Neighborhood HyperRough Set and Neighborhood SuperHyperRough Set

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Abstract

Fuzzy sets,[20] rough sets,[14] intuitionistic fuzzy sets,[3] neutrosophic sets,[15] soft sets,[13] hesitant fuzzy set,[17] plithogenic sets,[16] and other uncertainty-handling frameworks have been the focus of intensive and ongoing research. Rough set theory provides a mathematical framework for approximating subsets through lower and upper approximations defined by equivalence relations, effectively capturing uncertainty in classification and data analysis.[5,10] Building upon these foundational concepts, further generalizations such as Hyperrough Sets[8] and Superhyperrough Sets have been introduced. In this paper, we investigate the concepts of Neighborhood Hyperrough Sets and Neighborhood Superhyperrough Sets. These models extend the classical Neighborhood Rough Set framework by incorporating the structural richness of Hyperrough Sets and Superhyperrough Sets.

Keywords: Rough set; Hyperrough Set; Neighborhood Rough Set; SuperHyperRough set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets under consideration are assumed to be finite.

1.1 Rough Set, HyperRough Set, and Superhyperrough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership.[14] The following definitions formalize these concepts.

Definition 1.1 (Universal Set)**.** A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

Definition 1.2 (Rough Set Approximation)**.** [14] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

1. *Lower Approximation*:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

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2. *Upper Approximation*:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 1.3 (Real-life Example of a Rough Set)**.** Consider a small company with a set of employees

$$U = \{\text{Alice, Bob, Charlie, Diana, Eve, Frank}\}.$$

Each employee is assigned to a department according to the function

$$\begin{aligned} d(\text{Alice}) &= \text{Sales}, \\ d(\text{Bob}) &= \text{Engineering}, \\ d(\text{Charlie}) &= \text{Sales}, \\ d(\text{Diana}) &= \text{HR}, \\ d(\text{Eve}) &= \text{Engineering}, \\ d(\text{Frank}) &= \text{HR}. \end{aligned}$$

Define the equivalence relation R by "having the same department." Its induced equivalence classes are:

$$[\text{Alice}]_R = \{\text{Alice, Charlie}\}, \quad [\text{Bob}]_R = \{\text{Bob, Eve}\}, \quad [\text{Diana}]_R = \{\text{Diana, Frank}\}.$$

Let the target concept be the set of employees in either Sales or HR:

$$X = \{\text{Alice, Charlie, Diana, Frank}\}.$$

Then the *lower approximation* of X is

$$\underline{X} = \{x \in U \mid [x]_R \subseteq X\} = \{\text{Alice, Charlie, Diana, Frank}\},$$

since both Sales and HR equivalence classes are entirely contained in X. The *upper approximation* of X is

$$\overline{X} = \{x \in U \mid [x]_R \cap X \neq \varnothing\} = \{\text{Alice, Charlie, Diana, Frank}\}.$$

Thus, in this case the rough set $(\underline{X}, \overline{X})$ is crisp.

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below.[8]

Definition 1.4 (HyperRough Set)**.** [8] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as
$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

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- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 1.5 (Real-life Example of a Hyperrough Set)**.** Consider a customer segmentation scenario with six customers:

$$U = \{C_1,\, C_2,\, C_3,\, C_4,\, C_5,\, C_6\}.$$

Suppose we have two attributes: *Age* and *Income*. Let the domain for *Age* be

$$J_1 = \{\text{Young},\ \text{Old}\},$$

and for *Income* be

$$J_2 = \{\text{Low},\ \text{High}\}.$$

The Cartesian product is

$$J = J_1 \times J_2 = \{(\text{Young}, \text{Low}),\ (\text{Young}, \text{High}),\ (\text{Old}, \text{Low}),\ (\text{Old}, \text{High})\}.$$

Define a mapping $F : J \to \mathcal{P}(U)$ by

$$\begin{aligned} F(\text{Young}, \text{Low}) &= \{C_1,\, C_2\}, \\ F(\text{Young}, \text{High}) &= \{C_3\}, \\ F(\text{Old}, \text{Low}) &= \{C_4,\, C_5\}, \\ F(\text{Old}, \text{High}) &= \{C_6\}. \end{aligned}$$

Assume an equivalence relation R on U induced solely by the *Age* attribute, so that customers are equivalent if they are in the same age group. Then the equivalence classes are:

$$[C_1]_R = [C_2]_R = [C_3]_R = \{C_1,\, C_2,\, C_3\}, \quad [C_4]_R = [C_5]_R = [C_6]_R = \{C_4,\, C_5,\, C_6\}.$$

For the attribute combination $(\text{Young}, \text{Low})$, we compute the rough approximations:

$$\underline{F(\text{Young}, \text{Low})} = \{x \in U \mid [x]_R \subseteq F(\text{Young}, \text{Low})\},$$

$$\overline{F(\text{Young}, \text{Low})} = \{x \in U \mid [x]_R \cap F(\text{Young}, \text{Low}) \neq \emptyset\}.$$

