Neutrosophic regular and normal subspaces in Neutrosophic Topological Spaces

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Abstract

The main purpose of this paper is to define the notion of neutrosophic based normal and regular spaces. This study investigates and open new class and conception of generalization of classical regular and normal spaces. The hereditary and topological properties of neutrosophic based normal and regular spaces have been analyzed and investigated. It also examines neutrosophic topological subspaces, providing insights into their characteristics. Furthermore, the paper investigates neutrosophic regular spaces and demonstrates their hereditary nature, specifically focusing on R_1, R_2, R_3 and R_4. Additionally, we explain some example of a neutrosophic-regular based space X which is a neutrosophic based normal-space but it is not necessary to neutrosophic - T_1 spaces. Eventually, it is shown that under certain conditions that the images are preserved in neutrosophic based normal and regular spaces.

Keywords: Neutrosophic space; Neutrosophic subspace; Neutrosophic regular and normal space; Neutrosophic regular subspace

1. Introduction

In recent decades, the field of fuzzy set (uncertain set) theory has been emerged as a powerful framework for dealing with uncertainty and vagueness in various domains that deals with real-life problems in Business, Finance, Artificial Intelligence, Computer Science, Engineering, Medical Sciences, and Social Sciences, etc. Fuzzy sets, that uncovered and introduced by Zadeh L. A. [7] in 1965, and reformulated by him in 1978. Çoker's (1997) [3] has defined intuitionistic fuzzy topological spaces, and later in (2000) he extended his framework to intuitionistic topological spaces. Smarandache [6] studied and developed a new branch of philosophy, called now neutrosophic sets as a good method to solve issues that cover untrustworthy, indeterminacy, and persistent data, offers an extended perspective on uncertainty by incorporating an additional measure called the indeterminacy degree to determine membership level as a third membership function to account for indeterminacy, alongside truth and falsity. Salama & Alblowi [5] had revealed in 2012 the idea of neutrosophic topological space as an extension of intuitionistic fuzzy topological space, which was developed by Coker [3] in 1997, and this opens a broad extent of investigation in terms of neutrosophic topology and its application in decision-making algorithms. Neutrosophic topology is a framework that extends traditional topology to accommodate neutrosophic sets. Applications of neutrosophic topology by Al-Omeri [2] depend upon the characteristics of neutrosophic-closed sets, neutrosophic interior and closure operators, and neutrosophic open sets. Neutrosophic topology provides a powerful tool for analysing topological spaces under conditions of ambiguity, uncertainty, and indeterminacy, enabling the exploration of topological properties in highly uncertain environments. In this paper, we will try to build a new space and subspace in neutrosophic topology, called neutrosophic regular and normal space, and determine its properties, and its relationship with other spaces.

2. Basic Definitions

Definition 2.1. Let $S = \langle x, \mu_S(x), \sigma_S(x), \nu_S(x)\rangle : x \in X\}$, be a Ns on X **[6].** The complement of the set $S(C(S)$, for short) can be defined as follows:

(1) $C(S) = \{\langle x, \nu_S(x), \sigma_S(x), \mu_S(x)\rangle : x \in X\}$
(2) $C(S) = \{\langle x, \nu_S(x), 1 - \sigma_S(x), \mu_S(x)\rangle : x \in X\}$
(3) $C(S) = \{\langle x, 1 - \mu_S(x), \sigma_S(x), 1 - \nu_S(x)\rangle : x \in X\}$

Neutrosophic sets (NSs for short) 0_N and 1_N in X are submitted as follows:

(1) 0_N can be defined as three types:
(i) $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$
(ii) $0_N = \{\langle x, 0, 1, 1\rangle : x \in X\}$
(iii) $0_N = \{\langle x, 0, 1, 0\rangle : x \in X\}$

(2) 1_N can be defined as three types:

(i) $1_N = \{\langle x, 1, 0, 0\rangle : x \in X\}$
(ii) $1_N = \{\langle x, 1, 0, 1\rangle : x \in X\}$
(iii) $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

Definition 2.2 [6]. Let X be a non-empty set, and $S = \{\langle x, \mu_S(x), \sigma_S(x), \nu_S(x)\rangle : x \in X\}$, $T = \{\langle x, \mu_T(x), \sigma_T(x), \nu_T(x)\rangle : x \in X\}$ are NSs. Then

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(1) $S \cap T$ can be defined as :
(i) $S \cap T = \{\langle x, \mu_S(x) \wedge \mu_T(x), \sigma_S(x) \wedge \sigma_T(x), \nu_S(x) \vee \nu_T(x)\rangle : x \in X\}$.
(ii) $S \cap T = \{\langle x, \mu_S(x) \wedge \mu_T(x), \sigma_S(x) \vee \sigma_T(x), \nu_S(x) \vee \nu_T(x)\rangle : x \in X\}$.

(2) $S \cup T$ can be defined as :

(i) $S \cup T = \{\langle x, \mu_S(x) \vee \mu_T(x), \sigma_S(x) \vee \sigma_T(x), \nu_S(x) \wedge \nu_T(x)\rangle : x \in X\}$.
(ii) $S \cup T = \{\langle x, \mu_S(x) \vee \mu_T(x), \sigma_S(x) \wedge \sigma_T(x), \nu_S(x) \wedge \nu_T(x)\rangle : x \in X\}$.
(3) $[\ ]\, S = \{\langle x, \mu_S(x), \sigma_S(x), 1 - \mu_S(x)\rangle : x \in X\}$.
(4) $<>\, S = \{\langle x, 1 - \nu_S(x), \sigma_S(x), \nu_S(x)\rangle : x \in X\}$.

Definition 2.3 [4]. A neutrosophic topology (NT for short) a non-empty set X is a family of neutrosophic subsets in X satisfying the following axioms:

$(\boldsymbol{NT_1})$ $0_N, 1_N \in \mathcal{T}$.
$(\boldsymbol{NT_2})$ $Q_1 \cap Q_2 \in \mathcal{T}$ for any $Q_1, Q_2 \in \mathcal{T}$.
$(\boldsymbol{NT_3})$ $\cup Q_i \in \mathcal{T}$ $\forall \{Q_i : i \in J\} \subseteq \mathcal{T}$.

In this case the pair $(X, \mathcal{T})$ is called a neutrosophic topological space (NTS for short) and any neutrosophic set $\mathcal{T}$ is known as a neutrosophic open set (NOS for short) in X. The elements of $\mathcal{T}$ are called open neutrosophic sets, a neutrosophic set F is closed if and only if $C(F)$ is neutrosophic open.

Definition 2.4 [4]. Let $(X, \mathcal{T}_1)$, $(X, \mathcal{T}_2)$ be two neutrosophic topological spaces on X. Then $\mathcal{T}_1$ *is* said be contained in $\mathcal{T}_2$ (in symbols $\mathcal{T}_1 \subseteq \mathcal{T}_2$) if $Q \in \mathcal{T}_2$ if for each $Q \in \mathcal{T}_1$ in this case, we also say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

Defnition 2.5 [4]. [Neutrosophic interior] Let $(X. \mathcal{T})$ be a neutrosophic topological space over X and $S \in N(x)$. Then, the neutrosophic interior of S, denoted by *Int* (S) is the union of all neutrosophic open subsets of S.

Clearly that *Int* (S) is the biggest neutrosophic open set over X which containing S.

Defnition 2.6 [6]. [Neutrosophic closure] Let $(X. \mathcal{T})$ be a neutrosophic topological space over X and $S \in N(X)$. Then, the neutrosophic closure of S, denoted by *Cl* (S) is the intersection of all neutrosophic closed super sets of S.

