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An Introduction to Symbolic Neutrosophic Algebras

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Abstract

The objective of this paper is to study for the first time the concept of symbolic neutrosophic algebra defined over a neutrosophic ring R(I), where we derive a strict definition of this concept as an expansion of neutrosophic modules. In addition, we study some of its elementary properties such as neutrosophic subalgebras, neutrosophic homomorphisms and kernels through many theorems and mathematical proofs.

Keywords: Neutrosophic ring; Neutrosophic module; Neutrosophic algebra; Neutrosophic homomorphism

1. Introduction

Neutrosophic algebraic structures are generalizations of classical structures. The main idea of neutrosophic extension of classical structures is built on adding a logical element I with an algebraic property I*I=I [1]. The previous approach was used widely in the study of neutrosophic algebraic structures, where we can see examples about neutrosophic rings, neutrosophic groups, matrices and spaces [2-6]. In [9], the concept of neutrosophic module was obtained, and then many authors from different special sides, such as substructures and homomorphisms [7-8], studied it.

This has motivated us to study for the first time the concept of symbolic neutrosophic algebra defined over a neutrosophic ring R(I), where we derive a strict definition of this concept as an expansion of neutrosophic modules. In addition, we study some of its elementary properties such as neutrosophic subalgebras, neutrosophic homomorphisms and kernels through many theorems and mathematical proofs. For basic definitions in neutrosophic rings and modules, see [7,8,9].

2. Main Discussion

Neutrosophic algebra

In symbolic neutrosophic algebra, the classical algebraic structure is generalized by adding an element (I) with the multiplicative property $I^2 = I$. Here, squaring represents a multiplication operation that must be present in the algebraic structure before expansion.

This construction method uses definition before structure. We will construct the neutrosophic algebra in two ways and then prove their equivalence algebraically.

Definition 1:

Let $(M(I), +, .)$ be a neutrosophic module over the neutrosophic ring $R(I)$, and let

$$\circ : M(I) \times M(I) \to M(I)$$

Be a binary operation, with:

1-) $(x_0 + x_1 I) \circ [(y_0 + y_1 I) + (z_0 + z_1 I)] = [(x_0 + x_1 I) \circ (y_0 + y_1 I)] + [(x_0 + x_1 I) \circ (z_0 + z_1 I)]$; $x_i, y_i, z_i \in M$

2-) $(a + bI).[(x_0 + x_1 I) \circ (y_0 + y_1 I)] = [(a + bI).(x_0 + x_1 I)] \circ (y_0 + y_1 I)$; $a, b \in R$

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We call $(M(I),+,.,\circ)$ a neutrosophic algebra over the neutrosophic ring $R(I)$.

This definition means that a neutrosophic algebra is a classical algebra constructed by a neutrosophic module $M(I)$ over a neutrosophic ring $R(I)$.

Definition 2:

Let M be a non-empty set, and R, with operations

$$\begin{cases} +: M \times M \to M \\ .: R \times M \to M \\ \circ: M \times M \to M \end{cases}$$

such that $(M(I),+,.,\circ)$ be an algebra over R. We define the neutrosophic algebra in the following way

$$M(I) = \{x + yI \,; x, y \in M \,, I^2 = I\}$$

With operations

$$+: M(I) \times M(I) \to M(I)$$
$$.: R(I) \times M(I) \to M(I)$$
$$\circ: M(I) \times M(I) \to M(I)$$

such that

$$\begin{cases} (x + yI) + (z + tI) = (x + z) + (y + t)I, \;\; x, y, z, t \in M \\ (a + bI).(z + tI) = az + (at + bz + bt)I \,; a, b \in R \\ (x + yI) \circ (z + tI) = x \circ z + I(x \circ t + y \circ z + y \circ t), \;\; x, y, z, t \in M \end{cases}$$

Remark:

The first definition depends on the constructing a neutrosophic algebra from a neutrosophic module $M(I)$, while the second definition constructs a neutrosophic algebra from a classical $(M,+,.,\circ)$ algebra over the ring R.

Through the following theorem, we will prove the equivalence of the two definitions, i.e., a neutrosophic algebra can be constructed from a classical algebra or from a neutrosophic module at the same time.