Notice that $F(\text{Young}, \text{Low}) = \{C_1,\, C_2\}$, but for any $x \in \{C_1,\, C_2,\, C_3\}$ (the Young group), we have $[x]_R = \{C_1,\, C_2,\, C_3\}$, which is not a subset of $\{C_1,\, C_2\}$. Thus,

$$\underline{F(\text{Young}, \text{Low})} = \varnothing.$$

However, every Young customer has an equivalence class that intersects $\{C_1,\, C_2\}$; hence,

$$\overline{F(\text{Young}, \text{Low})} = \{C_1,\, C_2,\, C_3\}.$$

This example illustrates how a Hyperrough Set uses a mapping from a Cartesian product of attribute domains to yield rough approximations based on multiple attributes.

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty. The definition of n-SuperHyperRough Sets is described as follows.

Definition 1.6 (n-SuperHyperRough Set)**.** [9] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

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- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 1.7 (Real-life Example of a 2-SuperHyperRough Set)**.** Consider a medical-screening scenario with five patients:

$$U = \{P_1, P_2, P_3, P_4, P_5\}.$$

We use two attributes:

- *Symptom Group* with domain
$$J_1 = \{\text{Fever, Cough, Fatigue}\},$$
- *Test Results* with domain
$$J_2 = \{\text{Positive, Negative}\}.$$

Form the power-sets $\mathcal{J}_1 = \mathcal{P}(J_1)$, $\mathcal{J}_2 = \mathcal{P}(J_2)$, and let

$$J = \mathcal{J}_1 \times \mathcal{J}_2.$$

Define a mapping $F : J \to \mathcal{P}(U)$ by, for instance,

$$F\big(\{\text{Fever}, \text{Cough}\}, \{\text{Positive}\}\big) = \{P_1, P_2\},$$
$$F\big(\{\text{Fatigue}\}, \{\text{Negative}\}\big) = \{P_3, P_4\},$$
$$F\big(\{\text{Cough}\}, \{\text{Positive}, \text{Negative}\}\big) = \{P_5\}.$$

Let R be the equivalence relation "same ward" on U with classes

$$[P_1]_R = [P_2]_R = \{P_1, P_2\}, \quad [P_3]_R = \{P_3, P_4\}, \quad [P_5]_R = \{P_5\}.$$

Take the attribute combination

$$A = \big(\{\text{Fatigue}\}, \{\text{Negative}\}\big).$$

Then

$$\underline{F(A)} = \{x \in U \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in U \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Since $F(A) = \{P_3, P_4\}$ and $[P_3]_R = \{P_3, P_4\} \subseteq F(A)$, we have

$$\underline{F(A)} = \{P_3, P_4\}.$$

Also, because $[P_4]_R \cap F(A) \neq \emptyset$, the upper approximation is the same:

$$\overline{F(A)} = \{P_3, P_4\}.$$

Thus this 2-SuperHyperRough Set captures precisely the group of patients in ward $\{P_3, P_4\}$ who all exhibit fatigue and test negative.

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Example 1.8 (Real-life Example of a 3-SuperHyperRough Set)**.** Imagine a movie-streaming service with six subscribers:

$$U = \{U_1, U_2, U_3, U_4, U_5, U_6\}.$$

We choose three attributes:

- *Preferred Genres* with $J_1 = \{\text{Action}, \text{Drama}, \text{Comedy}\}$,
- *Preferred Languages* with $J_2 = \{\text{English}, \text{Spanish}\}$,
- *Watching Time* with $J_3 = \{\text{Morning}, \text{Evening}\}$.