Clearly, *Cl* (S) is the smallest neutrosophic closed set over X which contains S.

Definition 2.7 [5] Let $S = \{\langle \mu_S(x), \sigma_S(x), \nu_S(x)\rangle : x \in X\}$ be a neutrosophic open set and

$T = \{\langle \mu_T(x), \sigma_T(x), \nu_T(x)\rangle : x \in X\}$ a neutrosophic set on a neutrosophic topological space $(X, \mathcal{T})$. Then,

(1) S is called neutrosophic regular open if and only if S = *NInt* (*NCl* (S)).

(2) If $T \in NCS(x)$, then T is called neutrosophic regular closed if and only if S = *NCl* (*NInt* (S)).

Definition 2.8 [5]. Let X and Z be two nonempty sets and $g: X \to Z$ be a mapping.
(1) If $S = \langle \mu_S, \sigma_S, \nu_S\rangle$ is a neutrosophic set in Z, then the preimage of S under g denoted by $g^{-1}(S)$, is a neutrosophic set in X defined by $g^{-1}(S) = \langle g^{-1}(\mu_S), g^{-1}(\sigma_S), g^{-1}(\nu_S)\rangle$.

(2) If $S = \langle \mu_S, \sigma_S, \nu_S\rangle$ is a neutrosophic set in X, then the image of S under g denoted by $g(S)$, is the a neutrosophic set in Z defined by $g(S) = \langle g(\mu_S), g(\sigma_S), g(\nu_S)\rangle$.

Definition 2.9 [1]. Suppose $(X, \mathcal{T})$ is a NTS. Then $(X, \mathcal{T})$ is said to be
(1) A NTS $(X, \mathcal{T})$ is called a N-R_0-space if and only if for any two NP $x_{\alpha,\beta,\nu}$ and $z_{\alpha'\beta'\nu'}$, if $x_{\alpha,\beta,\nu}\ \tilde{q}\ z_{\alpha'\beta'\nu'}$ then $x_{\alpha,\beta,\nu}\ \tilde{q}\ \overline{z_{\alpha'\beta'\nu'}}$.

(2) A NTS $(X, \mathcal{T})$ is called a N-R_1-space if and only if for any two NP $x_{\alpha,\beta,\nu}$ and $z_{\alpha'\beta'\nu'}$, if $x_{\alpha,\beta,\nu}\ \tilde{q}\ \overline{z_{\alpha'\beta'\nu'}}$ then there exists two N-open sets F and G in $(X, \mathcal{T})$ such that $x_{\alpha,\beta,\nu} \in F$, $z_{\alpha'\beta'\nu'} \in G$ and $F\tilde{q}G$.

(3) A NTS $(X, \mathcal{T})$ is called a neutrosophic regular space (N-R_2-space for short) space if and only if, for any NP $x_{\alpha,\beta,\nu}$ and any N-closed set H in $(X, \mathcal{T})$ such that $x_{\alpha,\beta,\nu}\tilde{q}H$, there exists two N-open sets F and G in $(X, \mathcal{T})$ such that $x_{\alpha,\beta,\nu} \in F$, $H \subset G$ and $F\tilde{q}G$.

(4) A NTS $(X, \mathcal{T})$ is called a neutrosophic normal space (N-R_3-space for short) space if and only if, for any two N-closed sets H and K in $(X, \mathcal{T})$ such that $H\tilde{q}K$, there exists two N-open sets F and G in $(X, \mathcal{T})$ such that $H \subset F$, $K \subset G$ and $F\tilde{q}G$.

Defnition 2.10 [1]. Suppose $(X, \mathcal{T})$ is a NTS. Then $(X, \mathcal{T})$ is said to be

(1) A NTS $(X, \mathcal{T})$ is called a N-T_0-space if for every pair of NP $x_{\alpha,\beta,\nu}$, $z_{\alpha'\beta'\nu'}$, whose supports are different, there exists NOSs F, G such that $x_{\alpha,\beta,\nu} \in F$, $z_{\alpha'\beta'\nu'} \in F^c$ or $x_{\alpha,\beta,\nu} \in G^c$, $z_{\alpha'\beta'\nu'} \in G$.

(2) A NTS $(X, \mathcal{T})$ is called a N-T_1-space if for every pair of NP $x_{\alpha,\beta,\nu}$, $z_{\alpha'\beta'\nu'}$, whose supports are different, there exists NOSs F, G such that $x_{\alpha,\beta,\nu} \in F$, $z_{\alpha'\beta'\nu'} \in F^c$ and $x_{\alpha,\beta,\nu} \in G^c$, $z_{\alpha'\beta'\nu'} \in G$.

(3) A NTS $\mathcal{T}$ is called a N-T_2-space, if, for every pair of NP $x_{\alpha,\beta,\nu}$, $z_{\alpha'\beta'\nu'}$, whose supports are different, there exists NOSs F, G such that $x_{\alpha,\beta,\nu} \in F$, $z_{\alpha'\beta'\nu'} \in F^c$, $z_{\alpha'\beta'\nu'} \in G$, $x_{\alpha,\beta,\nu} \in G^c$ and $F\tilde{q}G$. For a NTS $(X, \mathcal{T})$ we have the following diagram:

N-T_2-space
↓
N-T_1-space
↓
N-T_0-space
Converse statements may not be true.
(4) A NTS $(X, \mathcal{T})$ is called a N-T_3-space if and only if, it is both a N-R_2 and N-T_1-space.
(5) A NTS $(X, \mathcal{T})$ is called a N-T_4-space if and only if, it is both a N-R_3 and N-T_1-space.

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3. Regular and normal spaces in neutrosophic based topological spaces

Definition 3.1. Assume $(X, \widetilde{\mathfrak{T}})$ be a neutrosophic based topological space and $S \subseteq X$. If $\widetilde{\mathfrak{T}}_S$ is a topology created by $\langle U \cap \langle S, \phi, \phi\rangle : U \in \widetilde{\mathfrak{T}}\rangle$, then $\widetilde{\mathfrak{T}}_S$ is a neutrosophic based topology on S, and the space $(S, \widetilde{\mathfrak{T}}_S)$ is a subspace of $(X, \widetilde{\mathfrak{T}})$.

Definition 3.2. A neutrosophic based space $(X, \widetilde{\mathfrak{T}})$ is regular if for all $r \in X$ and $U = \langle U_1, U_2, U_3\rangle \in \widetilde{\mathfrak{T}}$ with $r \notin U_3$ then there exists two disjoint open sets $S = \langle S_1, S_2, S_3\rangle$ and $T = \langle T_1, T_2, T_3\rangle$ such that $U^C = \langle U_3, U_2, U_1\rangle \subset S$ and $r \in T_1$.

Definition 3.3. A neutrosophic based space $(X, \widetilde{\mathfrak{T}})$ is normal if $U = \langle U_1, U_2, U_3\rangle \in \widetilde{\mathfrak{T}}$, $Q = \langle Q_1, Q_2, Q_3\rangle \in \widetilde{\mathfrak{T}}$ with $U_3 \cap Q_3 = \phi$ then there exists two disjoint open sets $S = \langle S_1, S_2, S_3\rangle$ and $T = \langle T_1, T_2, T_3\rangle$ such that $U^C = \langle U_3, U_2, U_1\rangle \subset S$ and $Q^C = \langle Q_3, Q_2, Q_1\rangle \subset T$.

