}\}.$$

All elements used in constructions like $\mathcal{P}(S)$ or $P_n(S)$ are drawn from the base set S.

Definition 1.3 (Powerset)**.** [9] The *powerset* of a set S, denoted by $\mathcal{P}(S)$, is the collection of all subsets of S, including both the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.4 (n-th Powerset)**.** (cf.[21]) The n-th powerset of a set H, denoted by $P_n(H)$, is defined recursively as follows:

$$P_1(H) = \mathcal{P}(H), \quad P_{n+1}(H) = \mathcal{P}(P_n(H)) \quad \text{for } n \geq 1.$$

Similarly, the n-th nonempty powerset, denoted by $P_n^*(H)$, is given by

$$P_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(P_n^*(H)),$$

where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$ is the set of all nonempty subsets of H.

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We define the concept of a Weighted Set as follows. Related concepts include the Weighted Rough Set,[15,17] Weighted Soft Set,[1] Weighted Fuzzy Set,[18] and Weighted Neutrosophic Set,[12,22] which are well-studied in the literature.

Definition 1.5 (Weighted Set)**.** (cf.[23,24]) A *Weighted Set* is an ordered pair (S, w) where:

- S is a non-empty set.
- $w : S \to \mathbb{R}$ is a function that assigns a unique real number to each element $s \in S$; that is, for every $s \in S$, there exists a uniquely determined weight $w(s) \in \mathbb{R}$.

Example 1.6 (Example of a Weighted Set)**.** Consider the set $S = \{a, b, c\}$ and define the weight function w by

$$w(a) = 2.5, \quad w(b) = 4.0, \quad w(c) = 3.7.$$

Then, the pair (S, w) forms a weighted set.

Next, we describe our definition of a weighted graph. Although there are various ways to introduce weights—assigning them to vertices, edges, or both—in this paper we adopt the following definition. In this paper, we restrict our attention to finite, undirected, and simple graphs, unless otherwise specified.

Definition 1.7 (Weighted Graph)**.** (cf.[5,16]) A *Weighted Graph* augments the structure of a graph by assigning a numerical weight to each edge. Formally, a weighted graph is defined as a triple $G = (V, E, w)$ where:

- V is a non-empty set of vertices.
- $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ is a set of edges.
- $w : E \to \mathbb{R}$ is a weight function that assigns a unique real number to each edge $e \in E$.

Example 1.8 (Example of a Weighted Graph)**.** Let the vertex set be $V = \{v_1, v_2, v_3\}$ and the edge set be

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Define the weight function w by

$$w(\{v_1, v_2\}) = 10 \quad \text{and} \quad w(\{v_2, v_3\}) = 15.$$

Then, the triple $G = (V, E, w)$ constitutes a weighted graph.

2 Result of this paper

As a result of this paper, we introduce new definitions such as the Hyperweighted Graph.

2.1 Hyperweighted Graphs

Before introducing Hyperweighted Graphs, we extend the idea of a weight from a single real number to a set of weights. In a *Hyperweighted Set*, each element is associated with not just one weight, but a non-empty set of weights, allowing for multiple attributes to be recorded simultaneously.

Definition 2.1 (Hyperweighted Set)**.** [10] Let S be a non-empty set. A *Hyperweighted Set* is a pair (S, W) where

$$W : S \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$$

is a function that assigns to each element $s \in S$ a non-empty subset $W(s) \subseteq \mathbb{R}$. In other words, every element s is associated with multiple weights.

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Example 2.2 (Real-Life Example of a Hyperweighted Set)**.** Consider a set S of restaurants in a metropolitan area. In this scenario, each restaurant is evaluated on several performance metrics such as food quality, service, ambiance, and cleanliness. Let

$$S = \{\text{Restaurant A}, \text{Restaurant B}, \text{Restaurant C}\}.$$

Define the hyperweight function $W : S \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ as follows:

- $W(\text{Restaurant A}) = \{8.5, 7.0, 9.0, 8.0\}$, where the numbers represent scores (on a scale of 0 to 10) for food quality, service, ambiance, and cleanliness, respectively.
- $W(\text{Restaurant B}) = \{7.0, 8.0, 8.5, 7.5\}$.
- $W(\text{Restaurant C}) = \{9.0, 8.5, 8.0, 9.0\}$.

Here, the hyperweighted set (S, W) encapsulates multiple evaluation criteria for each restaurant, providing a richer and more nuanced representation of their overall performance.