Form $\mathcal{J}_1 = \mathcal{P}(J_1)$, $\mathcal{J}_2 = \mathcal{P}(J_2)$, $\mathcal{J}_3 = \mathcal{P}(J_3)$, and

$$J = \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3.$$

Define $F : J \to \mathcal{P}(U)$, for example,

$$F\big(\{\text{Action}, \text{Comedy}\}, \{\text{English}\}, \{\text{Evening}\}\big) = \{U_1, U_2\},$$
$$F\big(\{\text{Drama}\}, \{\text{English}, \text{Spanish}\}, \{\text{Morning}\}\big) = \{U_3, U_4\},$$
$$F\big(\{\text{Comedy}\}, \{\text{Spanish}\}, \{\text{Evening}\}\big) = \{U_5\}.$$

Let R be the relation "same subscription tier" with classes

$$\{U_1, U_2\}, \quad \{U_3, U_4\}, \quad \{U_5, U_6\}.$$

Choose

$$A = \big(\{\text{Drama}\}, \{\text{English}, \text{Spanish}\}, \{\text{Morning}\}\big).$$

Then

$$\underline{F(A)} = \{x \in U \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in U \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Since $F(A) = \{U_3, U_4\}$ and $[U_3]_R = \{U_3, U_4\} \subseteq F(A)$, we get $\underline{F(A)} = \{U_3, U_4\}$. Moreover, $[U_4]_R \cap F(A) \neq \emptyset$, so $\overline{F(A)} = \{U_3, U_4\}$. This 3-SuperHyperRough Set example shows how combining three layers of preference subsets allows the service to target exactly the subscribers who share the same tier and match the chosen genre, language, and time-of-day profile.

1.2 Neighborhood Rough Sets

Neighborhood Rough Sets generalize classical rough sets by defining approximate regions of a subset using a distance threshold, thereby effectively handling numerical or hybrid data.[1,11,18,19] The definition of Neighborhood Rough Sets is described as follows.

Definition 1.9 (Neighborhood Rough Set)**.** [11] Let U be a nonempty finite set (called the universe) and let A be a set of attributes. For any subset of attributes $B \subseteq A$, assume there exists a distance function

$$D_B : U \times U \to \mathbb{R}_{\geq 0},$$

which satisfies the usual metric properties (nonnegativity, symmetry, and the triangle inequality). Given a fixed threshold $d \geq 0$, the *neighborhood* of an element $x \in U$ with respect to B is defined by

$$d_B(x) = \{y \in U \mid D_B(x, y) \leq d\}.$$

The pair (U, d_B) is called a *neighborhood approximation space*. For any subset $X \subseteq U$, the *lower approximation* and *upper approximation* of X are defined as

$$\underline{X}_B = \{x \in U \mid d_B(x) \subseteq X\},$$
$$\overline{X}_B = \{x \in U \mid d_B(x) \cap X \neq \varnothing\}.$$

The *boundary region* of X is then given by

$$BN_B(X) = \overline{X}_B \setminus \underline{X}_B.$$

Note that when $d = 0$, the neighborhood $d_B(x)$ reduces to the equivalence class of x induced by the indiscernibility relation on B, and the above model coincides with Pawlak's classical rough set model.

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Example 1.10. Consider the universe $U = \{x_1, x_2, x_3, x_4\}$ and a single attribute a with associated values:

$$f(x_1, a) = 2, \quad f(x_2, a) = 3, \quad f(x_3, a) = 5, \quad f(x_4, a) = 8.$$

Define the distance function $D_{\{a\}}$ by the absolute difference:

$$D_{\{a\}}(x_i, x_j) = |f(x_i, a) - f(x_j, a)|.$$

Let the threshold be $d = 2$. Then, the neighborhoods with respect to $\{a\}$ are computed as follows:

$$d_{\{a\}}(x_1) = \{x_1, x_2\} \quad \big(|2-3| = 1 \leq 2\big),$$

$$d_{\{a\}}(x_2) = \{x_1, x_2, x_3\} \quad \big(|3-2| = 1,\ |3-5| = 2\big),$$

$$d_{\{a\}}(x_3) = \{x_2, x_3\} \quad \big(|5-3| = 2\big),$$

$$d_{\{a\}}(x_4) = \{x_4\} \quad \big(|8-5| = 3 > 2,\ |8-3| = 5 > 2\big).$$

Now, let $X = \{x_1, x_2, x_3\}$. Then:

$$\underline{X}_{\{a\}} = \{x \in U \mid d_{\{a\}}(x) \subseteq X\}.$$

We observe that:

- $d_{\{a\}}(x_1) = \{x_1, x_2\} \subseteq X$,
- $d_{\{a\}}(x_2) = \{x_1, x_2, x_3\} \subseteq X$, and
- $d_{\{a\}}(x_3) = \{x_2, x_3\} \subseteq X$.

Thus, $\underline{X}_{\{a\}} = \{x_1, x_2, x_3\}$. Next, the upper approximation is

$$\overline{X}_{\{a\}} = \{x \in U \mid d_{\{a\}}(x) \cap X \neq \varnothing\}.$$

Since $d_{\{a\}}(x_4) = \{x_4\}$ does not intersect X, we have $\overline{X}_{\{a\}} = \{x_1, x_2, x_3\}$. Consequently, the boundary region is

$$BN_{\{a\}}(X) = \overline{X}_{\{a\}} \setminus \underline{X}_{\{a\}} = \emptyset.$$

This indicates that X is crisp (i.e., exactly defined) within the neighborhood rough set framework.