Theorem 3.1. Every classical regular space can be represented as a neutrosophic based regular space but the converse is not true.

Proof. Assume $(X, \mathfrak{T})$ be a classical regular space. Thus, based on the presupposition of regular space we have, if any $p \in X$, $U \in \mathfrak{T}$ with $p \notin U^C$, then there exists two open sets S and T are disjoint such that $U^C \subset S$ and $p \in T$.

Let $W = \langle U, \phi, U^C\rangle$, $F = \langle S, \phi, S^C\rangle$, $G = \langle T, \phi, T^C\rangle$, then it is explicit that $p \in X$ and $W = \langle U, \phi, U^C\rangle \in \widetilde{\mathfrak{T}}$, $p \notin U^C$, also we have two disjoint sets $F, G \in \widetilde{\mathfrak{T}}$ (since S, T disjoint) such that $W = \langle U, \phi, U^C\rangle \subset F = \langle S, \phi, S^C\rangle$ (as $U^C \subset S$) and $p \in T$. Therefore, $(X, \widetilde{\mathfrak{T}})$ is a neutrosophic based regular space.

Example 3.1. This example shows that the converse is not true.

Consider $X = \{s, t, k\}$, $U = \langle\{t, k\}, \phi, \{s\}\rangle$, $S = \langle\{s, k\}, \phi, \{t\}\rangle$, $T = \langle\{s, t\}, \phi, \{k\}\rangle$, $V = \langle\{s\}, \{k\}, \{t\}\rangle$, $W = \langle\{t\}, \{s\}, \{k\}\rangle$, $M = \langle\{k\}, \{t\}, \{s\}\rangle$. Assume $\widetilde{\mathfrak{T}}$ be a neutrosophic based topology created by $\{U, S, V, W, M\}$. Therefore, $(X, \widetilde{\mathfrak{T}})$ is a neutrosophic based topological space.

Thus, $(X, \widetilde{\mathfrak{T}})$ is a neutrosophic based topological space created by $\{U, S, T, V, W, M\}$. Now, choose $t \in X$, with $t \notin \{s\}$, where $U = \langle\{t, k\}, \phi, \{s\}\rangle \in \widetilde{\mathfrak{T}}$, then there exists two disjoint open sets S and W such that $U^C \subset S$ and $t \in W$. In the same way, it can be proven that the points s and k achieve the same result.

Thus $(X, \widetilde{\mathfrak{T}})$ is a neutrosophic based regular space but can't be shown as classical regular space.

Theorem 3.2. Every neutrosophic based regular subspace is also regular space.

Proof. Let $S \subseteq X$ and $(S, \widetilde{\mathfrak{T}}_S)$ be a subspace of the regular space $(X, \widetilde{\mathfrak{T}})$. Consider $M = \langle M_3, M_2, M_1\rangle$ is $\widetilde{\mathfrak{T}}_S$ -closed set in S. Therefore $M^C = \langle M_1, M_2, M_3\rangle$ is $\widetilde{\mathfrak{T}}_S$ -open set in S. Consider $r \in S$ with $r \notin M_3$. Since $M^C = \langle M_1, M_2, M_3\rangle \in \widetilde{\mathfrak{T}}_S$, then by the definition of subspace there exists $W = \langle W_1, W_2, W_3\rangle \in \widetilde{\mathfrak{T}}$ such that $M^C = W \cap \langle S, \phi, \phi\rangle$, then $\langle M_1, M_2, M_3\rangle = \langle W_1, W_2, W_3\rangle \cap \langle S, \phi, \phi\rangle = \langle W_1 \cap S, \phi, W_3\rangle$, i.e., $M_1 = W_1 \cap S$ and $M_2 = \phi$ and $M_3 = W_3$.

Since $r \notin M_3$ then $r \notin W_3$. So we get a $\widetilde{\mathfrak{T}}$ -closed set $W^C = \langle W_3, W_2, W_1\rangle$ such that $r \notin W_3$.

Since $(X, \widetilde{\mathfrak{T}})$ is a neutrosophic based regular space, hence there exists two open sets $U = \langle U_1, U_2, U_3\rangle$ and $Q = \langle Q_1, Q_2, Q_3\rangle$ are disjoint, such that $W^C \subset U$ and $r \in W_1$. Now, $W^C = \langle W_3, W_2, W_1\rangle \subset \langle U_1, U_2, U_3\rangle$ then $W_3 \subset U_1$, $U_2 \subset W_2$ and $W_1 \supset U_3$. Also, $W_3 \subset U_1$ then $M_3 \subset U_1$ then $M_3 \cap S \subset U_1 \cap S$ then $M_2 = M_2 \cap S \subset U_1 \cap S$. Again, $W_1 \supset U_3$ then $W_1 \cap S \supset U_3 \cap S$ then $M_1 \supset U_3 \cap S$. Therefore, $\langle M_3, M_2, M_1\rangle \subset \langle U_1 \cap S, U_2 \cap S, U_3 \cap S\rangle = U \cap S$, as $M_1 \cap M_2 \cap M_3 = \phi$ and $U_1 \cap U_2 \cap U_3 = \phi$.

Again, since $r \in S$ and $r \in Q_1$ then $r \in Q_1 \cap S$. Also, since $U \cap E = \phi$ then $U \cap S = \phi$ and $W \cap S = \phi$.

Eventually, $r \in S$ with $r \notin M_3$, there exists two disjoint open sets $U \cap S \in \widetilde{\mathfrak{T}}_S$ and $Q \cap S \in \widetilde{\mathfrak{T}}_S$ such that $M^C \subset U \cap S$ and $r \in Q_1 \cap S$. Thus $(S, \widetilde{\mathfrak{T}}_S)$ is a neutrosophic based regular space.

Theorem 3.3. Assume $(X, \widetilde{\mathfrak{T}})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, continuous and open map. If $(X, \widetilde{\mathfrak{T}})$ is neutrosophic based regular space then $(Z, \widetilde{\delta})$ is also neutrosophic based regular.

Proof. Assume $(X, \widetilde{\mathfrak{T}})$ and $(Z, \widetilde{\delta})$ be two neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, and open map. Assume $(X, \widetilde{\mathfrak{T}})$ is neutrosophic based regular space, we can proof that $(Z, \widetilde{\delta})$ is also neutrosophic based regular.

Assume $p \in Z$ with $p \notin U_3$ where $U = \langle U_1, U_2, U_3\rangle \in \widetilde{\delta}$, therefore there exists $s \in X$ such that $g(s) = p$ as g is onto. Once more, p is unique for s as g is one-one. Also, $U \in \widetilde{\delta}$ then $g^{-1}(U) \in \widetilde{\mathfrak{T}}$ as g is continuous. And $p \notin U_3$ then $s = g^{-1}(p) \notin g^{-1}(U_3)$. Since $(X, \widetilde{\mathfrak{T}})$ is neutrosophic based regular space, thus for any $s \in X$ with $g^{-1}(U_3)$ where $g^{-1}(U) = \langle g^{-1}(U_1), g^{-1}(U_2), g^{-1}(U_3)\rangle \in \widetilde{\mathfrak{T}}$ then there exists two open sets S and T that are disjoint and defined by $S = \langle S_1, S_2, S_3\rangle$, $T = \langle T_1, T_2, T_3\rangle \in \widetilde{\mathfrak{T}}$ such that $g^{-1}(U^C) \subset S$ and $s \in T_1$.

