Improvement to the Gradient Projection Method Used to Find the Optimal Solution for Neutrosophic Nonlinear Models Constrained by Equality Constraints

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Abstract

A mathematical model consists of decision variables, a goal function, and constraints. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints. The optimal solution is the vector whose components satisfy all constraints, and at which the function reaches an optimal value (maximum or minimum). Nonlinear programming constitutes an important and fundamental part of operations research and is more comprehensive than linear programming. Its applications have spread across all branches of science, including engineering, physics, chemistry, management, economics, and military fields, among others. Nonlinear programming can also be used in forecasting, estimation, applied statistics, and determining the costs resulting from the production, purchase, and storage of goods. Given this importance, and in order to obtain a more accurate solution that takes into account all the changes that the system under study may be exposed to, we have previously presented a neutrosophic study of nonlinear models and some of the methods used to find the optimal solution. In addition to what we have previously done, in a research we present an improvement to the gradient projection method used to find the optimal solution for nonlinear models constrained by equal constraints, enabling us to obtain the optimal solution in fewer steps. We will then apply it to find the solution. Optimization of nonlinear neutrosophic models.

Keywords: Nonlinear models; Neutrosophic logic; Neutrosophic nonlinear models constrained by equality constraints; Gradient projection method

1. Introduction

Operations research is concerned with the scientific determination of how to achieve the best design and operation of human-machine systems in conditions that usually require the allocation of scarce resources. The essence of this science is centered on the creation and use of models, because models help us deal with an alternative to the real thing without having to deal with the real thing itself, thus avoiding the risks associated with tampering with the real thing and accompanied by loss. This importance of operations research has prompted us to reformulate its methods using the concepts of neutrosophic logic, which is the latest achievement of logical thought in its relentless pursuit to comprehend reality and expresses the mind’s uprising when it collides with the unreasonableness of reality or with the false rationality that covers fossilized systems seeking stability in a world in which the pace of development and change is accelerating. To view the stages of development of this science from its inception until today and the most important research and studies that have been published using the concepts of this logic, which included most sciences, see [1-7]. Among the important methods of operations research is the method of nonlinear programming. A mathematical model is a nonlinear model if any component of the objective function or

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constraints is a nonlinear expression, and the nonlinear expressions may be in both. The region of possible solutions for a nonlinear mathematical model is the set of vectors whose components satisfy all constraints, while the optimal solution is the vector that satisfies all constraints and where the function reaches an optimal value (maximum or minimum). In classical studies, the components of the solution vector have specific values and are proportional to the data collected about the system under study in its current conditions. Any change that occurs Based on these data during the operation of the system, this solution would be inappropriate and would not achieve the ambition of the decision makers responsible for the operation of the system under study. To avoid any loss and provide an optimal solution that suits all conditions, we presented in previous research a study of nonlinear models and some of the methods used to find the optimal solution using the concepts of neutrosophic logic, see [1,6,7]. In this research, we present an improvement to the gradient projection method used to find the optimal solution for nonlinear models restricted by equal constraints that enables us to obtain the optimal solution in fewer steps. Then we will apply it to find the optimal solution for neutrosophic nonlinear models.

2. Discussion

Mathematical example problems rely on constructing mathematical models consisting of an objective function and constraints. These models may be linear, nonlinear..., The objective function is either a maximization function or a minimization function for a given quantity. This quantity depends on a number of decision variables, which may be independent of each other or related to each other through a set of constraints. We obtain the values. Of these variables by solving the mathematical model. In previous research, we presented a neutrosophic study of some methods used to find the optimal solution for constrained and unconstrained nonlinear models, see [1,6,7].

In this research, we present a study of the gradient projection method used to find the optimal solution for nonlinear models constrained by equality constraints. This study is divided into three sections:

Section One: A study of the gradient projection method used to find the optimal solution for nonlinear models constrained by equality constraints, as presented in classical references.

Section Two: In this section, we propose an improvement to this method that helps obtain the optimal solution more accurately and with fewer steps.

Section Three: Using this method to obtain the optimal solution for neutrosophic nonlinear models.

- **Section One: A study of the gradient projection method used to find the optimal solution for nonlinear models constrained by equality constraints as presented in classical references.**

Through the study in references, $[8-11]$ we present the following formulation of the gradient projection method used to find the optimal solution for nonlinear models constrained by equality constraints.