Theorem:

Let M be a non-empty set, and R be an arbitrary ring. Then, there exist operations

$$\begin{cases} +: M \times M \to M \\ .: R \times M \to M \\ \circ: M \times M \to M \end{cases}$$

such that $(M,+,.,\circ)$ be an algebra over R if and only if there exist the operations

$$\begin{cases} +': M(I) \times M(I) \to M(I) \\ .': R(I) \times M(I) \to M(I) \\ \circ': M(I) \times M(I) \to M(I) \end{cases}$$

such that $(M(I),+',.',\circ')$ be an algebra over $R(I)$ where

$$I^2 = I, I \circ I = I.$$

Proof:

Let us assume that $(M,+,.,\circ)$ is an algebra over the ring R, and define

$$\begin{cases} +': M(I) \times M(I) \to M(I) \\ .': R(I) \times M(I) \to M(I) \\ \circ': M(I) \times M(I) \to M(I) \end{cases}$$

Where, for $a, b \in R, x_i, y_i \in M$

$$(x_0 + x_1 I) +' (y_0 + y_1 I) = (x_0 + y_0) + (x_1 + y_1)I,$$
$$(a + bI).' (x_0 + x_1 I) = ax_0 + I(ax_1 + bx_0 + bx_1)\,,$$
$$(x_0 + x_1 I) \circ' (y_0 + y_1 I) = x_0 \circ y_0 + I(x_0 \circ y_1 + x_1 \circ y_0 + x_1 \circ y_1),$$

We note that $(M(I),+',.')$ is a neutrosophic module according to []

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On the other hand,

let $x_0 + x_1 I, y_0 + y_1 I, z_0 + z_1 I \in M(I)$. Then

$$\begin{aligned}(x_0 + x_1 I) \circ [(y_0 + y_1 I) + (z_0 + z_1 I)] &= (x_0 + x_1 I) \circ [(y_0 + z_0) + (y_1 + z_1)I] \\ &= x_0 \circ (y_0 + z_0) + I[x_0 \circ (y_1 + z_1) + x_1 \circ (y_0 + z_0) + x_1 \circ (y_1 + z_1)] \\ &= x_0 \circ y_0 + I[x_0 \circ y_1 + x_1 \circ y_0 + x_1 \circ y_1] + x_0 \circ z_0 + I[x_0 \circ z_1 + x_1 \circ z_0 + x_1 \circ z_1] \\ &= (x_0 + x_1 I) \circ' (y_0 + y_1 I) +' (x_0 + x_1 I) \circ' (z_0 + z_1 I)\end{aligned}$$

$$\begin{aligned}(x_0 + x_1 I) \circ [(a + bI).(y_0 + y_1 I)] &= (x_0 + x_1 I) \circ [(ay_0) + (ay_1 + by_0 + by_1)] \\ &= x_0 \circ (ay_0) + I[x_0 \circ (ay_1 + by_0 + by_1) + x_1 \circ (ay_0) + x_1 \circ (ay_1 + by_0 + by_1)] \\ &= a(x_0 \circ y_0) \\ &\quad + I[a(x_0 \circ y_1) + b(x_0 \circ y_0) + b(x_0 \circ y_1) + a(x_1 \circ y_0) + a(x_1 \circ y_1) + b(x_1 \circ y_0) \\ &\quad + b(x_1 \circ y_1)] = (a + bI).' [(x_0 + x_1 I) \circ' (y_0 + y_1 I)],\end{aligned}$$

Thus $(M(I), +', .', \circ')$ is a neutrosophic algebra over $R(I)$.

Now, let us assume that $(M(I), +', .', \circ')$ is a neutrosophic algebra over $R(I)$, we prove that $(M, +, ., \circ)$ is an algebra over R.

Consider

$$\begin{cases} +: M \times M \to M \\ .: R \times M \to M \\ \circ: M \times M \to M \end{cases}$$

where

$$\begin{gathered} x + y = (x + 0.I) +' (y + 0.I), \\ a.x = (a + 0.I).' (x + 0.I), \\ x \circ y = (x + 0.I) \circ' (y + 0.I); x, y \in M, a \in R \end{gathered}$$

The operations $(\circ, ., +)$ are the restrictions of $(\circ', .', +')$ over M. Thus $(M, +, .)$ is a module over R.

On the other hand

$$x \circ (y + z) = (x + 0.I) \circ' [(y + 0.I) +' (z + 0.I)] = x \circ y + x \circ z$$

$$x \circ (a.y) = (x + 0.I) \circ' [(a + 0.I).' (y + 0.I)] = a(x \circ y)$$

Thus $(M, +, ., \circ)$ is an algebra over R.