Example 2.3 (Hyperweighted Set in Project Management)**.** Project management is the process of planning, executing, and controlling tasks to achieve specific goals within time, scope, and budget constraints. Consider a set S of projects:

$$S = \{\text{Project Alpha}, \text{Project Beta}, \text{Project Gamma}\}.$$

In project management, each project can be evaluated based on several key performance metrics. For instance, suppose each project is assigned the following numerical evaluations:

- **Project Alpha:** Estimated cost $1,000,000, risk level 0.3 (on a scale from 0 to 1), and expected duration 12 months. Thus,
$$W(\text{Project Alpha}) = \{1000000,\ 0.3,\ 12\}.$$
- **Project Beta:** Estimated cost $750,000, risk level 0.2, and duration 10 months. Hence,
$$W(\text{Project Beta}) = \{750000,\ 0.2,\ 10\}.$$
- **Project Gamma:** Estimated cost $1,250,000, risk level 0.5, and duration 14 months. Therefore,
$$W(\text{Project Gamma}) = \{1250000,\ 0.5,\ 14\}.$$

This hyperweighted set (S, W) captures multiple critical metrics for each project, enabling comprehensive evaluations in project management.

Example 2.4 (Hyperweighted Set in Weather Prediction)**.** Weather prediction is the scientific process of forecasting atmospheric conditions like temperature, rainfall, and wind using data and models (cf.[2, 14, 19]). Consider a set S of weather monitoring stations:

$$S = \{\text{Station 1}, \text{Station 2}, \text{Station 3}\}.$$

In weather prediction, each station is associated with several measurements that are crucial for forecasting. For example:

- **Station 1:** Records a temperature of 22.5°C, humidity of 60%, and wind speed of 5.2 m/s, so that
$$W(\text{Station 1}) = \{22.5,\ 60,\ 5.2\}.$$
- **Station 2:** Records a temperature of 18.3°C, humidity of 75%, and wind speed of 3.8 m/s, hence
$$W(\text{Station 2}) = \{18.3,\ 75,\ 3.8\}.$$

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- **Station 3:** Records a temperature of $25.0°$C, humidity of 55%, and wind speed of 4.5 m/s, therefore

$$W(\text{Station 3}) = \{25.0,\ 55,\ 4.5\}.$$

This hyperweighted set (S, W) consolidates essential weather parameters from each station, supporting detailed and robust weather prediction models.

Definition 2.5 (Hyperweighted Graph)**.** [10] A *Hyperweighted Graph* is a triple $G = (V, E, W)$ where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $W : E \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ is a *hyperweight function* that assigns to each edge $e \in E$ a non-empty set of real numbers.

Thus, each edge e is associated with a collection of weights $W(e)$, where these weights may represent different metrics (for instance, distance, time, cost, etc.).

Example 2.6 (Real-Life Example of a Hyperweighted Graph)**.** Consider a transportation network (cf.[6,7,20]) within a small town, where the intersections are represented as vertices and the roads connecting them as edges. Let the set of vertices be

$$V = \{v_1,\ v_2,\ v_3,\ v_4\},$$

with each vertex representing a major intersection. The edges E represent the roads between these intersections. Define the hyperweight function $W : E \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ to assign multiple attributes to each road. These attributes are:

- **Distance** in kilometers,
- **Average Travel Time** in minutes,
- **Toll Cost** in dollars.

For example, define the following edges and their associated weights:

- Let $e_1 = (v_1, v_2)$ with $W(e_1) = \{2.5,\ 5.0,\ 0\}$. This indicates a road of 2.5 km in length, an average travel time of 5 minutes, and no toll.
- Let $e_2 = (v_2, v_3)$ with $W(e_2) = \{3.0,\ 6.0,\ 1.5\}$. Here, the road has a distance of 3.0 km, takes about 6 minutes to traverse on average, and incurs a toll of 1.5 dollars.
- Let $e_3 = (v_3, v_4)$ with $W(e_3) = \{4.0,\ 8.0,\ 0\}$.
- Let $e_4 = (v_4, v_1)$ with $W(e_4) = \{3.5,\ 7.0,\ 2.0\}$.

This hyperweighted graph (V, E, W) effectively models the transportation network by providing multiple quantitative measures for each road, thus enabling more detailed analyses for route planning, traffic management, and infrastructure development.

Theorem 2.7 (Weighted Graph as a Special Case of a Hyperweighted Graph)**.** *(cf.[10]) Every Weighted Graph is a special case of a Hyperweighted Graph, where each edge is associated with a singleton set of weights.*

Proof. Let $G' = (V', E', w')$ be a weighted graph, where:

- V' is the set of vertices.
- $E' \subseteq V' \times V'$ is the set of edges.