The mathematical model takes the following form:

Find the maximum (minimum) value of the function:

$$y = f(x) \quad (1)$$

Taking into account the following constraints:

$$g_i(x) = b_i \quad ; i = 1,2,\dots,m \quad (2)$$

$$X = (x_1, x_2, \dots, x_n)$$

In this problem, the gradient projection method mentioned in [7] cannot be used, because it is a constrained problem, and changes in the X-ray can lead to unacceptable regions (that do not satisfy one or more of the constraints set (2)), in other words, any change in the solution vector X must satisfy the constraints set (2).

Therefore, for any small transition dx_i, the following must be satisfied at the optimal solution:

$$dg_k = \sum_{i=1}^{n} \left(\frac{\partial g_k}{\partial x_i}\right) dx_i = 0 \quad ; k = 1,2,\dots,m$$

That is, we must look for acceptable changes that make the rate of change of the objective function as large (small) as possible, and give the rate the change in a specific direction is given by the following relation:

$$\frac{dy}{ds} = \sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right) \frac{dx_i}{ds}$$

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Provided that ds satisfy the following relation:

$$ds = \sqrt{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2}$$

Which also satisfies:

$$\frac{dg_k}{ds} = \sum_{i=1}^{n} \left(\frac{\partial g_k}{\partial x_i}\right)\frac{dx_i}{ds} = 0 \quad ; k = 1,2,\dots,m$$

In this formulation, the optimal rates of change $\frac{dx_i}{ds}$ are those that make the following Lagrange function take a stable value:

$$L = \frac{dy}{ds} + \lambda_0\left[1 - \sum_{i=1}^{n} \left(\frac{dx_i}{ds}\right)^2\right] + \sum_{k=1}^{m} \lambda_k\left[\sum_{i=1}^{n} \left(\frac{dg_k}{dx_i}\right)\frac{dx_i}{ds}\right]$$

The necessary conditions are given by:

$$\frac{\partial L}{\partial\left[\frac{dx_i}{ds}\right]} = \frac{\partial y}{\partial x_i} - 2\lambda_0\left(\frac{dx_i}{ds}\right) + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right) = 0 \quad ; \quad i = 1,2,\dots,n \qquad (3)$$

$$\frac{\partial L}{\partial \lambda_0} = \left[1 - \sum_{i=1}^{n} \left(\frac{dx_i}{ds}\right)^2\right] = 0 \qquad (4)$$

$$\frac{\partial L}{\partial \lambda_k} = \left[\sum_{i=1}^{n} \left(\frac{dg_k}{dx_i}\right)\frac{dx_i}{ds}\right] \quad ; k = 1,2,\dots,m \quad (5)$$

From equation (3) we get:

$$\frac{dx_i}{ds} = \frac{1}{2\lambda_0}\left[\frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right] \quad ; \quad i = 1,2,\dots,n \;\; (6)$$

Equation (6) is the rate of change in x_i those results from the largest change in $f(x)$, subject to the equality constraint of m.If we substitute Equation (6) into Equation (5), we get:

$$\sum_{i=1}^{n} \left(\frac{dg_j}{dx_i}\right)\left[\frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right]\frac{1}{2\lambda_0} = 0 \;\Rightarrow$$

$$\sum_{i=1}^{n} \left(\frac{dg_j}{dx_i}\right)\left(\frac{\partial y}{\partial x_i}\right) = -\sum_{i=1}^{n} \frac{dg_j}{dx_i}\sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right) \qquad ; j = 1,2,\dots,m \quad (7)$$

Now if we substitute equation (6) into equation (4) we find:

$$\sum_{i=1}^{n} \left[\frac{1}{2\lambda_0}\left(\frac{\partial y}{\partial x_i}\right) + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right]^2 = 1$$

Or:

$$4\lambda_0^2 = \sum_{i=1}^{n} \left[\left(\frac{\partial y}{\partial x_i}\right) + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right]^2 \Rightarrow$$

$$2\lambda_0 = \pm\sqrt{\sum_{i=1}^{n} \left[\left(\frac{\partial y}{\partial x_i}\right)^2 + \frac{\partial y}{\partial x_i}\sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right]^2} \qquad (8)$$

$2\lambda_0$ is a restricted derivative. When λ_0 becomes very small, we are necessarily close to an optimal solution. This means that as we approach the optimal solution, the positive component of Equation (8) represents the maximum rate of increase of the objective function, and the negative component represents the maximum rate of decrease.