Because $S, T \in \widetilde{\mathfrak{T}}$ then $g(S), g(T) \in \widetilde{\mathfrak{T}}$ as g is open. We have $g(S) = \langle g(S_1), g(S_2), g(S_3)\rangle$, $g(T) = \langle g(T_1), g(T_2), g(T_3)\rangle$. Also, since $s \in T_1$ then $g(s) = p \in g(T_1)$ and $g^{-1}(U^C) \subset S$ then $gg^{-1}(U^C) \subset g(S)$ then $U^C \subset g(U)$. Furthermore, $g(S) \cap g(T) = \phi$ as $S \cap T = \phi$.

Finally, we have, for any $p \in Z$ with $p \notin U_3$ there exists two disjoint and open sets $g(S), g(T) \in \widetilde{\delta}$ such that $g(U^C) \subset g(S)$ and $p \in g(T_1)$. Therefore, $(Z, \widetilde{\delta})$ is a neutrosophic based regular space.

Theorem 3.4. Assume $(X, \widetilde{\mathfrak{T}})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, continuous, and open map. If $(Z, \widetilde{\delta})$ is neutrosophic based regular space then $(X, \widetilde{\mathfrak{T}})$ is also neutrosophic based regular.

Proof. Assume $(X, \widetilde{\mathfrak{T}})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces $g : X \to Z$ be one-one, onto, and open map. Assume $(Z, \widetilde{\delta})$ be neutrosophic based regular space, we have to prove that $(X, \widetilde{\mathfrak{T}})$ is also neutrosophic based regular.

Assume $r \in X$ with $r \notin U_3$, where $U = \langle U_1, U_2, U_3\rangle \in \widetilde{\mathfrak{T}}$ then there exists $s \in Z$ such that $s = g(r)$, for unique r as g is one-one. Also, $U \in \widetilde{\mathfrak{T}}$ then $(U) \in \widetilde{\delta}$ as g is open, and $r \notin U_3$ then $s = g(r) \notin g(U_3)$. Since $(Z, \widetilde{\delta})$ is neutrosophic based regular space, thus for any $s \in Z$ and $s \notin g(U_3)$, where $U = \langle U_1, U_2, U_3\rangle \in \widetilde{\delta}$ then there exists $S, T \in \widetilde{\delta}$ that are open and disjoint defined by $S = \langle S_1, S_2, S_3\rangle$ and $T = \langle T_1, T_2, T_3\rangle$ such that $g(U^C) \subset S$ and $s \in T_1$. Once more since $S, T \in \widetilde{\delta}$ then $g^{-1}(S), g^{-1}(T) \in \widetilde{\mathfrak{T}}$, as g is continuous. Now we get, $g^{-1}(S) = \langle g^{-1}(S_1), g^{-1}(S_2), g^{-1}(S_3)\rangle$, $g^{-1}(T) = \langle g^{-1}(T_1), g^{-1}(T_2), g^{-1}(T_3)\rangle$.

Therefore, $g(r) \in T_1$ then $g^{-1}(r) \in g^{-1}(T_1)$ then $r \in g^{-1}(T_1)$ and $g(U^C) \subset S$ then $g^{-1}g(U^C) \subset g^{-1}(S)$ then $U^C \subset g^{-1}(S)$.

Therefore, $g^{-1}(S) \cap g^{-1}(T_1) = \phi$ as $S \cap T = \phi$.

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Eventually, we have, for any $r \in X$ and $r \notin U_3$, there exists $g^{-1}(S)$, $g^{-1}(T) \in \widetilde{\Upsilon}$ that are open sets and disjoint such that $U^C \subset g^{-1}(S)$ and $r \in g^{-1}(T_1)$. Thus $(X, \widetilde{\Upsilon})$ is a neutrosophic based regular space.

Theorem 3.5. Neutrosophic based regular space doesn't require to be neutrosophic based normal space.

Example 3.2. This example shows that neutrosophic based regular space doesn't require to be neutrosophic based normal space.

Consider $X = \{s, t, k, n\}$, $S = \langle\{s, k\}, \phi, \{t\}\rangle$, $U = \langle\{t, k\}, \phi, \{s\}\rangle$, $V = \langle\{s\}, \{k\}, \{t\}\rangle$, $T = \langle\{s, t\}, \phi, \{k\}\rangle$, $W = \langle\{t\}, \{s\}, \{k\}\rangle$, $M = \langle\{k\}, \{t\}, \{s\}\rangle$. Assume $\widetilde{\Upsilon}$ be a neutrosophic based topology created by $\{U, S, V, W, M\}$. Therefore $(X, \widetilde{\Upsilon})$ is a neutrosophic based regular space (see Theorem 3.1). But $(X, \widetilde{\Upsilon})$ isn't a neutrosophic based normal space. Since for two closed subsets, disjoint U^C and T^C, there are no open sets, disjoint, so that U^C and T^C satisfy the normal space's axioms.

Theorem 3.6. Every neutrosophic based regular space doesn't require to be neutrosophic based T_1 space.

Example 3.3. This example shows that neutrosophic based regular space doesn't require to be neutrosophic based T_1 space.

Consider $X = \{s, t, k, n\}$, $U = \langle\{t, k, n\}, \phi, \{s\}\rangle$, $S = \langle\{s, k, n\}, \phi, \{t\}\rangle$, $T = \langle\{s, t, n\}, \phi, \{k\}\rangle$, $V = \langle\{s\}, \{k\}, \{t\}\rangle$, $W = \langle\{t\}, \{s\}, \{k\}\rangle$, $M = \langle\{k\}, \{t\}, \{s\}\rangle$. Let $\widetilde{\Upsilon}$ be a neutrosophic based topology created by $\{U, S, V, W, M\}$. Therefore, $(X, \widetilde{\Upsilon})$ is a neutrosophic based regular space. Choose $t \in X$, and $t \notin \{s\}$, where $U = \langle\{t, k, n\}, \phi, \{s\}\rangle \in \widetilde{\Upsilon}$, there exists two disjoint open sets S and W such that $U^C \subset S$ and $t \in W$. In the same way, it can be proven that the points s, k and n achieve the same result.

But the space isn't a neutrosophic based T_1-space. Since for the points t, n there are no open sets such that t and n can be contained in the sets separately.

Theorem 3.7. Every classical normal space can be represented as a neutrosophic based normal space but the converse is not true.

Proof. Assume $(X, \widetilde{\Upsilon})$ be classical and normal space. Thus, based on the axioms of normal space we get if for any $U, Q \in \widetilde{\Upsilon}$ and $U^C \cap Q^C = \phi$, then there exists two sets S and T are disjoint and open such that $U^C \subset S$, $Q^C \subset T$.

Let $W = \langle U, \phi, U^C\rangle$, $M = \langle Q, \phi, Q^C\rangle$, $F = \langle S, \phi, S^C\rangle$, $G = \langle T, \phi, T^C\rangle$ then it is explicit that $W^C \cap M^C = \phi$ (Because $U^C \cap Q^C = \phi$), and we have two sets $F, G \in \widetilde{\Upsilon}$ that are disjoint (Because $S \cap T = \phi$) such that $W^C = \langle U^C, \phi, U\rangle \subset F = \langle S, \phi, S^C\rangle$, (as $U^C \subset S$) and $M^C = \langle Q^C, \phi, Q\rangle \subset G = \langle T, \phi, T^C\rangle$ as $Q^C \subset T$. Thus, corresponding $(X, \widetilde{\Upsilon})$ is neutrosophic based normal space.

Example 3.4. This example shows that the converse is not true.