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- $w' : E' \to \mathbb{R}$ is the weight function.

We construct a Hyperweighted Graph $G = (V, E, W)$ as follows:

$$V = V', \quad E = E', \quad \text{and} \quad W(e) = \{w'(e)\} \quad \text{for all } e \in E'.$$

That is, for each edge e, the hyperweight function W assigns the singleton set $\{w'(e)\}$.

To verify the validity of this construction:

$$W : E \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\},$$

since for every $e \in E$, $\{w'(e)\}$ is a non-empty subset of $\mathbb{R}$. Therefore, every weighted graph can be viewed as a hyperweighted graph with each edge's weight replaced by a singleton set containing that weight.

For example, if G' has an edge e with $w'(e) = 10$, then in G we have $W(e) = \{10\}$. This concrete transformation shows that the class of weighted graphs is contained in the class of hyperweighted graphs. □

Definition 2.8 (Minkowski Sum)**.** Let $A, B \subseteq \mathbb{R}$ be non-empty sets. Their *Minkowski sum* is

$$A \oplus B \;=\; \{\, a + b : a \in A,\; b \in B\}.$$

Definition 2.9 (Path Weight Set)**.** Let $G = (V, E, W)$ be a hyperweighted graph and let $P = (v_0, v_1, \ldots, v_k)$ be a simple path in G. The *path weight set* of P is

$$W(P) \;=\; W\big(v_0, v_1\big) \;\oplus\; W\big(v_1, v_2\big) \;\oplus\; \cdots \;\oplus\; W\big(v_{k-1}, v_k\big).$$

Theorem 2.10 (Non-emptiness and Boundedness of Path Weights)**.** *For any simple path P in a hyperweighted graph $G = (V, E, W)$, the set $W(P)$ is non-empty. If each $W(e)$ is bounded above and below, then $W(P)$ is a bounded interval in $\mathbb{R}$.*

Proof. Since each $W(e) \neq \emptyset$, the recursive Minkowski sum of finitely many non-empty sets is non-empty, so $W(P) \neq \emptyset$. If $\inf W(e) \geq m_e$ and $\sup W(e) \leq M_e$ for each e on P, then

$$\inf W(P) \;=\; \sum_{e \in P} m_e, \qquad \sup W(P) \;=\; \sum_{e \in P} M_e,$$

hence $W(P) \subseteq [\sum m_e,\; \sum M_e]$, a bounded interval. □

Theorem 2.11 (Supremum Decomposition)**.** *Let P be any simple path in $G = (V, E, W)$. Then*

$$\sup W(P) \;=\; \sum_{e \in P} \sup W(e).$$

Proof. For two bounded sets $A, B \subset \mathrm{R}$, one has $\sup(A \oplus B) = \sup A + \sup B$. By induction over the edges of P, the formula extends to any finite Minkowski sum along P, yielding

$$\sup W(P) = \sup\Big(\bigoplus_{e \in P} W(e)\Big) = \sum_{e \in P} \sup W(e).$$

□

Theorem 2.12 (Reduction to Classical Shortest Path)**.** *Define the* supremal weight function $w_{\sup} : E \to \mathrm{R}$ *by $w_{\sup}(e) = \sup W(e)$. Then any path P^* minimizing* $\sup W(P)$ *is exactly a shortest path in the weighted graph $(V, E, w_{\sup})$.*

Proof. By the Supremum Decomposition theorem,

$$\sup W(P) = \sum_{e \in P} w_{\sup}(e).$$

Hence minimizing $\sup W(P)$ over all u–v paths is equivalent to minimizing the classical path length $\sum_{e \in P} w_{\sup}(e)$, so P^* solves the ordinary shortest-path problem in $(V, E, w_{\sup})$. □

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Theorem 2.13 (Infimum Decomposition)**.** *Let $G = (V, E, W)$ be a hyperweighted graph and let $P = (v_0, v_1, \ldots, v_k)$ be any simple path in G. Then*

$$\inf W(P) \;=\; \sum_{i=1}^{k} \inf W\big(v_{i-1}, v_i\big).$$

Proof. For two non-empty bounded sets $A, B \subset \mathrm{R}$, the Minkowski sum satisfies $\inf(A \oplus B) = \inf A + \inf B$. By induction on the edges of P,

$$\inf W(P) = \inf\Big(W(v_0, v_1) \oplus \cdots \oplus W(v_{k-1}, v_k)\Big) = \sum_{i=1}^{k} \inf W\big(v_{i-1}, v_i\big).$$