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Our goal is to start from an acceptable rule $x_j^{(\delta)}$ (i.e., one that satisfies all constraints). Using this rule, we construct a new rule $x_j^{(\delta+1)}$ in a way that makes the objective function $y = f(x)$ increase or decrease (depending on the type of nonlinear mathematical model under study), until we reach the desired optimal solution. The new base point is a point at a distance s along the components of the solution vector $(\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ and is written algebraically as:

$$x^{(\delta+1)} = x_j^{(\delta+1)} = x_j^{(\delta)} + sm_j \quad ; j = 1,2,\dots,n$$

Using Equation (3), it becomes:

$$x^{(\delta+1)} = x_j^{(\delta+1)} = x_j^{(\delta)} + s\left\{\frac{1}{2\lambda_0}\left[\frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \left(\frac{dg_k}{dx_i}\right)\right]\right\} \quad ; \; j = 1,2,\dots,n \quad (9)$$

Thus, we obtain an improved solution vector for small transitions s.

The solution steps can be summarized as follows:

1. We define an acceptable initial base point $x^{(0)} = x_j^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right)$ and calculate the value of the objective function $f\left(x^{(0)}\right)$.
2. We assign an arbitrary value to step s.
3. We calculate the partial derivatives of the objective function $y = f(x)$, i.e., $\frac{\partial y}{\partial x_i}$, where $i = 1,2,\dots,n$, and the partial derivatives of the constraint functions $\frac{dg_k}{dx_i}$, where $i = 1,2,\dots,n$ and $k = 1,2,\dots,m$, at the point $x^{(0)} = x_j^{(0)}$.
4. We calculate the Lagrange factors λ_k where $k = 1,2,\dots,m$ from equation (7).
5. We calculate λ_0 from equation (8).
6. If the value of λ_0 in the step becomes zero, this means that the last base point represents the optimal solution. If it is not zero, we move to the next step.
7. We calculate a new acceptable base point $x_j^{(1)}$ using Equation (9).
8. We verify that this point satisfies the constraint. If it is satisfied, we calculate the value of the objective function at that point and compare the result with the previous value.

➢ If it is smaller than that, this means we have achieved an improvement. We continue the procedure from step (3), using point $x_j^{(1)}$ as the base point.

➢ If it is larger than that, we reduce the step size s and use point $x_j^{(1)}$ as the initial base point. We perform the steps starting from step (3).

We repeat the process until we obtain the optimal solution.

We illustrate the above through the following example:

Example 1:

Find the minimum value of the function:

$$f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$

Taking into account the constraint:

$$g(x) = 2x_1 + x_2 = 3$$

Stage One: We randomly choose a valid initial solution point $x^{(0)}$ and a step size s.

1. The constraint is a linear function that represents a straight line graphically. Therefore, we have an infinite number of valid solutions, which are all the points on the line. We choose a starting point, for example, the point

$$x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$$

2. We take a step size $s = 2$.

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3. We calculate the value of the objective function $f\left(x^{(0)}\right)$.

$$f\left(x^{(0)}\right) = 13$$

4. We calculate the value of the partial derivatives of the objective function and the constraint function at the point $x^{(0)}$.

$$\left(\frac{\partial f}{\partial x_1}\right)_{x^{(0)}} = -4 \quad , \quad \left(\frac{\partial f}{\partial x_2}\right)_{x^{(0)}} = -6$$

$$\left(\frac{\partial g}{\partial x_1}\right)_{x^{(0)}} = 2 \quad , \quad \left(\frac{\partial g}{\partial x_2}\right)_{x^{(0)}} = 1$$