Neutrosophic Subalgebra:

Let $(M(I), +, ., \circ)$ be a neutrosophic algebra over $R(I)$, and let B is a non-empty subset from $M(I)$, we call B is a neutrosophic subalgebra from $M(I)$ if it holds

1) $aI \in B$ for some $a \in M$ and $a \neq 0$.

2) $(B, +, .)$ is a submodule from $M(I)$ over $R(I)$.

3) $x \circ y \in B$ for any $x, y \in B$.

Remark:

Since $M(I)$ is an algebra in the classical meaning, we can say

1) $M(I)$ associative if $(\circ)$ is associative

2) $M(I)$ commutative if $(\circ)$ is commutative

3) $M(I)$ unitary if $(\circ)$ has a unity (1).

Theorem:

Let M be an algebra over R, and let $M(I)$ is the corresponding neutrosophic algebra over $R(I)$. Then

1) $M(I)$ is associative if and only if M is associative

2) $M(I)$ is commutative if and only if M is commutative

3) $M(I)$ unitary if and if M is unitary

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Proof:

Since M *is contained in* $M(I)$, then any property which is true in $M(I)$ still true M.

Now, let us assume that is associativity/unity existence/ commutativity, then

$$\begin{aligned}(x_0+x_1I)\circ'[(y_0+y_1I)\circ'(z_0+z_1I)] &= (x_0+x_1I)\circ'[(y_0\circ z_0)+I(y_0\circ z_1+y_1\circ z_0+y_1\circ z_1)]\\ &= x_0\circ(y_0\circ z_0)\\ &\quad+I[x_1\circ(y_0\circ z_0)+x_0\circ(y_0\circ z_1+y_1\circ z_0+y_1\circ z_1)+x_1\circ(y_0\circ z_1+y_1\circ z_0+y_1\circ z_1)]\\ &= (x_0\circ y_0)\circ z_0\\ &\quad+I[(x_1\circ y_0)\circ z_0+(x_0\circ y_0)\circ z_1+(x_0\circ y_1)\circ z_0+(x_0\circ y_1)\circ z_1+(x_1\circ y_0)\circ z_1\\ &\quad+(x_1\circ y_1)\circ z_0+(x_1\circ y_1)\circ z_1]=[(x_0+x_1I)\circ'(y_0+y_1I)]\circ'(z_0+z_1I)\end{aligned}$$

$$\begin{aligned}(x_0+x_1I)\circ'(y_0+y_1I) &= x_0\circ y_0+I[x_0\circ y_1+x_1\circ y_0+x_1\circ y_1]\\ &= y_0\circ x_0+I[y_1\circ x_0+y_0\circ x_1+y_1\circ x_1]=(y_0+y_1I)\circ'(x_0+x_1I)\end{aligned}$$

$1\circ(x+yI)=1\circ x+1\circ yI=x+yI$

Where $x_i, y_i, z_i \ , x, y \in M$. Therefore $M(I)$ is associative/ unitary/ commutative.

Theorem:

Let B be a subalgebra of M. Then $B(I)$ is a neutrosophic subalgebra of $M(I)$, where

$$B(I)=\{x+yI\,;x,y\in B\}$$

Proof:

It is clear that $yI\in B(I)$ and $(B(I),+',.')$ is a neutrosophic submodule from $M(I)$.

$$\forall\ x_0+x_1I, y_0+y_1I\in B(I)\,,(x_0+x_1I)\circ'(y_0+y_1I)=x_0\circ y_0+I[x_0\circ y_1+x_1\circ y_0+x_1\circ y_1]\in B(I)$$

because $x_i\circ y_j\in B$. Thus, B(I) is a neutrosophic subalgebra of $M(I)$.

Definition:

Let $M(I)$ be a is neutrosophic algebra over $R(I)$, and N be a non-empty subset from $M(I)$‹. We call that N is an ideal of $M(I)$, if

1) $aI\in B$ for some $a\in M$ and $a\neq 0$.

2) $(N,+,.)$ is a neutrosophic submodule of $M(I)$.

3) $(x+yI)\circ n\in N$ for any $x+yI\in M(I), n\in N$.