2 Results of This Paper

This section presents the results obtained in this paper.

2.1 Neighborhood Hyperrough Sets

The definition of Neighborhood Hyperrough Sets is described as follows.

Definition 2.1 (Neighborhood Hyperrough Set)**.** Let U be a nonempty finite universe, B a nonempty set of attributes with metric D_B and threshold $d \geq 0$, and let

$$d_B(x) = \{y \in U \mid D_B(x, y) \leq d\}.$$

Let $T_1, \ldots, T_n$ be n distinct attributes with domains $J_1, \ldots, J_n$ and $J = J_1 \times \cdots \times J_n$. A *Neighborhood Hyperrough Set* is a pair (F, J) where

$$F \colon J \to \mathcal{P}(U),$$

and for each $a \in J$,

$$\underline{F(a)}_B = \{x \in U \mid d_B(x) \subseteq F(a)\}, \quad \overline{F(a)}_B = \{x \in U \mid d_B(x) \cap F(a) \neq \varnothing\}.$$

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Theorem 2.2 (Approximation Inclusion). *For every $a \in J$,*

$$\underline{F(a)}_B \subseteq F(a) \subseteq \overline{F(a)}_B.$$

Proof. If $x \in \underline{F(a)}_B$ then $d_B(x) \subseteq F(a)$. Since $x \in d_B(x)$, we conclude $x \in F(a)$. Conversely, if $x \in F(a)$ then $x \in d_B(x)$ and thus $d_B(x) \cap F(a) \neq \varnothing$, so $x \in \overline{F(a)}_B$. This yields $\underline{F(a)}_B \subseteq F(a) \subseteq \overline{F(a)}_B$. □

Theorem 2.3 (Monotonicity and Distributivity). *For any $a, b \in J$:*

$$\underline{F(a \cap b)}_B \subseteq \underline{F(a)}_B \cap \underline{F(b)}_B, \qquad \overline{F(a \cup b)}_B \supseteq \overline{F(a)}_B \cup \overline{F(b)}_B.$$

Proof. Suppose $x \in \underline{F(a \cap b)}_B$. Then $d_B(x) \subseteq F(a \cap b) = F(a) \cap F(b)$. Hence $d_B(x) \subseteq F(a)$ and $d_B(x) \subseteq F(b)$, so $x \in \underline{F(a)}_B \cap \underline{F(b)}_B$.

Next, if $x \in \overline{F(a)}_B \cup \overline{F(b)}_B$, then either $d_B(x) \cap F(a) \neq \varnothing$ or $d_B(x) \cap F(b) \neq \varnothing$. In either case $d_B(x) \cap \big(F(a) \cup F(b)\big) \neq \varnothing$, so $x \in \overline{F(a \cup b)}_B$. Thus $\overline{F(a \cup b)}_B \supseteq \overline{F(a)}_B \cup \overline{F(b)}_B$. □

Theorem 2.4 (Boundary and Crispness). *Define the boundary region $\mathrm{BN}_B(a) = \overline{F(a)}_B \setminus \underline{F(a)}_B$. Then:*

1. $F(a) = \varnothing \implies \underline{F(a)}_B = \overline{F(a)}_B = \varnothing$.
2. $F(a) = U \implies \underline{F(a)}_B = \overline{F(a)}_B = U$.
3. $F(a)$ *is* crisp *under B if and only if* $\mathrm{BN}_B(a) = \varnothing$.

Proof. (1) If $F(a) = \varnothing$, then no x can satisfy $d_B(x) \subseteq F(a)$ or $d_B(x) \cap F(a) \neq \varnothing$, so both approximations are empty.

(2) If $F(a) = U$, then $d_B(x) \subseteq U$ and $d_B(x) \cap U \neq \varnothing$ for all x, so both approximations equal U.