5. We calculate the Lagrange factors λ_k using Equation(7).

$$2(-4) + 1(-6) = -\{(2)\lambda(2) + (1)\lambda(1)\}$$

We obtain the value of λ in the first step:

$$\lambda_1 = \frac{14}{5}$$

6. We calculate $\lambda_{0(x^{(0)})}$ from Equation(8).

$$4\lambda^2_{0(x^{(0)})} = \left\{(-4)^2 + (-4)\left(\frac{14}{5}\right)(2)\right\}^2 + \left\{(-6)^2 + (-6)\left(\frac{14}{5}\right)(1)\right\}^2 = 409.6$$

$$2\lambda_{0(1)} = \pm 20.24$$

➢ $\lambda_{0(x^{(0)})} \neq 0$ This means that the accepted solution is not optimal. Since we are searching for the minimum value of the objective function, we take:

$$2\lambda_{0(x^{(0)})} = -20.24$$

7. We calculate a new accepted base point $x_j^{(1)}$ using Equation(9).

$$x_1^{(1)} = 1 + 2\left\{\frac{1}{-20.24}\left[-4 + \frac{14}{5}(2)\right]\right\} = 0.84$$

$$x_2^{(1)} = 1 + 2\left\{\frac{1}{-20.24}\left[-6 + \frac{14}{5}(1)\right]\right\} = 1.32$$

We find:

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.84{,}1.32)$$

➢ Substitute into the constraint:

$$2(0.84) + 1.32 = 1.68 + 1.32 = 3$$

The constraint is satisfied.

➢ Substitute the objective function:

$$f\left(x^{(1)}\right) = 11.848$$

Note that:

$11.848 < 13$

i.e.:

$$f\left(x^{(1)}\right) < f\left(x^{(0)}\right)$$

➢ Since $\lambda_{0(x^{(0)}} \neq 0$, this solution is not optimal. We continue searching for the optimal solution using the new solution:

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.84{,}1.32)$$

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Stage Two: We start this stage from step four:

➢ We calculate $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ at the point $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.84,1.32)$ We find:

$$\left(\frac{\partial f}{\partial x_1}\right)_{x^{(1)}} = -4.32 \quad , \quad \left(\frac{\partial f}{\partial x_2}\right)_{x^{(1)}} = -5.36$$

$$\left(\frac{\partial g}{\partial x_1}\right)_{x^{(1)}} = 2 \quad , \quad \left(\frac{\partial g}{\partial x_2}\right)_{x^{(1)}} = 1$$

➢ We calculate λ:

$$(-4.32\ ) + 1(-5.36) = -\{(2)\lambda(2) + (1)\lambda(1)\}$$

$$\lambda_2 = \frac{14}{5}$$

➢ We calculate $\lambda_{0(x^{(1)})}$:

$$4\lambda_{0(x^{(1)})}^2 = \left\{(-4.32\ )^2 + (-4.32\ )\left(\frac{14}{5}\right)(2)\right\}^2 + \left\{(-5.36)^2 + (-5.36)\left(\frac{14}{5}\right)(1)\right\}^2 = 218.86$$

$$2\lambda_{0(x^{(1)})} = -14.79$$

➢ Calculate a new acceptable base point: $s = 2$

$$x_1^{(2)} = 0.84 + \ 2\left\{\frac{1}{-14.79}\left[-4.32\ + \left(\frac{14}{5}\right)(2)\right]\right\} = 0.67$$

$$x_2^{(2)} = 1.32 + \ 2\left\{\frac{1}{-14.79}\left[-5.36 + \left(\frac{14}{5}\right)(1)\right]\right\} = 1.67$$

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.67,1.67)$$

➢ Substitute into the constraint:

$$2(0.67) + 1.67 = 3$$

The constraint is satisfied.

➢ Substitute into the objective function:

$$f\left(x^{(2)}\right) = (0.67 - 3)^2 + (1.67 - 4)^2 = 10.86$$

$$f\left(x^{(2)}\right) = 10.86$$

Note that: $10.895 < 11.848$

i.e.:

$$f\left(x^{(2)}\right) < f\left(x^{(1)}\right)$$

➢ Since $\lambda_{0(x^{(1)})} \neq 0$, this solution is not optimal. We continue searching for the optimal solution using the new solution:

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.67,1.67)$$

We repeat what we did in the second stage and stop when the resulting values do not improve the objective function.

Stage third: We take $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.67,1.67)$, and $s = 2$, we find:

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$$\lambda_3 = \frac{14}{5}$$

$$2\lambda_{0(x^{(2)})} = -9.7$$

$$x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.48, 2.05)$$

$$f\left(x^{(3)}\right) = 10.15$$

$$f\left(x^{(3)}\right) < f\left(x^{(2)}\right)$$

➢ Since $\lambda_{0(x^{(2)})} \neq 0$, this solution is not optimal.