Consider $X = \{s, t, u, v, w, r, k, m\}$, $S = \langle\{u, v, w, r, k, m\}, \phi, \{s, t\}\rangle$, $T = \langle\{s, t, w, r, k, m\}\ \phi, \{u, v\}\rangle$, $U = \langle\{s, t, w, r, k\}, \phi, \{u, v, m\}\rangle$, $Q = \langle\{u, v, m\}, \phi, \{s, t, w, r, k\}\rangle$.

Let $\widetilde{\Upsilon}$ be a neutrosophic based topology created by $\{S, T, U, Q\}$. Therefore, $(X, \widetilde{\Upsilon})$ is a neutrosophic based topological space. Therefore $(X, \widetilde{\Upsilon})$ is also Neutrosophic based normal space since for any two disjoint and closed sets $S^C = \langle\{s, t\}, \phi, \{u, v, w, r, k, m\}\rangle$ and $T^C = \langle\{u, v\}, \phi, \{s, t, w, r, k, m\}\rangle$ there exists two disjoint open sets $U = \langle\{s, t, w, r, k\}, \phi, \{u, v, m\}\rangle$, $Q = \langle\{u, v, m\}, \phi, \{s, t, w, r, k\}\rangle \in \widetilde{\Upsilon}$ such that $S^C \subset U$ and $T^C \subset Q$. So, this finding is completely true for any others pair of disjoint and closed sets. Then $(X, \widetilde{\Upsilon})$ is a neutrosophic based normal space. But this space can't be considered as classical and normal space.

Theorem 3.8. Let $(X, \widetilde{\Upsilon})$ be a neutrosophic based normal space and $S \subseteq X$, then $(S, \widetilde{\Upsilon}_S)$ is neutrosophic based normal too.

Proof. Assume $(S, \widetilde{\Upsilon}_S)$ be a subspace of the neutrosophic based regular space $(X, \widetilde{\Upsilon})$. Consider $W = \langle W_3, W_2, W_1\rangle$ and $M = \langle M_3, M_2, M_1\rangle$ are disjoint $\widetilde{\Upsilon}_S$ -closed set in S then $W^C = \langle W_1, W_2, W_3\rangle$ and $M^C = \langle M_1, M_2, M_3\rangle$ are $\widetilde{\Upsilon}_S$ −open sets in S.

By the definition of $\widetilde{\Upsilon}_S$, we have , $F = \langle F_1, F_2, F_3\rangle \in \widetilde{\Upsilon}$ such that $W^C = F \cap \langle S, \phi, \phi\rangle$,

$\Rightarrow \langle W_1, W_2, W_3\rangle = \langle F_1, F_2, F_3\rangle \cap \langle S, \phi, \phi\rangle$,

$= \langle F_1 \cap S, \phi, F_3\rangle$.

Thus, $W_1 = F_1 \cap S$ and $W_2 = \phi$, $W_3 = F_3$, $G = \langle G_1, G_2, G_3\rangle \in \widetilde{\Upsilon}$ such that $M^C = G \cap \langle S, \phi, \phi\rangle$,

$\Rightarrow \langle M_1, M_2, M_3\rangle = \langle G_1, G_2, G_3\rangle \cap \langle S, \phi, \phi\rangle$,

$= \langle G_1 \cap S, \phi, G_3\rangle$.

i.e., $M_1 = G_1 \cap S$ and $G_2 = \phi$ and $M_3 = G_3$.

Now $F^C \cap G^C = \langle F_3, F_2, F_1\rangle \cap \langle G_3, G_2, G_1\rangle = \langle F_3 \cap G_3, F_2 \cap G_2, F_1 \cup G_1\rangle = \langle W_3 \cap M_3, W_2 \cap M_2, F_1 \cup G_1\rangle = \langle \phi, \phi, F_1 \cup G_1\rangle$ as $W \cap M = \phi$, so $F^C \cap G^C = \phi$.

We get F^C and G^C disjoint $\widetilde{\Upsilon}$ -closed sets. Because $(X, \widetilde{\Upsilon})$ is a neutrosophic based normal space then there exists disjoint $\widetilde{\Upsilon}$ -open sets $U = \langle U_1, U_2, U_3\rangle$ and $E = \langle E_1, E_2, E_3\rangle$: $F^C \subset U$ and $G^C \subset Q$. Now, $F^C = \langle F_3, F_2, F_1\rangle \subset \langle U_1, U_2, U_3\rangle$ implies that $F_3 \subset U_1$, $F_2 \subset U_2$ and $F_1 \supset U_3$.

Also, $F_3 \subset U_1$,

$\Rightarrow W_3 \subset U_1$,

$\Rightarrow W_3 \cap S \subset U_1 \cap S$,

$\Rightarrow W_3 = W_3 \cap S \subset U_1 \cap S$.

Again, $F_2 \subset U_2$,

$\Rightarrow W_2 \subset U_2$,

$\Rightarrow W_2 \cap S \subset U_2 \cap S$,

$\Rightarrow W_2 = W_2 \cap S \subset U_2 \cap S$.

Again, $F_1 \supset U_3$,

$\Rightarrow F_1 \cap S \supset U_3 \cap S$,

$\Rightarrow W_1 \supset U_3 \cap S$.

Thus, $\langle W_3, W_2, W_1\rangle = W \subset \langle U_1 \cap S, U_2 \cap S, U_3 \cap S\rangle = U \cap S$, as $W_1 \cap W_2 \cap W_3 = \phi$ and $U_1 \cap U_2 \cap U_3 = \phi$. In the same way, we can proof that $\langle M_3, M_2, M_1\rangle = M \subset \langle Q_1 \cap S, Q_2 \cap S, Q_3 \cap S\rangle = Q \cap S$, as $M_1 \cap M_2 \cap M_3 = \phi$ and $Q_1 \cap Q_2 \cap Q_3 = \phi$. Further, since $U \cap Q = \phi$, then $U \cap S = \phi$ and $Q \cap S = \phi$.

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Eventually, we have, for any two disjoint and closed sets $W = \langle W_3, W_2, W_1\rangle$ and $M = \langle M_3, M_2, M_1\rangle$, where $W^C = \langle W_1, W_2, W_3\rangle \in \widetilde{\Upsilon}_S$ and $M^C = \langle M_1, M_2, M_3\rangle \in \widetilde{\Upsilon}_S$ then there exist two disjoint open sets $U \cap S \in \widetilde{\Upsilon}_S$ and $Q \cap S \in \widetilde{\Upsilon}_S$ such that $W \subset U \cap S$ and $M \subset Q \cap S$. Therefore $(S, \widetilde{\Upsilon}_S)$ is a neutrosophic based normal space.

Theorem 3.9. Assume $(X, \widetilde{\Upsilon})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, continuous and open map. If $(X, \widetilde{\Upsilon})$ is neutrosophic based normal space then $(Z, \widetilde{\delta})$ is also neutrosophic based normal.

Proof. Assume $(X, \widetilde{\Upsilon})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, and open map. Let $(X, \widetilde{\Upsilon})$ is neutrosophic based normal space, we can proof that $(Z, \widetilde{\delta})$ is also neutrosophic based normal.