Theorem:

Let N be an ideal of M over R, then $N(I)=\{n_0+n_1I\,;n_i\in N\}$ is a neutrosophic ideal in $M(I)$.

Proof:

It is clear that $n_1I\in N(I)$, and $(N(I),+',.')$ is a neutrosophic module of $M(I)$.

$\forall\, x+yI\in M(I), n_0+n_1I\in N(I)$:

$$(x+yI)\circ(n_0+n_1I)=(x\circ n_0)+I[x\circ n_1+y\circ n_0+y\circ n_1]\in N(I)$$

That is because $x\circ n_i, y\circ n_i\in N$, thus $N(I)$ is an ideal of $M(I)$.

Definition:

Let $M(I)$ be a neutrosophic algebra over $R(I)$. We define

$$z\big(M(I)\big)=\left\{\begin{array}{r}x+yI\ \in M(I);(x+yI)\circ(z+tI)=(z+tI)\circ(x+yI);\\ \forall\ z+tI\in M(I)\end{array}\right.$$

We call $z\big(M(I)\big)$ is the center of $M(I)$.

Theorem:

Let $M(I)$ be a neutrosophic algebra over $R(I)$. Then

1) $M(I)$ is commutative if and only if $z\big(M(I)\big)=M(I)$.

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2) If $M(I)$ is associative, then $z\big(M(I)\big)$ is a subalgebra from $M(I)$.

Proof:

1) If $M(I)$ is commutative, then it is clear that $z\big(M(I)\big) = M(I)$.

Let us assume $z\big(M(I)\big) = M(I)$, then

$$\forall\, x + yI, z + tI \in M(I) \Longrightarrow (x + yI) \circ (z + tI) = (z + tI) \circ (x + yI)$$

Thus it is commutative.

2) Let assume that $M(I)$ is associative. Then

$$\forall\, x_0 + x_1I, y_0 + y_1I \in z\big(M(I)\big), z_0 + z_1I \ \in M(I)\ , a + bI \in R(I)$$

We have

$$\begin{aligned}(z_0 + z_1I) \circ [(x_0 + x_1I) - (y_0 + y_1I)] &= (z_0 + z_1I) \circ (x_0 + x_1I) - (z_0 + z_1I) \circ (y_0 + y_1I)\\ &= (x_0 + x_1I) \circ (z_0 + z_1I) - (y_0 + y_1I) \circ (z_0 + z_1I)\\ &= [(x_0 + x_1I) - (y_0 + y_1I)] \circ (z_0 + z_1I) \Longrightarrow (x_0 + x_1I) - (y_0 + y_1I) \in z\big(M(I)\big)\end{aligned}$$

$$[(a + bI).(x_0 + x_1I)] \circ (z_0 + z_1I) = (a + bI).[(z_0 + z_1I) \circ (x_0 + x_1I)] = (z_0 + z_1I) \circ [(a + bI).(x_0 + x_1I)] \Longrightarrow (a + bI).(x_0 + x_1I) \in z\big(M(I)\big)$$

$$\begin{aligned}[(x_0 + x_1I) \circ (y_0 + y_1I)] \circ (z_0 + z_1I) &= (x_0 + x_1I) \circ [(y_0 + y_1I) \circ (z_0 + z_1I)]\\ &= (x_0 + x_1I) \circ [(z_0 + z_1I) \circ (y_0 + y_1I)] = [(x_0 + x_1I) \circ (z_0 + z_1I)] \circ (y_0 + y_1I)\\ &= (z_0 + z_1I) \circ [(x_0 + x_1I) \circ (y_0 + y_1I)] \Longrightarrow (x_0 + x_1I) \circ (y_0 + y_1I) \in z\big(M(I)\big)\end{aligned}$$

Thus $z\big(M(I)\big)$, is neutrosophic subalgebra from $M(I)$.

Definition:

Let $M(I)$ be a neutrosophic algebra over $R(I)$, and $B(I)$ be a neutrosophic subalgebra where is a subalgebra from M. We define

$$B^{\perp}(I) = \{x + yI \ \in M(I); (x + yI) \circ (z + tI) = 0\ ; \forall\ \ z + tI \in B(I)\}$$

Theorem:

Let be a neutrosophic algebra over $R(I)$, and $B(I)$ be a neutrosophic subalgebra of $M(I)$. Then

1) $B^{\perp}(I)$ is a neutrosophic subalgebra of $M(I)$ if $M(I)$ is associative.