We continue searching for the optimal solution using the new solution:

$$x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.48, 2.05)$$

Stage Four: We take $x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.48, 2.05)$ and $s = 2$, we find:

$$\lambda_4 = \frac{14}{5}$$

$$2\lambda_{0(x^{(3)})} = -5.14$$

$$x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.26, 2.48)$$

$$f\left(x^{(4)}\right) = 9.82$$

$$f\left(x^{(4)}\right) < f\left(x^{(3)}\right)$$

➢ Since $\lambda_{0(x^{(3)})} \neq 0$, this solution is not optimal.

We continue searching for the optimal solution using the new solution:

$$x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.26, 2.48)$$

Stage Five: We take $x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.26, 2.48)$ and $s = 2$, we find:

$$\lambda_5 = \frac{14}{5}$$

$$2\lambda_{0(x^{(4)})} = -0.98$$

$$x^{(5)} = \left(x_1^{(5)}, x_2^{(5)}\right) = (0.02, 3)$$

$$f\left(x^{(5)}\right) = 9.88$$

➢ We notice that at this point we have not obtained an improvement for the objective function in the fifth stage because we are looking for a minimization of the objective function and the value, we obtained in this stage is greater than the value we obtained in the previous stage:

$$f\left(x^{(5)}\right) > f\left(x^{(4)}\right)$$

To continue searching for the optimal solution, we take the solution resulting from the fourth step:

$$x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.26, 2.48)$$

We consider it an initial solution, then reduce the step size and continue working as we did before.

First attempt: Taking the step size $s = 1.5$ instead of $s = 2$, we get:

$s = 1.5$ and the initial solution $x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.26, 2.48)$.

$$\frac{\partial f}{\partial x_1} = 2(0.26 - 3) \qquad \frac{\partial f}{\partial x_2} = 2(2.48 - 4)$$

$$\left(\frac{\partial f}{\partial x_1}\right)_{x^{(4)}} = -5.48 \quad , \quad \left(\frac{\partial f}{\partial x_2}\right)_{x^{(4)}} = -3.04$$

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$$\left(\frac{\partial g}{\partial x_1}\right)_{x^{(4)}} = 2 \quad , \quad \left(\frac{\partial g}{\partial x_2}\right)_{x^{(4)}} = 1$$

$$2(-5.48) + 1(-3.04) = -\{(2)\lambda(2) + (1)\lambda(1)\}$$

$$\lambda = \frac{14}{5}$$

$$4\lambda_0^2 = \left\{(-5.48\ )^2 + (-5.48\ )\left(\frac{14}{5}\right)(2)\right\}^2 + \left\{(-3.04)^2 + (-3.04)\left(\frac{14}{5}\right)(1)\right\}^2 = 1.17$$

$$2\lambda_{0(x^{(4)})} = -1.08$$

$$x_j^{(\delta+1)} = x_j^{(\delta)} + s\left\{\frac{1}{2\lambda_0}\left[\frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k\left(\frac{dg_k}{dx_i}\right)\right]\right\} \quad ; \ j = 1,2,\dots,n \quad (11)$$

$$x_1^{(5)} = 0.48 + 1.5\left\{\frac{1}{-1.08}\left[-5.04 + \left(\frac{14}{5}\right)(2)\right]\right\} = 0.30$$

$$x_2^{(5)} = 2.04 + 1.5\left\{\frac{1}{-1.08}\left[-3.92 + \left(\frac{14}{5}\right)(1)\right]\right\} = 2.04$$

$$x^{(5)} = \left(x_1^{(5)}, x_2^{(5)}\right) = (0.30,2.04)$$

- We compensate in the restriction:

$$(x) = 2x_1 + x_2 = 32(0.30) + 2.04 = 2.64$$

The restriction is not verified.

Second attempt: Taking the step size $s = 1$,we do the necessary calculations:

$$\lambda = \frac{14}{5}$$

$$2\lambda_{0(x^{(4)})} = -1.08$$

$$x^{(5)} = \left(x_1^{(5)}, x_2^{(5)}\right) = (0.03,3.08)$$

- **We compensate in the restriction:**

$$g(x) = 2x_1 + x_2 = 32(0.30) + 3.08 = 3.68$$

The restriction is not verified.

We repeat these attempts and choose a step size less than $s = 2$ until we get the optimal solution.