(3) By definition $\mathrm{BN}_B(a) = \varnothing$ means $\underline{F(a)}_B = \overline{F(a)}_B$, which—together with Theorem 1—forces $F(a) = \underline{F(a)}_B = \overline{F(a)}_B$, i.e. the concept is crisply represented. □

Theorem 2.5 (Complement Duality). *For each $a \in J$, writing $F(a)^c = U \setminus F(a)$:*

$$\underline{F(a)^c}_B = U \setminus \overline{F(a)}_B, \qquad \overline{F(a)^c}_B = U \setminus \underline{F(a)}_B.$$

Proof. By definitions,

$$x \in \underline{F(a)^c}_B \iff d_B(x) \subseteq F(a)^c \iff d_B(x) \cap F(a) = \varnothing \iff x \notin \overline{F(a)}_B,$$

hence $\underline{F(a)^c}_B = U \setminus \overline{F(a)}_B$. The second identity follows by dual reasoning. □

Theorem 2.6. *A Neighborhood Hyperrough Set generalizes both a Neighborhood Rough Set and a Hyperrough Set.*

Proof. We prove that by suitable choices of the parameter set J and the threshold d, a Neighborhood Hyperrough Set reduces exactly to (i) a Neighborhood Rough Set and (ii) a Hyperrough Set.

(i) Reduction to Neighborhood Rough Set. Assume J contains exactly one element, say $J = \{a_0\}$. Then the mapping

$$F : J \to \mathcal{P}(U)$$

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is completely determined by the single subset $A := F(a_0) \subseteq U$. By definition, for each $x \in U$ the *neighborhood-based* approximations are

$$\underline{F(a_0)}_B = \{x \in U \mid d_B(x) \subseteq A\}, \qquad \overline{F(a_0)}_B = \{x \in U \mid d_B(x) \cap A \neq \varnothing\},$$

where $d_B(x) = \{y \in U \mid D_B(x, y) \leq d\}$ is exactly the neighborhood used in the classical Neighborhood Rough Set model with threshold d. Hence $\underline{F(a_0)}_B$ and $\overline{F(a_0)}_B$ coincide with the usual lower and upper approximations $\underline{A}_B$ and $\overline{A}_B$ of the subset A. Thus, when $|J| = 1$, the Neighborhood Hyperrough Set (F, J) is precisely the classical Neighborhood Rough Set on A.

(ii) Reduction to Hyperrough Set. Now assume the threshold is zero, $d = 0$. Then for each $x \in U$,

$$d_B(x) = \{\, y \in U \mid D_B(x, y) \leq 0\} = \{\, y \in U \mid D_B(x, y) = 0\}.$$

If we further assume that D_B induces an indiscernibility relation

$$x \sim y \iff D_B(x, y) = 0,$$

then the "neighborhood" of x becomes its equivalence class $[x]_\sim$. Therefore the approximations

$$\underline{F(a)}_B = \{x \in U \mid d_B(x) \subseteq F(a)\} = \{x \in U \mid [x]_\sim \subseteq F(a)\},$$

$$\overline{F(a)}_B = \{x \in U \mid d_B(x) \cap F(a) \neq \varnothing\} = \{x \in U \mid [x]_\sim \cap F(a) \neq \varnothing\},$$

match exactly the lower and upper approximations in the Hyperrough Set model (where $\sim$ plays the role of the equivalence R).

In both cases, the Neighborhood Hyperrough Set framework specializes to the respective classical models. Hence it indeed generalizes both the Neighborhood Rough Set and the Hyperrough Set. □

Example 2.7. Let $U = \{1, 2, 3, 4\}$ and consider a single attribute with a distance function defined by

$$D(x, y) = |x - y|,$$

with threshold $d = 1$. Then, for each $x \in U$,

$$d(x) = \{y \in U \mid |x - y| \leq 1\},$$

so that:

$$d(1) = \{1, 2\}, \quad d(2) = \{1, 2, 3\}, \quad d(3) = \{2, 3, 4\}, \quad d(4) = \{3, 4\}.$$

Let there be two attributes T_1 and T_2 with domains $J_1 = \{\text{red}, \text{blue}\}$ and $J_2 = \{\text{circle}, \text{square}\}$, respectively. Then,

$$J = J_1 \times J_2 = \{(\text{red}, \text{circle}), (\text{red}, \text{square}), (\text{blue}, \text{circle}), (\text{blue}, \text{square})\}.$$

Define a mapping $F : J \to \mathcal{P}(U)$ by

$$\begin{aligned} F(\text{red}, \text{circle}) &= \{1, 2\}, \\ F(\text{red}, \text{square}) &= \{1\}, \\ F(\text{blue}, \text{circle}) &= \{3, 4\}, \\ F(\text{blue}, \text{square}) &= \varnothing. \end{aligned}$$

For instance, for $a = (\text{red}, \text{circle})$ with $F(a) = \{1, 2\}$:

- $d(1) = \{1, 2\} \subseteq \{1, 2\}$ so $1 \in \underline{F(a)}_B$,
- $d(2) = \{1, 2, 3\} \not\subseteq \{1, 2\}$ but $d(2) \cap \{1, 2\} \neq \varnothing$, so $2 \in \overline{F(a)}_B$.