□

Theorem 2.14 (Monotonicity of Path Weights)**.** *Let $G_1 = (V, E, W_1)$ and $G_2 = (V, E, W_2)$ be two hyperweighted graphs on the same vertex and edge sets. If*

$$W_1(e) \;\subseteq\; W_2(e) \quad \forall e \in E,$$

then for every simple path P in G_1, G_2,

$$W_1(P) \;\subseteq\; W_2(P).$$

Proof. Minkowski sum is monotone with respect to set inclusion: whenever $A \subseteq A'$ and $B \subseteq B'$, one has $A \oplus B \subseteq A' \oplus B'$. Thus, if $W_1(e) \subseteq W_2(e)$ for each edge of P, repeated application of this property yields

$$W_1(P) = \bigoplus_{e \in P} W_1(e) \;\subseteq\; \bigoplus_{e \in P} W_2(e) = W_2(P).$$

□

2.2 Superhyperweighted Graphs

In many applications, it is useful to further generalize the concept by allowing an edge to be associated with multiple hyperweights, each being a set of real numbers. This motivates the definition of a *Superhyperweighted Graph*.

Definition 2.15 (n-Superhyperweighted Set)**.** (cf.[10]) Let S be a non-empty set and let $n \geq 0$ be an integer. An *n-Superhyperweighted Set* is an ordered pair

$$(S, \mathcal{W}_n),$$

where the function

$$\mathcal{W}_n : S \to P_{n+1}(\mathbb{R}) \setminus \{\varnothing\}$$

assigns to each element $s \in S$ a nonempty nested collection of weights. In particular:

- For $n = 0$: $\mathcal{W}_0(s) \in P_1(\mathbb{R}) = \mathcal{P}(\mathbb{R})$ represents a nonempty set of real numbers (a standard Hyperweighted Set).
- For $n = 1$: $\mathcal{W}_1(s) \in P_2(\mathbb{R}) = \mathcal{P}(\mathcal{P}(\mathbb{R}))$ represents a collection of nonempty subsets of $\mathbb{R}$ (a Superhyperweighted Set).
- For $n = 2$: $\mathcal{W}_2(s) \in P_3(\mathbb{R})$ represents a nested structure with three layers.

Example 2.16 (Real-Life Example of a 2-Superhyperweighted Set)**.** Consider a set S of hospitals in "MetroCity" evaluated by two independent agencies (Agency Alpha and Agency Beta). Each agency conducts two rounds of evaluations: an initial assessment and a follow-up assessment. The evaluations are based on three criteria:

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1. Quality of care (score from 0 to 10),
2. Patient satisfaction (score from 0 to 10),
3. Operational efficiency (score from 0 to 10).

Thus, each evaluation is a nonempty subset of $\mathbb{R}$ with three scores.

Let

$$S = \{\text{MetroCity Hospital}, \text{Central Health Clinic}, \text{Eastside Medical Center}\}.$$

For each hospital $s \in S$, define the 2-superhyperweight function $\mathcal{W}_2 : S \to P_3(\mathbb{R}) \setminus \{\varnothing\}$ by assigning:

$$\mathcal{W}_2(s) = \Big\{\{H_{A1}(s), H_{A2}(s)\}, \{H_{B1}(s), H_{B2}(s)\}\Big\},$$

where:

- $H_{A1}(s)$ and $H_{A2}(s)$ are the initial and follow-up evaluations from Agency Alpha.
- $H_{B1}(s)$ and $H_{B2}(s)$ are the initial and follow-up evaluations from Agency Beta.

Concrete Data for MetroCity Hospital:

- Agency Alpha:
  - Initial evaluation: $H_{A1}(\text{MetroCity Hospital}) = \{8.2, 7.9, 8.5\}$,
  - Follow-up evaluation: $H_{A2}(\text{MetroCity Hospital}) = \{8.3, 8.0, 8.6\}$.
- Agency Beta:
  - Initial evaluation: $H_{B1}(\text{MetroCity Hospital}) = \{8.0, 8.1, 8.4\}$,
  - Follow-up evaluation: $H_{B2}(\text{MetroCity Hospital}) = \{8.1, 8.2, 8.5\}$.

Thus, the 2-superhyperweight for MetroCity Hospital is:

$$\mathcal{W}_2(\text{MetroCity Hospital}) = \Big\{\{\{8.2, 7.9, 8.5\}, \{8.3, 8.0, 8.6\}\}, \{\{8.0, 8.1, 8.4\}, \{8.1, 8.2, 8.5\}\}\Big\}.$$

This three-layered structure captures, in order, the agency grouping, the evaluation round, and finally the actual metric scores.