2) $B^{\perp}(I)$ is an ideal of $M(I)$ if $M(I)$ is associative.

3) $\forall x + yI \in M^{\perp}(I) \Longrightarrow (x + yI)^2 = 0$.

Proof:

1) Let $x_0 + x_1I, y_0 + y_1I \in B^{\perp}(I)$ and $z_0 + z_1I \in B(I)$. Then

$$[(x_0 + x_1I) - (y_0 + y_1I)] \circ (z_0 + z_1I) = (x_0 + x_1I) \circ (z_0 + z_1I) - (y_0 + y_1I) \circ (z_0 + z_1I) = 0 \Longrightarrow (x_0 + x_1I) - (y_0 + y_1I) \in B^{\perp}(I)$$

$$\forall\ (a + bI) \in R(I): [(a + bI).(x_0 + x_1I)] \circ (z_0 + z_1I) = (a + bI).[(x_0 + x_1I) \circ (z_0 + z_1I)] = (a + bI).(0) = 0 \Longrightarrow (a + bI).(x_0 + x_1I) \in B^{\perp}(I)$$

$$[(x_0 + x_1I) \circ (y_0 + y_1I)] \circ (z_0 + z_1I) = (x_0 + x_1I) \circ [(y_0 + y_1I) \circ (z_0 + z_1I)] = (x_0 + x_1I) \circ (0) = 0 \Longrightarrow (x_0 + x_1I) \circ (y_0 + y_1I) \in B^{\perp}(I)$$

Therefore, $B^{\perp}(I)$ is a subalgebral from $M(I)$.

2) We found that $B^{\perp}(I)$ is a neutrosophic submodule of $M(I)$, hence

$$[(z_0 + z_1I) \circ (x_0 + x_1I)] \circ (y_0 + y_1I) = (z_0 + z_1I) \circ [(x_0 + x_1I) \circ (y_0 + y_1I)] = (z_0 + z_1I) \circ (0) = 0 \Longrightarrow (z_0 + z_1I) \circ (x_0 + x_1I) \in B^{\perp}(I)$$

Therefore, $B^{\perp}(I)$ is an ideal in $M(I)$.

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3) We have

$$\forall\, x+yI \in M^{\perp}(I) \Longrightarrow (x+yI)^2 = (x+yI)\circ(x+yI) = 0.$$

Homomorphisms of neutrosophic algebras

Let $M(I), N(I)$ be two neutrosophic algebras over the same neutrosophic ring $R(I)$, and let $f: M(I) \longrightarrow N(I)$be a mapping. We call it a homomorphism of neutrosophic algebras, if

1) $f(x+y) = f(x) + f(y)$

2) $f(ax) = af(x)$

3) $f(x\circ y) = f(x)\circ f(y)$ for any $x, y \in M(I)$.

Remark:

If $f: M(I) \longrightarrow N(I)$is a one-to-one (bijective) homomorphism of neutrosophic algebras, then we call f an isomorphism of neutrosophic algebras, and we write $M(I) \cong N(I)$.

Theorem:

Let $f: M(I) \longrightarrow N(I)$ be a homomorphism of neutrosophic algebras, then

1) $f(O) = O$

2) $f(-x) = -f(x)$

3) $ker(f)$ is an ideal of $M(I)$.

4) $Im(f)$ is a subalgebra of $N(I)$

5) If B is a neutrosophic subalgebra of $M(I)$, then $f(B)$ is a neutrosophic subalgebra of $N(I)$.

6) If T is a neutrosophic ideal of $M(I)$, then $f(T)$ is a neutrosophic ideal of $f(M(I))$.

7) If T is an ideal of $f(M(I))$, then $f^{-1}(T)$ is an ideal of $M(I)$.

Proof:

1) For any $x \in M(I)$, we have

$$f(0\circ x) = f(0) = f(0)\circ f(x)$$

Thus, $f(0) = 0$.

2) Follows from the definition.

3) Let $x, y \in ker(f)$, then

$$f(x-y) = f(x) - f(y) = 0 \Longrightarrow x - y \in ker(f)$$

$$\forall\, a \in R(I): f(ax) = af(x) = a(0) = 0 \Longrightarrow ax \in ker(f)$$

$$\forall\, z \in M(I): f(xoz) = f(x)of(z) = (0)of(z) = 0 \Longrightarrow xoz \in ker(f)$$

Thus, $ker(f)$ is an ideal of $M(I)$.