Hence, $\underline{F(\text{red}, \text{circle})}_B = \{1\}$ and $\overline{F(\text{red}, \text{circle})}_B = \{1, 2, 3\}$.

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2.2 Neighborhood n-Superhyperrough Sets

The definition of Neighborhood n-Superhyperrough Sets is described as follows.

Definition 2.8 (Neighborhood n-Superhyperrough Set)**.** Let U be a nonempty finite universe, and let $T_1, \ldots, T_n$ be n distinct attributes with domains $J_1, \ldots, J_n$. Set

$$J = \mathcal{P}(J_1) \times \cdots \times \mathcal{P}(J_n).$$

Let $B \subseteq \{T_1, \ldots, T_n\}$ admit a metric

$$D_B : U \times U \to \mathbb{R}_{\geq 0},$$

and fix $d \geq 0$. For each $x \in U$, let

$$d_B(x) = \{y \in U \mid D_B(x, y) \leq d\}.$$

A *Neighborhood n-Superhyperrough Set* is a pair (F, J) where

$$F : J \to \mathcal{P}(U),$$

and for each $A \in J$ the approximations are

$$\underline{F(A)}_B = \{x \in U \mid d_B(x) \subseteq F(A)\}, \quad \overline{F(A)}_B = \{x \in U \mid d_B(x) \cap F(A) \neq \varnothing\}.$$

Theorem 2.9 (Approximation Inclusion)**.** *For every $A \in J$,*

$$\underline{F(A)}_B \subseteq F(A) \subseteq \overline{F(A)}_B.$$

Proof. Let $A \in J$ be arbitrary.

(i) $\underline{F(A)}_B \subseteq F(A)$**.** Take any $x \in \underline{F(A)}_B$. By definition,

$$d_B(x) \subseteq F(A).$$

Since $x \in d_B(x)$, it follows immediately that $x \in F(A)$.

(ii) $F(A) \subseteq \overline{F(A)}_B$**.** Now let $x \in F(A)$. Again $x \in d_B(x)$, so

$$d_B(x) \cap F(A) \neq \varnothing.$$

Hence $x \in \overline{F(A)}_B$.

Combining (i) and (ii) yields $\underline{F(A)}_B \subseteq F(A) \subseteq \overline{F(A)}_B$. □

Theorem 2.10 (Monotonicity)**.** *If $A, B \in J$ satisfy $A \subseteq B$ (component-wise inclusion), then*

$$\underline{F(A)}_B \subseteq \underline{F(B)}_B, \quad \overline{F(A)}_B \subseteq \overline{F(B)}_B.$$

Proof. Assume $A = (A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n) = B$. Then $F(A) \subseteq F(B)$.

Lower approximations: If $x \in \underline{F(A)}_B$, then

$$d_B(x) \subseteq F(A) \subseteq F(B),$$

so $x \in \underline{F(B)}_B$.

Upper approximations: If $x \in \overline{F(A)}_B$, then

$$d_B(x) \cap F(A) \neq \varnothing.$$

Since $F(A) \subseteq F(B)$, we also have

$$d_B(x) \cap F(B) \neq \varnothing,$$

hence $x \in \overline{F(B)}_B$. □

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Theorem 2.11 (Distributivity). *For any $A, B \in J$:*

$$\underline{F(A \cap B)}_B \subseteq \underline{F(A)}_B \cap \underline{F(B)}_B, \quad \overline{F(A \cup B)}_B \supseteq \overline{F(A)}_B \cup \overline{F(B)}_B.$$

Proof. **(i) Lower approximations:** Suppose $x \in \underline{F(A \cap B)}_B$. Then

$$d_B(x) \subseteq F(A \cap B) = F(A) \cap F(B),$$

so $d_B(x) \subseteq F(A)$ and $d_B(x) \subseteq F(B)$. Thus $x \in \underline{F(A)}_B \cap \underline{F(B)}_B$.

(ii) Upper approximations: Suppose $x \in \overline{F(A)}_B \cup \overline{F(B)}_B$. Without loss of generality, $x \in \overline{F(A)}_B$, so

$$d_B(x) \cap F(A) \neq \varnothing.$$

Hence

$$d_B(x) \cap \big(F(A) \cup F(B)\big) \neq \varnothing,$$

which implies $x \in \overline{F(A \cup B)}_B$. The argument is symmetric if $x \in \overline{F(B)}_B$. □

Theorem 2.12 (Boundary and Crispness). *Define the boundary region $\mathrm{BN}_B(A) = \overline{F(A)}_B \setminus \underline{F(A)}_B$. Then:*

1. *If $F(A) = \varnothing$, then $\underline{F(A)}_B = \overline{F(A)}_B = \varnothing$.*
2. *If $F(A) = U$, then $\underline{F(A)}_B = \overline{F(A)}_B = U$.*
3. *$F(A)$ is* crisp *under B if and only if $\mathrm{BN}_B(A) = \varnothing$.*

Proof. **(1)** If $F(A) = \varnothing$, then no x satisfies $d_B(x) \subseteq \varnothing$ or $d_B(x) \cap \varnothing \neq \varnothing$, so both approximations are empty.