Theorem 2.17. *For any integer $n \geq 0$, every n-Superhyperweighted Set is a special case of an $(n+1)$-Superhyperweighted Set.*

Proof. Let $(S, \mathcal{W}_n)$ be an n-Superhyperweighted Set with

$$\mathcal{W}_n : S \to P_{n+1}(\mathbb{R}) \setminus \{\varnothing\}.$$

Define a new function

$$\mathcal{W}_{n+1} : S \to P_{n+2}(\mathbb{R}) \setminus \{\varnothing\}$$

by

$$\mathcal{W}_{n+1}(s) = \{\mathcal{W}_n(s)\} \quad \text{for each } s \in S.$$

Since $\mathcal{W}_n(s)$ is nonempty in $P_{n+1}(\mathbb{R})$, the singleton $\{\mathcal{W}_n(s)\}$ is nonempty in $P_{n+2}(\mathbb{R})$. Therefore, $(S, \mathcal{W}_{n+1})$ is an $(n+1)$-Superhyperweighted Set, embedding the n-level structure into the $(n+1)$-level structure. □

Definition 2.18 (n-Superhyperweighted Graph)**.** Let $n \geq 0$ be an integer. An *n-Superhyperweighted Graph* is a triple

$$G = (V, E, \mathcal{W}_n),$$

where:

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- V is a nonempty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $\mathcal{W}_n : E \to P_{n+1}(\mathbb{R}) \setminus \{\varnothing\}$ is an n-*superhyperweight function* that assigns to each edge $e \in E$ a nonempty nested collection of weights of depth $n+1$.

Thus, $\mathcal{W}_n(e)$ encodes a hierarchical set of numerical attributes associated with edge e.

Example 2.19 (Real-Life Example of a 2-Superhyperweighted Graph)**.** Consider a transportation network in "Urbanopolis" where intersections are vertices and roads are edges. Let

$$V = \{\text{Intersection } A, \text{ Intersection } B, \text{ Intersection } C, \text{ Intersection } D\}.$$

Each road $e \in E$ is evaluated under two traffic conditions: **Rush Hour** and **Off-Peak Hours**. Moreover, for each condition, measurements are taken in two sessions: **Morning** and **Evening**. The metrics recorded for each evaluation include:

1. Road length (km) – a constant value,
2. Average travel time (minutes),
3. Fuel consumption (liters per 100 km),
4. Congestion index (scale from 0 to 10).

For each edge e, define the 2-superhyperweight function

$$\mathcal{W}_2 : E \to P_3(\mathbb{R}) \setminus \{\varnothing\}$$

by:

$$\mathcal{W}_2(e) = \Big\{ \{H_{\text{Rush,Morning}}(e), H_{\text{Rush,Evening}}(e)\}, \{H_{\text{OffPeak,Morning}}(e), H_{\text{OffPeak,Evening}}(e)\} \Big\}.$$

Concrete Data for Road e_{AB} (connecting Intersection A and B):

- **Rush Hour Evaluations:**
  - Morning: $H_{\text{Rush,Morning}}(e_{AB}) = \{3.2, 10.5, 8.2, 9.0\}$,
  - Evening: $H_{\text{Rush,Evening}}(e_{AB}) = \{3.2, 11.0, 8.5, 9.2\}$.
- **Off-Peak Evaluations:**
  - Morning: $H_{\text{OffPeak,Morning}}(e_{AB}) = \{3.2, 6.0, 7.0, 3.5\}$,
  - Evening: $H_{\text{OffPeak,Evening}}(e_{AB}) = \{3.2, 6.5, 7.2, 3.8\}$.

Thus, the 2-superhyperweight for road e_{AB} is:

$$\mathcal{W}_2(e_{AB}) = \Big\{ \{\{3.2, 10.5, 8.2, 9.0\}, \{3.2, 11.0, 8.5, 9.2\}\}, \{\{3.2, 6.0, 7.0, 3.5\}, \{3.2, 6.5, 7.2, 3.8\}\} \Big\}.$$

This structure clearly distinguishes between traffic conditions (outer set), time sessions (middle set), and the specific metric values (innermost set).

Additional roads (e.g., e_{BC}, e_{CD}, e_{DA}) can be defined similarly with their own concrete evaluations.