4) Let $x', y' \in Im(f)$, then there exists $x, y \in M(I)$, such that

$f(x) = x', f(y) = y'$ and

$x' - y' = f(x) - f(y) = f(x-y) \in Im(f)$

$x' \circ y' = f(x)\circ f(y) = f(x\circ y) \in Im(f)$

$\forall\, a' \in R(I): a.x' = af(x) = f(ax) \in Im(f)$

Thus, $Im(f)$ is a subalgebra.

Now, since there exists $x \in M$ such that $xI \in M$, then $f(x).I \in Im(f)$. Thus, $Im(f)$is a neutrosophic subalgebra.

5) Let us assume that B is a neutrosophic subalgebra from $M(I)$. Then, there is $a \in M$, such that $aI \in M$. Thus, $f(a).I \in f(B)$ and

$$\forall\, x', y' \in f(B)\ \exists\, x, y \in B; x' = f(x), y' = f(y)$$

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$$x' - y' = f(x-y) \in f(B), x' \circ y' = f(x \circ y) \in f(B), ax' = f(ax) \in f(B)$$

For all $a \in R(I)$. Hence $f(B)$ is neutrosophic subalgebra from $N(I)$.

6) Let us assume that T is a neutrosophic ideal in $M(I)$, then there is $x \in M$, such that $xI \in T$, thus $f(x).I \in f(T)$ and

$$\forall\, x', y' \in f(T)\ \exists\, x, y \in T; x' = f(x), y' = f(y)$$

$$\forall\, b \in f(M(I))\ \exists\, a \in M(I); b = f(a)$$

$$x' - y' = f(x-y) \in f(T), b \circ x' = f(a \circ x) \in f(T)$$

For all $m \in R(I)$. Thus, $mx' = f(mx) \in f(T)$. Hence $f(T)$ is a neutrosophic ideal in $M(I)$.

7) Let us assume that T is a neutrosophic ideal in $f(M(I))$, then there is $b \in f(M)$, such that $bI \in T$, thus there is $a \in M$, such that $f(a) = b$ and $f(aI) = bI \in T$, therefore $aI \in f^{-1}(T)$, and

$$\forall\, x, y \in f^{-1}(T), z \in M(I)\ ;\ f(x), f(y) \in T, f(z) \in f(M(I))$$

and

$$f(x-y) = f(x) - f(y) \in T \Longrightarrow x - y \in f^{-1}(T),$$

$$f(x \circ z) = f(x) \circ f(z) \in T \Longrightarrow x \circ z \in f^{-1}(T),$$

$$f(ax) = af(x) \in T \Longrightarrow ax \in f^{-1}(T)\ \forall\, a \in R(I)$$

Thus $f^{-1}(T)$, is a neutrosophic ideal in $M(I)$.

Theorem:

Let $M(I), N(I), S(I)$ be three neutrosophic algebras over $R(I)$, and let

$f: M(I) \longrightarrow N(I), g: N(I) \longrightarrow S(I)$ be mappings. Then

1) If f, g are homomorphisms then $g \circ f$ is a homomorphism.

2) If f, g are isomorphisms then $g \circ f$ is a isomorphism.

Proof:

At first, we have $g \circ f: M(I) \longrightarrow S(I)$.

1) $\forall\, x, y \in M(I), a \in R(I)$, we have

$$g \circ f(x+y) = g\big(f(x) + f(y)\big) = g \circ f(x) + g \circ f(y),$$

$$g \circ f(ax) = g\big(af(x)\big) = a. g \circ f(x),$$

$$g \circ f(x \circ y) = g\big(f(x) \circ f(y)\big) = g \circ f(x) \circ g \circ f(y),$$

Thus, $g \circ f$ is a homomorphism.

2) Follows from (1) and from the fact that the composition of two bijective mappings is a bijection.

3. Conclusion

In this paper is we have studied for the first time the concept of symbolic neutrosophic algebra defined over a neutrosophic ring R(I), where we derived a strict definition of this concept as an expansion of neutrosophic modules. In addition, we studied some of its elementary properties such as neutrosophic subalgebras, neutrosophic homomorphisms and kernels through many theorems and mathematical proofs.

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