(2) If $F(A) = U$, then for all x, $d_B(x) \subseteq U$ and $d_B(x) \cap U \neq \varnothing$, hence both approximations equal U.

(3) By definition, $\mathrm{BN}_B(A) = \varnothing$ means $\overline{F(A)}_B = \underline{F(A)}_B$. Combined with Theorem 1, this forces $\underline{F(A)}_B = F(A) = \overline{F(A)}_B$, i.e. the concept is represented without uncertainty. □

Theorem 2.13 (Generalization). *A Neighborhood n-Superhyperrough Set reduces to*

- *a Neighborhood Hyperrough Set if J is a singleton,*
- *an n-Superhyperrough Set if $d = 0$ (so $d_B(x)$ becomes an equivalence class).*

Proof. We verify each specialization in turn, by showing that the definitions of the approximations coincide with the respective models.

(i) Reduction to Neighborhood Hyperrough Set. Assume $J = \{A_0\}$ is a singleton. Then the mapping

$$F : J \to \mathcal{P}(U)$$

is determined uniquely by the single set $S := F(A_0) \subseteq U$. By the definition of Neighborhood n-Superhyperrough Set, for each $x \in U$ the approximations are

$$\underline{F(A_0)}_B = \{\, x \in U \mid d_B(x) \subseteq S\}, \qquad \overline{F(A_0)}_B = \{\, x \in U \mid d_B(x) \cap S \neq \varnothing\},$$

where $d_B(x) = \{y \in U \mid D_B(x, y) \leq d\}$. But this is exactly the definition of the lower and upper approximations in the *Neighborhood Hyperrough Set* model applied to the single concept S. Thus $\underline{F(A_0)}_B$ and $\overline{F(A_0)}_B$ coincide term-for-term with the approximations of the Neighborhood Hyperrough Set $(S, \{A_0\})$.

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(ii) Reduction to n-Superhyperrough Set. Now assume $d = 0$ and that the metric D_B induces the equivalence relation

$$x \sim y \iff D_B(x, y) = 0.$$

Then for each $x \in U$,

$$d_B(x) = \{ y \in U \mid D_B(x, y) \leq 0\} = \{ y \in U \mid D_B(x, y) = 0\} = [x]_\sim.$$

Hence the "neighborhood-based" approximations become

$$\underline{F(A)}_B = \{ x \in U \mid [x]_\sim \subseteq F(A)\}, \qquad \overline{F(A)}_B = \{ x \in U \mid [x]_\sim \cap F(A) \neq \varnothing\},$$

which are exactly the lower and upper approximations in the definition of an *n-Superhyperrough Set* (where $\sim$ plays the role of the indiscernibility relation).

In both cases the specialization of J or d forces the Neighborhood n-Superhyperrough Set approximations to agree identically with those of the simpler models. Therefore, the Neighborhood n-Superhyperrough Set indeed generalizes both the Neighborhood Hyperrough Set and the n-Superhyperrough Set. □

Example 2.14 (Real-life Example of a Neighborhood 2-SuperHyperrough Set)**.** Consider a small real-estate market with five houses:

$$U = \{H_1, H_2, H_3, H_4, H_5\}.$$

Each house has two attributes:

- *Price Tier* T_1 with domain $J_1 = \{\text{Affordable, Expensive}\}$,
- *Area Type* T_2 with domain $J_2 = \{\text{Urban, Suburban}\}$.