Theorem 2.20. *For any integer $n \geq 0$, every n-Superhyperweighted Graph is a special case of an $(n+1)$-Superhyperweighted Graph.*

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Proof. Let $G = (V, E, \mathcal{W}_n)$ be an n-Superhyperweighted Graph, with

$$\mathcal{W}_n : E \to P_{n+1}(\mathbb{R}) \setminus \{\varnothing\}.$$

Define a function

$$\mathcal{W}_{n+1} : E \to P_{n+2}(\mathbb{R}) \setminus \{\varnothing\}$$

by

$$\mathcal{W}_{n+1}(e) = \{\mathcal{W}_n(e)\} \quad \text{for every edge } e \in E.$$

Since $\mathcal{W}_n(e)$ is nonempty in $P_{n+1}(\mathbb{R})$, the singleton $\{\mathcal{W}_n(e)\}$ is nonempty in $P_{n+2}(\mathbb{R})$. Thus, $G' = (V, E, \mathcal{W}_{n+1})$ is an $(n+1)$-Superhyperweighted Graph, confirming the embedding. □

Definition 2.21 (Iterated Supremum). Let $X \in P_{n+1}(\mathbb{R})$. Define

$$\overset{(1)}{\sup} X = \sup X, \quad \overset{(k)}{\sup} X = \sup\big\{ \overset{(k-1)}{\sup} Y : Y \in X \big\} \quad (2 \le k \le n+1).$$

Definition 2.22 (Nested Minkowski Sum). For $X, Y \in P_{n+1}(\mathrm{R})$, their *nested Minkowski sum* is

$$X \oplus Y = \big\{ A \oplus B : A \in X,\ B \in Y \big\},$$

where each $A, B \in P_n(\mathrm{R})$ are themselves added by the same rule recursively.

Theorem 2.23 (Iterated Supremum Decomposition). *Let $G = (V, E, \mathcal{W}_n)$ be an n-superhyperweighted graph, and let*

$$P = (v_0, v_1, \ldots, v_k)$$

be a simple path in G. Define the path superhyperweight by

$$\mathcal{W}_n(P) = \mathcal{W}_n(v_0, v_1) \ \oplus\ \mathcal{W}_n(v_1, v_2) \ \oplus\ \cdots\ \oplus\ \mathcal{W}_n(v_{k-1}, v_k).$$

Then

$$\overset{(n+1)}{\sup} \mathcal{W}_n(P) \ = \ \sum_{i=1}^{k} \overset{(n+1)}{\sup} \mathcal{W}_n\big(v_{i-1}, v_i\big).$$

Proof. We proceed by induction on the number of edges k. For $k = 1$, $\mathcal{W}_n(P) = \mathcal{W}_n(v_0, v_1)$ and the statement is trivial. Assume it holds for paths of length $k-1$. Write

$$P' = (v_0, \ldots, v_{k-1}), \quad e_k = (v_{k-1}, v_k).$$

Then

$$\mathcal{W}_n(P) = \mathcal{W}_n(P') \ \oplus\ \mathcal{W}_n(e_k).$$

By the property of nested Minkowski sum and iterated supremum for two sets A, B,

$$\overset{(n+1)}{\sup} (A \oplus B) = \overset{(n+1)}{\sup} A \ +\ \overset{(n+1)}{\sup} B.$$

Applying this with $A = \mathcal{W}_n(P')$ and $B = \mathcal{W}_n(e_k)$, and using the induction hypothesis,

$$\overset{(n+1)}{\sup} \mathcal{W}_n(P) = \overset{(n+1)}{\sup} \mathcal{W}_n(P') \ +\ \overset{(n+1)}{\sup} \mathcal{W}_n(e_k) = \sum_{i=1}^{k-1} \overset{(n+1)}{\sup} \mathcal{W}_n(v_{i-1}, v_i) \ +\ \overset{(n+1)}{\sup} \mathcal{W}_n(e_k),$$

which is the desired decomposition for k. □

Theorem 2.24 (Reduction to Classical Shortest Path). *With notation as above, define a real-valued* flattened weight

$$w_{\text{flat}}(e) \ = \ \overset{(n+1)}{\sup} \mathcal{W}_n(e) \quad \forall e \in E.$$

Then a path P^* that minimizes $\sup^{(n+1)} \mathcal{W}_n(P)$ among all u–v paths is exactly a shortest path in the weighted graph (V, E, w_{flat}).