Let $U^C = \langle U_3, U_2, U_1\rangle$, $Q^C = \langle Q_3, Q_2, Q_1\rangle$ are disjoint and closed set $\widetilde{\delta}$ closed set, that is, $U = \langle U_1, U_2, U_3\rangle, Q = \langle Q_1, Q_2, Q_3\rangle$ then $g^{-1}(U), g^{-1}(Q) \in \widetilde{\Upsilon}$, is continuous. Now,

$g^{-1}(U) = \langle g^{-1}(U_1), g^{-1}(U_2), g^{-1}(U_3)\rangle$, $g^{-1}(Q) = \langle g^{-1}(Q_1), g^{-1}(Q_2), g^{-1}(Q_3)\rangle$, that is, $g^{-1}(U^C) = (g^{-1}(U))^C = \langle g^{-1}(U_3), g^{-1}(U_2), g^{-1}(U_1)\rangle$, and $g^{-1}(Q^C) = (g^{-1}(Q))^C = \langle g^{-1}(Q_3), g^{-1}(Q_2), g^{-1}(Q_1)\rangle$, are disjoint $\widetilde{\Upsilon}$ closed set (as $U^C \cap Q^C = \phi$, again g is one-one and onto).

Because $(X, \widetilde{\Upsilon})$ is normal space, so for any disjoint and closed sets $g^{-1}(U^C), g^{-1}(Q^C)$, where $g^{-1}(U), g^{-1}(Q) \in \widetilde{\Upsilon}$ there exists two disjoint and open sets $S = \langle S_1, S_2, S_3\rangle$, $T = \langle T_1, T_2, T_3\rangle, \in \widetilde{\Upsilon}$ such that $g^{-1}(U^C) \subset S$ and $g^{-1}(Q^C) \subset T$. Because $S, T \in \widetilde{\Upsilon}$ then $g(S), g(T) \in \widetilde{\delta}$ as g is open. Now we get,

$g(S) = \langle g(S_1), g(S_2), g(S_3)\rangle$, $g(T) = \langle g(T_1), g(T_2), g(T_3)\rangle$. Also, $g^{-1}(U^C) \subset S$,

$\Rightarrow gg^{-1}(U^C) \subset g(S)$,

$\Rightarrow U^C \subset g(S)$.

And $g^{-1}(Q^C) \subset T$,

$\Rightarrow gg^{-1}(Q^C) \subset g(T)$,

$\Rightarrow Q^C \subset g(T)$.

Again, $g(S) \cap g(T) = \phi$ as $S \cap T = \phi$, g is one-one.

Eventually, we have, for any two disjoint and closed sets U^C, Q^C where $U = \langle U_1, U_2, U_3\rangle$, $Q = \langle Q_1, Q_2, Q_3\rangle \in \widetilde{\delta}$ there exists disjoint and open sets $g(S), g(T) \in \widetilde{\delta}$ such that $U^C \subset g(S)$, $Q^C \subset g(T)$. Therefore $(Z, \widetilde{\delta})$ is a neutrosophic based normal space.

Theorem 3.10. Assume $(X, \widetilde{\Upsilon})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, onto, continuous and closed map. If $(Z, \widetilde{\delta})$ is neutrosophic based normal space then $(X, \widetilde{\Upsilon})$ is also neutrosophic based normal.

Proof. Assume $(X, \widetilde{\Upsilon})$ and $(Z, \widetilde{\delta})$ be neutrosophic based topological spaces and $g : X \to Z$ be one-one, continuous and open map. Assume $(Z, \widetilde{\delta})$ is neutrosophic based normal space, we can proof that $(X, \widetilde{\Upsilon})$ is also neutrosophic based normal.

Assume $r, m \in X$ and $r \neq m$ then $g(r), g(m) \in Z$ and $g(r) \neq g(m)$ as g is one-one.

Assume $Q^C = \langle Q_1, Q_2, Q_3\rangle, U^C = \langle U_3, U_2, U_1\rangle$, are disjoint $\widetilde{\Upsilon}$ closed set, that is, $Q = \langle Q_1, Q_2, Q_3\rangle$, $U = \langle U_1, U_2, U_3\rangle, \in \widetilde{\Upsilon}$ then $g(Q^C) \cap g(U^C) = \phi$, $\widetilde{\delta}$ closed set, as g is closed, one-one and onto.

Now we get, $g(Q^C) = \langle g(Q_3), g(Q_2), g(Q_1)\rangle$, $g(Q^C) = g(U_1), g(U_2), g(U_3)$. Since $(Z, \widetilde{\delta})$ is normal space, so for any two disjoint $\widetilde{\delta}$ closed sets $g(Q^C)$, and $g(U^C)$, there exists two disjoint and open sets $S = \langle S_1, S_2, S_3\rangle$, $T = \langle t_1, t_2, t_3\rangle \in \widetilde{\delta}$ such that $g(Q^C) \subset S$ and $g(U^C) \subset T$. Again because $S, T \in \widetilde{\delta}$ then $g^{-1}(S), g^{-1}(T) \in \widetilde{\Upsilon}$, as g is continuous. Now we get,
$g^{-1}(S) = \langle g^{-1}(S_1), g^{-1}(S_2), g^{-1}(S_3)\rangle$, $g^{-1}(T) = \langle g^{-1}(T_1), g^{-1}(T_2), g^{-1}(T_3)\rangle$.
Also, $g(Q^C) \subset S$,
$\Rightarrow g^{-1}g(g^{-1}(T)) \subset g^{-1}(S)$,
$\Rightarrow Q^C \subset g^{-1}(S)$.
And $g(U^C) \subset T$,
$\Rightarrow g^{-1}g(U^C) \subset g^{-1}(T)$,
$\Rightarrow U^C \subset g^{-1}(T)$.
Again, $g^{-1}(S) \cap g^{-1}(T) = \phi$ as $S \cap T = \phi$.

Eventually, we have, for any two disjoint and closed sets Q^C and U^C where $Q = \langle Q_1, Q_2, Q_3\rangle$, $U = \langle U_1, U_2, U_3\rangle, \in \widetilde{\Upsilon}$.

Then there exists two disjoint and open sets $g^{-1}(S) \in \widetilde{\delta}$ and $g^{-1}(T) \in \widetilde{\delta}$ such that $Q^C \subset g^{-1}(S)$ and $U^C \subset g^{-1}(T)$.

Hence, $(X, \widetilde{\Upsilon})$ is a neutrosophic based normal space.

Theorem 3.11. Neutrosophic based normal space doesn't require to be neutrosophic based regular space.

Example 3.5. This example shows that neutrosophic based normal space doesn't require to be neutrosophic based regular space.

Consider $X = \{s, t, u, v, w, r, k, m\}$, $S = \langle\{u, v, w, r, k, m\}, \phi, \{s, t\}\rangle$, $T = \langle\{s, t, w, r, k, m\}, \phi, \{u, v\}\rangle$, $U = \langle\{s, t, w, r, k\}, \phi, \{u, v, m\}\rangle$, $Q = \langle\{u, v, m\}, \phi, \{s, t, w, r, k\}\rangle$.

Let $\widetilde{\Upsilon}$ be a neutrosophic based topology created by $\{S, T, U, Q\}$. Therefore $(X, \widetilde{\Upsilon})$ is a neutrosophic based topological space. This $(X, \widetilde{\Upsilon})$ is also neutrosophic based normal space (see Theorem 3.7).

But the space isn't a neutrosophic based regular space. Since for any point $w \in X$ and $w \notin \{s, t\}$, where $S^C = \langle\{s, t\}, \phi, \{u, v, w, r, k, m\}\rangle$, then there are no disjoint and open sets such that they satisfy the neutrosophic based regular space's axioms.

Theorem 3.12. Every neutrosophic based normal space doesn't require to be neutrosophic based T_1 space.

Example 3.6. This example shows that neutrosophic based normal space doesn't require to be neutrosophic based T_1 space.