We form

$$\mathcal{P}(J_1) = \{\varnothing, \{\text{Affordable}\}, \{\text{Expensive}\}, J_1\}, \quad \mathcal{P}(J_2) = \{\varnothing, \{\text{Urban}\}, \{\text{Suburban}\}, J_2\},$$

and let

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Suppose the geographic coordinates (in km) of the houses are:

$$H_1 : (0, 0),\ H_2 : (1, 1),\ H_3 : (2, 2),\ H_4 : (5, 5),\ H_5 : (6, 6),$$

and define the distance

$$D_B(H_i, H_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$

With threshold $d = 2.5$, the neighborhoods are

$$\begin{aligned}
d_B(H_1) &= \{H_1, H_2\},\\
d_B(H_2) &= \{H_1, H_2, H_3\},\\
d_B(H_3) &= \{H_2, H_3\},\\
d_B(H_4) &= \{H_4, H_5\},\\
d_B(H_5) &= \{H_4, H_5\}.
\end{aligned}$$

Define $F : J \to \mathcal{P}(U)$ on the most relevant combinations by

$$\begin{aligned}
F(\{\text{Affordable}\}, \{\text{Urban}\}) &= \{H_1, H_2\},\\
F(\{\text{Expensive}\}, \{\text{Suburban}\}) &= \{H_4, H_5\},\\
F(J_1, J_2) &= U,\\
F(A) &= \varnothing \quad \text{for all other } A \in J.
\end{aligned}$$

Take $A = (\{\text{Affordable}\}, \{\text{Urban}\})$. Then

$$F(A) = \{H_1, H_2\},$$

and the neighborhood-based approximations are

$$\underline{F(A)}_B = \{x \in U \mid d_B(x) \subseteq \{H_1, H_2\}\} = \{H_1\},$$

since $d_B(H_1) = \{H_1, H_2\} \subseteq \{H_1, H_2\}$ but $d_B(H_2) = \{H_1, H_2, H_3\} \nsubseteq \{H_1, H_2\}$. Likewise,

$$\overline{F(A)}_B = \{x \in U \mid d_B(x) \cap \{H_1, H_2\} \neq \varnothing\} = \{H_1, H_2, H_3\}.$$

This illustrates how a Neighborhood 2-SuperHyperrough Set captures both definite and possible members under spatial proximity and attribute subsets.

Example 2.15 (Real-life Example of a Neighborhood 3-SuperHyperrough Set)**.** Imagine a hotel-booking platform with six hotels:

$$U = \{H_1, H_2, H_3, H_4, H_5, H_6\}.$$

Each hotel has three attributes:

- *Price Tier* T_1: $J_1 = \{\text{Budget}, \text{Standard}, \text{Premium}\}$,
- *Amenity* T_2: $J_2 = \{\text{Pool}, \text{Gym}\}$,
- *Location* T_3: $J_3 = \{\text{Downtown}, \text{Suburb}\}$.

Form $\mathcal{P}(J_1), \mathcal{P}(J_2), \mathcal{P}(J_3)$ and

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \mathcal{P}(J_3).$$

Suppose their coordinates are:

$$H_1 : (0,0),\ H_2 : (1,1),\ H_3 : (2,1),\ H_4 : (6,5),\ H_5 : (7,5),\ H_6 : (9,9),$$

with the same Euclidean distance and threshold $d = 3$. Then

$$\begin{aligned} d_B(H_1) &= \{H_1, H_2, H_3\}, \\ d_B(H_2) &= \{H_1, H_2, H_3\}, \\ d_B(H_3) &= \{H_1, H_2, H_3\}, \\ d_B(H_4) &= \{H_4, H_5\}, \\ d_B(H_5) &= \{H_4, H_5\}, \\ d_B(H_6) &= \{H_6\}. \end{aligned}$$

Define $F : J \to \mathcal{P}(U)$ on the key combination

$$A = \big(\{\text{Budget}, \text{Standard}\},\ \{\text{Pool}\},\ \{\text{Downtown}\}\big)$$

by

$$F(A) = \{H_1, H_2, H_3\},$$

and set $F(A) = \varnothing$ or U for other A as appropriate. Then

$$\underline{F(A)}_B = \{x \in U \mid d_B(x) \subseteq \{H_1, H_2, H_3\}\} = \{H_1, H_2, H_3\},$$

because each of H_1, H_2, H_3 has neighborhood $\{H_1, H_2, H_3\}$. Likewise,

$$\overline{F(A)}_B = \{x \in U \mid d_B(x) \cap \{H_1, H_2, H_3\} \neq \varnothing\} = \{H_1, H_2, H_3\},$$

so this target combination is crisply captured. This example shows how a Neighborhood 3-SuperHyperrough Set can precisely represent multi-attribute, proximity-based concepts in a hotel-booking context.

3 Conclusion and Future Work

In this paper, we have explored the concepts of Neighborhood Hyperrough Sets and Neighborhood Superhyperrough Sets. In future research, we plan to investigate additional applications and develop new algorithms, as well as consider extensions that incorporate Fuzzy Rough Sets,[12,21,23] Neutrosophic Rough Sets,[4,22] Soft Rough sets,[2,6,7] and related frameworks.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

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The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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