Proof. By the Iterated Supremum Decomposition theorem, for any path P,

$$\overset{(n+1)}{\sup} \mathcal{W}_n(P) = \sum_{e \in P} w_{\text{flat}}(e).$$

Hence minimizing the left-hand side over all u–v paths is equivalent to minimizing the classical path length $\sum_{e \in P} w_{\text{flat}}(e)$. Therefore any minimiser P^* for the superhyperweighted criterion is also a shortest path in (V, E, w_{flat}), and vice versa. □

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2.3 MultiWeighted Graph

We introduce the concepts of a MultiWeighted Set and a MultiWeighted Graph. These concepts generalize the ideas of a Weighted Set and a Weighted Graph by allowing each element or edge to be assigned a vector of weights rather than a single numerical weight.

Definition 2.25 (MultiWeighted Set)**.** A *MultiWeighted Set* is an ordered pair (S, w) where:

- S is a non-empty set.
- $w : S \to \mathbb{R}^d$ is a weight function that assigns to each element $s \in S$ a weight vector $w(s) = (w_1(s), w_2(s), \ldots, w_d(s)) \in \mathbb{R}^d$ for some positive integer d.

Example 2.26 (Example of a MultiWeighted Set)**.** Consider the set $S = \{a, b, c\}$. Let the dimension of the weight vector be $d = 3$ and define the weight function w by:

$$w(a) = (2.5,\ 1.0,\ 3.0), \quad w(b) = (4.0,\ 0.5,\ 2.2), \quad w(c) = (3.7,\ 1.5,\ 3.3).$$

Then, (S, w) is a MultiWeighted Set, where each element has an associated weight vector in $\mathbb{R}^3$.

Theorem 2.27 (Generalization of Weighted Set)**.** *Every Weighted Set is a special case of a MultiWeighted Set with* $d = 1$*.*

Proof. A Weighted Set is defined as an ordered pair (S, w) where S is non-empty and $w : S \to \mathbb{R}$. Notice that $\mathbb{R}$ can be identified with $\mathbb{R}^1$. If we take $d = 1$ in the definition of a MultiWeighted Set, then the weight function becomes $w : S \to \mathbb{R}^1$; that is, each element $s \in S$ is assigned a weight vector $(w(s))$ which is essentially just the scalar $w(s)$. Therefore, any Weighted Set is trivially a MultiWeighted Set with $d = 1$. □

Definition 2.28 (MultiWeighted Graph)**.** A *MultiWeighted Graph* is defined as a triple $G = (V, E, w)$ where:

- V is a non-empty set of vertices.
- E is a set of edges, where each edge is a two-element subset of V (i.e., for each $e \in E$, there exist distinct vertices $u, v \in V$ such that $e = \{u, v\}$).
- $w : E \to \mathbb{R}^d$ is a weight function that assigns to each edge $e \in E$ a weight vector $w(e) \in \mathbb{R}^d$ for some positive integer d.

Example 2.29 (Example of a MultiWeighted Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4\},$$

and consider the edge set

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}.$$

Suppose we want each edge to have a weight vector in $\mathbb{R}^2$ (i.e., $d = 2$). Define the weight function w as follows:

$$w(\{v_1, v_2\}) = (1.5, 2.0), \quad w(\{v_2, v_3\}) = (2.0, 1.8), \quad w(\{v_3, v_4\}) = (1.2, 2.5), \quad w(\{v_4, v_1\}) = (2.2, 1.5).$$

Then, $G = (V, E, w)$ is a MultiWeighted Graph where each edge's weight is given by a vector in $\mathbb{R}^2$.

Theorem 2.30 (Generalization of Weighted Graph)**.** *Every Weighted Graph is a special case of a Multi-Weighted Graph with* $d = 1$*.*

Proof. A Weighted Graph is defined as a triple $G = (V, E, w)$ where the weight function $w : E \to \mathbb{R}$ assigns each edge a unique real number. By identifying $\mathbb{R}$ with $\mathbb{R}^1$, we see that a weighted graph is simply a MultiWeighted Graph with $d = 1$. In other words, if we let the weight function $w : E \to \mathbb{R}^1$ be given by $w(e) = (w'(e))$ where $w'(e) \in \mathbb{R}$, then the weighted graph conforms to the definition of a MultiWeighted Graph with one-dimensional weight vectors. □