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Consider $X = \{s, t, u, v, w, r, k, m\}$, $S = \langle\{u, v, w, r, k, m\}, \phi, \{s, t\}\rangle$, $T = \langle\{s, t, w, r, k, m\}, \phi, \{u, v\}\rangle$, $U = \langle\{s, t, w, r, k\}, \phi, \{u, v, m\}\rangle$, $Q = \langle\{u, v, m\}, \phi, \{s, t, w, r, k\}\rangle$.

Let $\widetilde{\mathrm{T}}$ be a neutrosophic based topology created by $\{S, T, U, Q\}$. Therefore $(X, \widetilde{\mathrm{T}})$ is a neutrosophic based topological space. This $(X, \widetilde{\mathrm{T}})$ is also neutrosophic based normal space (see Theorem 3.7).

But the space is not a neutrosophic based T_1 space. Since for the points s, t there are no open sets such that t and u can be contained to the sets separately.

4. Regular and normal spaces in neutrosophic based topological spaces

Definition 4.1 Consider three subsets S_T and S_I and S_F of a non-empty set X such that $S_T \cap S_I = \phi$, $S_T \cap S_F = \phi$ and $S_I \cap S_F = \phi$. Let $S = \langle S_T, S_I, S_F\rangle$, then S is called a Ns of X. Here S_T is said to be set of membership, S_I is said to be set of indeterminacy, and S_F is said to be the set of nonmembers of S.

Definition 4.2 Consider a Ns S of a non-empty set X and $p_N \in X$.

(a) $p_N = \langle\{p\}, \phi, \{p\}^C\rangle$ is a NP and $p_{N_N} = \langle\phi, \{p\}, \{p\}^C\rangle$ is a vanishing point of X.

(b) If $p \in S_T$, then $p_N \in S$ and if $p \notin S_F$ then $p_{N_N} \in S$.

NP (X) is the set of all NP or neutrosophic vanishing points in X.

Example 4.1 let $X = \{s, t, u, v\}$ and $p = t \in X$.

Then $p_N = \langle\{t\}, \phi, \{s, u, v\}\rangle$,

$p_{N_N} = \langle\phi, \{t\}, \{s, u, v\}\rangle$,

$p = \langle\{t\}, \{s\}, \{v\}\rangle$.

Definition 4.3 Let (X, T) be a NTS and S be a subset of X. The class $\mathrm{T}_S = \{\langle U_T \cap S, U_I \cap S, U_F \cap S\} : U = \langle U_T, U_I, U_F\rangle \in \mathrm{T}\rangle$ is a NT on S. Then the NT T_S is called the relative NT on S or the relativization of T to S and the NTS (S, T_S) is called a neutrosophic topological subspace of (X, T).

Definition 4.4 Suppose (X, T) is a NTS. Then (X, T) is said to be

(a) R_1 if for each $p \in X$, M is a NCs of X and $p_N \notin M$ then there exists $A, B \in \mathrm{T}$ such that $p_N \in A$, $M \subseteq B$ and $A \cap B = \phi_N$.

(b) R_2 if for each $p \in X$, M is a NCs of X and $p_{N_N} \notin M$ then there exists $A, B \in \mathrm{T}$ such that $p_{N_N} \in A$, $M \subseteq B$ and $A \cap B = \phi_N$.

(c) R_3 if for each $p \in X$, M is a NCs of X and $p_N \notin M$ then there exists $A, B \in \mathrm{T}$ such that $p_N \in A$, $M \subseteq B$ and $A \subseteq B^C$.

(d) R_4 if for each $p \in X$, M is a NCs of X and $p_{N_N} \notin M$ then there exists $A, B \in \mathrm{T}$ such that $p_{N_N} \in A$, $M \subseteq B$ and $A \subseteq B^C$.

Theorem 4.1 Every neutrosophic subspace of a neutrosophic regular R_1 space is also a neutrosophic regular R_1 space. i.e., neutrosophic regular R_1 is hereditary.

Proof. Suppose (X, T) is a neutrosophic regular R_1 space and (S, T_S) is a subspace of (X, T).

Let $p \in S$ and $W = \langle W_T, W_I, W_F\rangle$ be a neutrosophic T_S-closed set of S and $p_N \notin W$. Therefore $W^C = \langle W_F, W_I, W_T\rangle$ is a neutrosophic T_S-open set of S .

Since (S, T_S) is a subspace of (X, T), then there exists a neutrosophic T-open set $U = \langle U_T, U_I, U_F\rangle$ of X such that $W^C = \langle W_F, W_I, W_T\rangle = \langle U_T \cap S, U_I \cap S, U_F \cap S\rangle$ implies that $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Since $p_N \notin W$, then $p \notin W_T$ implies $p \notin U_F \cap S$, which implies that $p \notin U_F$ since $p \in S$.

Therefore $p_N \notin U^C$.

Hence $p \in X$ and U^C is a neutrosophic T-closed set of X and $p_N \notin U^C$. By the definition 4.4 (a), there exists $A, B \in \mathrm{T} : p_N \in A$, $U^C \subseteq B$ and $A \cap B = \phi_N$.

Since $p_N \in A$ and $p \in S$ implies that $p_N \in \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle = K$, which is a neutrosophic T_S-open set of S.

Again, since $U^C \subseteq B$ then $\langle U_F, U_I, U_T\rangle \subseteq \langle B_T, B_I, B_F\rangle$ then $\langle U_F \cap S, U_I \cap S, U_T \cap S\rangle \subseteq \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = H$, which is a neutrosophic T_S-open set of S. Then, $W \subseteq H$, since $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Now, $K \cap H = \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle \cap \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = \langle(S_T \cap S) \cap (B_T \cap S), (S_I \cap S) \cap (B_I \cap S), (S_F \cap S) \cup (B_F \cap S)\rangle = \langle(S_T \cap B_T) \cap S, (S_I \cap B_I) \cap S, \langle(S_F \cap B_F) \cup S\rangle = \langle\phi \cap S, \phi \cap S, X \cap S\rangle = \langle\phi, \phi, S\rangle = \phi_{N_S}$, since $A \cap B = \phi_N$ then $\langle(S_T \cap B_T), (S_I \cap B_I), (S_F \cup B_F)\rangle = \langle\phi, \phi, X\rangle$. Therefore (S, T_S) is a neutrosophic regular R_1 space.

Proposition 4.1 Every neutrosophic subspace of a neutrosophic regular R_2 space is also a neutrosophic regular R_2 space. i.e., neutrosophic regular R_2 is hereditary.

Proof. Suppose (X, T) is a neutrosophic regular R_2 space and (S, T_S) is a subspace of (X, T).

Let $p \in S$ and $W = \langle W_T, W_I, W_F\rangle$ be a neutrosophic T_S-closed set of S and $p_{N_N} \notin W$. Therefore $W^C = \langle W_F, W_I, W_T\rangle$ is a neutrosophic T_S-open set of S .

Since (S, T_S) is a subspace of (X, T), then there exists a neutrosophic T-open set $U = \langle U_T, U_I, U_F\rangle$ of X such that $W^C = \langle W_F, W_I, W_T\rangle = \langle U_T \cap S, U_I \cap S, U_F \cap S\rangle$ implies that $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Since $p_{N_N} \notin W$, then $p \in W_F = U_T \cap S$ implies $p \in U_T$. Again since $p \in U_T$ and $U^C = \langle U_F, U_I, U_T\rangle$.

Therefore $p_{N_N} \notin U^C$.