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Notation 1. *Let $G = (V, E, w)$ be a MultiWeighted Graph with weight dimension d, i.e. $w : E \to \mathrm{R}^d$. For any simple path $P = (v_0, v_1, \ldots, v_k)$ define*

$$w(P) = \sum_{i=1}^{k} w(\{v_{i-1}, v_i\}) \in \mathrm{R}^d .$$

Theorem 2.31 (Path Weight Additivity)**.** *If P and Q are simple paths with $P = (v_0, \ldots, v_m)$ and $Q = (v_m, \ldots, v_{m+n})$, then*

$$w(P \circ Q) \;=\; w(P) + w(Q).$$

Proof. By definition,

$$w(P \circ Q) = \sum_{i=1}^{m+n} w(\{v_{i-1}, v_i\}) = \sum_{i=1}^{m} w(\{v_{i-1}, v_i\}) \;+\; \sum_{i=m+1}^{m+n} w(\{v_{i-1}, v_i\}) = w(P) + w(Q).$$

□

Example 2.32 (Additivity in R^2)**.** Let $V = \{v_1, v_2, v_3\}$, $E = \{\{v_1, v_2\}, \{v_2, v_3\}\}$, and $w(\{v_1, v_2\}) = (1, 2)$, $w(\{v_2, v_3\}) = (3, 4)$. Take $P = (v_1, v_2)$, $Q = (v_2, v_3)$. Then

$$w(P) = (1, 2), \quad w(Q) = (3, 4),$$

and

$$w(P \circ Q) = w(\{v_1, v_2\}) + w(\{v_2, v_3\}) = (1 + 3,\; 2 + 4) = (4, 6),$$

confirming additivity.

Definition 2.33 (Pareto Optimal Path)**.** A u–v path P is *Pareto optimal* if there is no other u–v path Q with

$$w(Q)_i \le w(P)_i \;\forall i, \quad w(Q)_j < w(P)_j \text{ for some } j.$$

Theorem 2.34 (Scalarization Implies Pareto Optimality)**.** *Given $\alpha = (\alpha_1, \ldots, \alpha_d)$ with each $\alpha_i > 0$, define*

$$w_\alpha(e) = \sum_{i=1}^{d} \alpha_i \, w(e)_i.$$

If P^* is a shortest u–v path in (V, E, w_α), then P^* is Pareto optimal in the multiweighted sense.

Proof. Suppose P^* is not Pareto optimal. Then some other path Q satisfies $w(Q)_i \le w(P^*)_i \,\forall i$ and $w(Q)_j < w(P^*)_j$ for at least one j. Multiplying by positive α_i and summing gives

$$w_\alpha(Q) < w_\alpha(P^*),$$

contradicting the minimality of P^*. □

Example 2.35 (Scalarization in R^2)**.** Graph as above plus direct edge $\{v_1, v_3\}$ with $w(\{v_1, v_3\}) = (4, 10)$. Then two paths from v_1 to v_3 are:

$$P_1 = v_1 - v_2 - v_3, \quad w(P_1) = (1 + 3, 2 + 4) = (4, 6),$$

$$P_2 = v_1 - v_3, \quad w(P_2) = (4, 10).$$

Choose $\alpha = (1, 1)$. Then

$$w_\alpha(P_1) = 4 + 6 = 10, \quad w_\alpha(P_2) = 4 + 10 = 14,$$

so P_1 is shortest and hence Pareto optimal, matching direct comparison $(4, 6) \le (4, 10)$.

Theorem 2.36 (Every Pareto Path Is Scalar-Optimal)**.** *If P is Pareto optimal, then there exists $\alpha \in \mathrm{R}^d_{>0}$ such that P is a shortest path in (V, E, w_α).*

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Proof. Let $\mathcal{P}$ be all simple u–v paths, and label the others $Q^1, \ldots, Q^N$. For each k, set $\Delta^k = w(Q^k) - w(P)$. Pareto optimality gives $\max_i \Delta_i^k > 0$. Choose $\alpha_i > 0$ (e.g. $\alpha_i = 2^{i-1}$) and scale so that for each k,

$$\sum_i \alpha_i \, \Delta_i^k > 0.$$

Then for every Q^k,

$$w_\alpha(Q^k) = w_\alpha(P) + \sum_i \alpha_i \, \Delta_i^k > w_\alpha(P),$$

making P strictly minimal. □

Example 2.37 (Constructing α in R^2)**.** Using the previous two-path example, P_1 has $w(P_1) = (4, 6)$, P_2 has $w(P_2) = (4, 10)$. Their difference is $\Delta = (0, 4)$. Any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 > 0$ yields $\alpha \cdot \Delta = \alpha_2 \cdot 4 > 0$. For instance $\alpha = (1, 1)$ again makes P_1 uniquely shortest.

3 Future Works

In future work, we aim to explore further generalizations by incorporating the frameworks of Bidirected Graphs[11,13] and Hypergraphs.[3,4,8]

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

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Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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