Hence $p \in X$ and U^C is a neutrosophic T-closed set of X and $p_{N_N} \notin U^C$. By the definition 4.4 (b), there exists $A, B \in \mathrm{T} :$ $p_{N_N} \in A$, $U^C \subseteq B$ and $A \cap B = \phi_N$.

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Since $p_{N_N} \in A$ and $p \in S$ implies that $p_{N_N} \in \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle = K$, which is a neutrosophic $\Tau_S$-open set of S .

Again, since $U^C \subseteq B$ then $\langle U_F, U_I, U_T\rangle \subseteq \langle B_T, B_I, B_F\rangle$ then $\langle U_F \cap S, U_I \cap S, U_T \cap S\rangle \subseteq \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = H$, which is a neutrosophic $\Tau_S$-open set of S . Then, $W \subseteq H$, since $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Now, $K \cap H = \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle \cap \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = \langle (S_T \cap S) \cap (B_T \cap S), (S_I \cap S) \cap (B_I \cap S), (S_F \cap S) \cup (B_F \cap S)\rangle = \langle (S_T \cap B_T) \cap S, (S_I \cap B_I) \cap S, (S_F \cup B_F) \cap S\rangle = \langle \phi \cap S, \phi \cap S, X \cap S\rangle = \langle \phi, \phi, S\rangle = \phi_{N_S}$, since $A \cap B = \phi_N$ then $\langle (S_T \cap B_T), (S_I \cap B_I), (S_F \cup B_F)\rangle = \langle \phi, \phi, X\rangle$. Therefore $(S, \Tau_S)$ is a neutrosophic regular R_2 space.

Theorem 4.2 Every neutrosophic subspace of a neutrosophic regular R_3 space is also a neutrosophic regular R_3 space. i.e., neutrosophic regular R_3 is hereditary.

Proof. Suppose $(X, \Tau)$ is a neutrosophic regular R_3 space and $(S, \Tau_S)$ is a subspace of $(X, \Tau)$. Let $p \in S$ and $W = \langle W_T, W_I, W_F\rangle$ be a neutrosophic $\Tau_S$-closed set of S and $p_N \notin W$. Therefore $W^C = \langle W_F, W_I, W_T\rangle$ is a neutrosophic $\Tau_S$-open set of S.

Since $(S, \Tau_S)$ is a subspace of $(X, \Tau)$, then there exists a neutrosophic $\Tau$-open set $U = \langle U_T, U_I, U_F\rangle$ of X such that $W^C = \langle W_F, W_I, W_T\rangle = \langle U_T \cap S, U_I \cap S, U_F \cap S\rangle$ implies that $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Since $p_N \notin W$, then $p \notin W_T$ implies $p \notin U_F \cap S$, which implies that $p \notin U_F$ since $p \in S$.

Therefore $p_N \notin U^C$.

Hence $p \in X$ and U^C is a neutrosophic $\Tau$-closed set of X and $p_N \notin U^C$. By the definition 4.4 (3), there exists $A, B \in \Tau$: $p_N \in A$, $U^C \subseteq B$ and $A \subseteq B^C$.

Since $p_N \in A$ and $p \in S$ implies that $p_N \in \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle = K$, which is a neutrosophic $\Tau_S$-open set of S .

Again, since $U^C \subseteq B$ then $\langle U_F, U_I, U_T\rangle \subseteq \langle B_T, B_I, B_F\rangle$ then $\langle U_F \cap S, U_I \cap S, U_T \cap S\rangle \subseteq \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = H$, which is a neutrosophic $\Tau_S$-open set of S . Then, $W \subseteq H$, since $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$. Since, $A \subseteq B^C$ then $\langle S_T, S_I, S_F\rangle \subseteq \langle B_F, B_I, B_T\rangle$ then $\langle S_T \cap S, S_I \cap S, S_F \cap S\rangle \subseteq \langle B_F \cap S, B_I \cap S, B_T \cap S\rangle$ then $K \subseteq H^C$.

Therefore $(S, \Tau_S)$ is a neutrosophic regular R_3 space.

Proposition 4.2 Every neutrosophic subspace of a neutrosophic regular R_4 space is also a neutrosophic regular R_4 space. i.e., neutrosophic regular R_4 is hereditary.

Proof. Suppose $(X, \Tau)$ is a neutrosophic regular R_4 space and $(S, \Tau_S)$ is a subspace of $(X, \Tau)$.

Let $p \in S$ and $W = \langle W_T, W_I, W_F\rangle$ be a neutrosophic $\Tau_S$-closed set of S and $p_{N_N} \notin W$. Therefore $W^C = \langle W_F, W_I, W_T\rangle$ is a neutrosophic $\Tau_S$-open set of S.

Since $(S, \Tau_S)$ is a subspace of $(X, \Tau)$, then there exists a neutrosophic $\Tau$-open set $U = \langle U_T, U_I, U_F\rangle$ of X such that $W^C = \langle W_F, W_I, W_T\rangle = \langle U_T \cap S, U_I \cap S, U_F \cap S\rangle$ implies that $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Since $p_{N_N} \notin W$, then $p \in W_F = U_T \cap S$ implies $p \in U_T$. Again since $p \in U_T$ and $U^C = \langle U_F, U_I, U_T\rangle$.

Therefore $p_{N_N} \notin U^C$.

Hence $p \in X$ and U^C is a neutrosophic $\Tau$-closed set of X and $p_{N_N} \notin U^C$. By the definition 4.4 (d), there exists $A, B \in \Tau$: at $p_{N_N} \in A$, $U^C \subseteq B$ and $A \subseteq B^C$.

Since $p_{N_N} \in A$ and $p \in S$ implies that $p_{N_N} \in \langle S_T \cap S, S_I \cap S, S_F \cap S\rangle = K$, which is a neutrosophic $\Tau_S$-open set of S.

Again, since $U^C \subseteq B$ then $\langle U_F, U_I, U_T\rangle \subseteq \langle B_T, B_I, B_F\rangle$ then $\langle U_F \cap S, U_I \cap S, U_T \cap S\rangle \subseteq \langle B_T \cap S, B_I \cap S, B_F \cap S\rangle = H$, which is a neutrosophic $\Tau_S$-open set of S. Then, $W \subseteq$ H, since $W = \langle W_T, W_I, W_F\rangle = \langle U_F \cap S, U_I \cap S, U_T \cap S\rangle$.

Since, $A \subseteq B^C$ then $\langle S_T, S_I, S_F\rangle \subseteq \langle B_F, B_I, B_T\rangle$ then $\langle S_T \cap S, S_I \cap S, S_F \cap S\rangle \subseteq \langle B_F \cap S, B_I \cap S, B_T \cap S\rangle$ then $K \subseteq H^C$.

Therefore $(S, \Tau_S)$ is a neutrosophic regular R_4 space.

5. Conclusion

The neutrosophic regular and normal subspaces in neutrosophic topological spaces are introduced and investigated, and their basic properties have been studied and analyzed in this thesis. We have studied the relation between the regular, normal spaces in neutrosophic based topological spaces, and regular subspace in neutrosophic based topological spaces with other sets. Furthermore, we introduced the basic definitions of regular and normal spaces in neutrosophic based and regular subspace in neutrosophic based in topological spaces, according to the conclusions derived from some of their characterizations. Because of thesis time constraints, a lot of modifications and remarks have been postponed to later times or to other new thesis, and we hope that we could explore and focus more light on more points about neutrosophic regular and normal subspaces in neutrosophic topological spaces.

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