We summarize the previous study in the following table:

starting point: $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$,and $s = 2$, we get: $f\left(x^{(0)}\right) = 13$

Table 1: Results of the previous study

Results stage	s	λ_k	$2\lambda_0$	$x_j^{(\delta+1)}$	$f\left(x_j^{(\delta+1)}\right)$
Stage One	2	$\frac{14}{5}$	-20.24	(0.84,1.32)	11.85
Stage Two	2	$\frac{14}{5}$	-14.79	(0.67,1.67)	10.86
Stage third	2	$\frac{14}{5}$	-9.7	(0.48,2.05)	10.15
Stage Four	2	$\frac{14}{5}$	-5.14	(0.26,2.48)	9.82
Stage Five	2	$\frac{14}{5}$	-0.98	(0.02,3)	9.88

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Attempts to reduce step size					
First attempt	1.5	$\frac{14}{5}$	-1.08	$(0.30,2.04)$ **The point does not meet the constraint**	$---$
Second attempt	1	$\frac{14}{5}$	-1.08	$(0.03,3.08)$ **The point does not meet the constraint**	$---$

- **Section Two: In it, we propose an improvement to this method that helps in obtaining the optimal solution more accurately and with fewer steps.**

From the study presented in the first section, we note that determining the step size s appropriately helps in quickly obtaining the optimal solution. We can obtain the step size s using the study presented in [7]:

The direction of greatest descent for vehicle i in the case of the target function being a minimization function is given by the following relation:

$$m_i = \frac{-\frac{\partial y}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial y}{\partial x_i}\right)^2}} \qquad (1)$$

1. We take an initial solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$
2. We calculate m_i from equation (1).
3. We substitute $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ and m_i into the equation:

$$x_{j+1}^{(1)} = x_j^{(0)} + m_i s$$

4. We substitute $x_{j+1}^{(1)}$ for the objective function expression, and we obtain a new variable function, s.
5. We search for the minimum value of this function.
6. We substitute this minimum value into the relation:

$$x_{j+1}^{(1)} = x_j^{(1)} + m_i s$$

We obtain a new point $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$. We substitute it into the objective function expression

$y = f(x)$. If the result is equal to zero, the point $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$ is a stable point. We determine its type by testing the convexity and concavity of the function. See [7].

Finding the optimal solution to a constrained problem:

1. We replace the coordinates of the point $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$ with the constraint.
2. If the constraint is satisfied, this point is an acceptable solution to the constrained problem.
3. If $\lambda_{0(x^{(0)})} = 0$, this solution is also an optimal solution to the constrained problem.
4. If $\lambda_{0(x^{(0)})} \neq 0$, this solution is acceptable but not optimal. We use it as an initial solution and apply the steps mentioned in the first section of the research to obtain the optimal solution.
5. If the constraint is not satisfied, we follow the study mentioned in the first section and take the step size s as the minimum value we obtained or any value close to it.

We illustrate the above through the following example:

Find the minimum value of the function:

$$f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$

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Taking into account the constraint:

$$g(x) = 2x_1 + x_2 = 3$$

1. We study the problem without a constraint to determine the step size s:

☐ We calculate the partial derivatives of the objective function and the constraint function.

$$\frac{\partial f}{\partial x_1} = 2(x_1 - 3) \qquad \frac{\partial f}{\partial x_2} = 2(x_2 - 4)$$

$$\frac{\partial y}{\partial x_1}|_{(1,1)} = -4 \qquad \frac{\partial y}{\partial x_2}|_{(1,1)} = -6$$

$$m_1 = \frac{4}{\sqrt{52}} \qquad m_2 = \frac{6}{\sqrt{52}}$$

☐ We substitute in the following relation:

$$x_{j+1}^{(i)} = x_j^{(i)} + \left[\frac{\partial y}{\partial x_i} . \frac{1}{2\lambda}\right] s = x_j^{(i)} + m_i s$$

$$x_1^{(1)} = 1 + s\left(\frac{4}{\sqrt{52}}\right)$$

$$x_2^{(1)} = 1 + s\left(\frac{6}{\sqrt{52}}\right)$$

☐ The following quantity is required to be minimized with respect to s:

$$(x) = \left(1 + \frac{4}{\sqrt{52}}s - 3\right)^2 + \left(1 + \frac{6}{\sqrt{52}}s - 4\right)^2$$

$$f(x) = \left(\frac{4}{\sqrt{52}}s - 2\right)^2 + \left(\frac{6}{\sqrt{52}}s - 3\right)^2$$

$$\frac{\partial y}{\partial s} = 2s - \sqrt{52} = 0 \Rightarrow s = \frac{\sqrt{52}}{2}$$

$$\Rightarrow s = \frac{\sqrt{52}}{2} = 3.61$$

$$x_1^{(1)} = 3 \qquad x_2^{(1)} = 4$$

$$x^{(1)} = (3,4)$$

$$y|_{x^{(1)}} = 0$$

$x^{(1)}$ is a stable point to determine whether it is a minimum value. We know that:

☐ The Hessian matrix for the given function is:

$$H = [2\ 0\ 0\ 2\ ]$$

The elements of the main diagonal are positive and the main principal minor determinants are positive because:

$$|2| > 0$$

$$|2\ 0\ 0\ 2\ | = 4 > 0$$

If the Hessian matrix is defined positive for the point where the partial derivatives are zero, then the point $(3,4)$ is a local minimum limit point of the function f.

☐ We replace the coordinates of this point with the constraint:

$$2x_1 + x_2 = 2 \times 3 + 4 = 10 \neq 3$$

☐ The constraint is not satisfied. This point is not an acceptable solution to the problem

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2. **We search for the optimal solution to the constrained problem, we take a value close to the minimum value of s that we obtained, let this value be $s = 3.65$.**

Example 2:

Find the minimum value of the function:

$$f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$

Taking into account the constraint:

$$g(x) = 2x_1 + x_2 = 3$$

Stage first: Initial acceptable solution $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$ and $s = 3.65$.

➢ We substitute $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$ into the objective function:

$$f\left(x^{(0)}\right) = 13$$

➢ We calculate the partial derivatives of the objective function and the constraint function.

$$\frac{\partial f}{\partial x_1} = 2(x_1 - 3) \qquad \frac{\partial f}{\partial x_2} = 2(x_2 - 4)$$

$$\frac{\partial g}{\partial x_1} = 2 \qquad \frac{\partial g}{\partial x_2} = 1$$

➢ We substitute $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$:

$$\left(\frac{\partial f}{\partial x_1}\right)_{x^{(0)}} = -4 \quad , \quad \left(\frac{\partial f}{\partial x_2}\right)_{x^{(0)}} = -6$$

$$\left(\frac{\partial g}{\partial x_1}\right)_{x^{(0)}} = 2 \quad , \quad \left(\frac{\partial g}{\partial x_2}\right)_{x^{(0)}} = 1$$

➢ To calculate λ_1, we substitute into the equation:

$$\sum_{i=1}^{n} \left(\frac{\partial g_j}{\partial x_i}\right)\left(\frac{\partial y}{\partial x_i}\right) = -\sum_{i=1}^{n} \frac{\partial g_j}{\partial x_i} \sum_{k=1}^{m} \lambda_k \left(\frac{\partial g_k}{\partial x_i}\right) \quad ; \quad j = 1,2,\dots,m$$

$$2(-4) + 1(-6) = -\{(2)\lambda(2) + (1)\lambda(1)\}$$

$$\lambda_1 = \frac{14}{5}$$

☐ To calculate $2\lambda_{0(\,x^{(0)})}$, we substitute into the equation:

$$4\lambda_0^2 = \sum_{i=1}^{n} \left[\left(\frac{\partial y}{\partial x_i}\right)^2 + \frac{\partial y}{\partial x_i} \sum_{k=1}^{m} \lambda_k \left(\frac{\partial g_k}{\partial x_i}\right)\right]^2$$

$$4\lambda_0^2 = \left\{(-4)^2 + (-4)\left(\frac{14}{5}\right)(2)\right\}^2 + \left\{(-6)^2 + (-6)\left(\frac{14}{5}\right)(1)\right\}^2 = 409.6$$

$$2\lambda_{0(x^{(0)})} = \pm 20.24$$

Since we are looking for the minimum value of the objective function, we take the negative component, i.e.:

$$2\lambda_{0(x^{(0)})} = -20.24$$

☐ To calculate the coordinates of the new point, we substitute the following relation:

$$x_j^{(\delta+1)} = x_j^{(\delta)} + s\left\{\frac{1}{2\lambda_0}\left[\frac{\partial y}{\partial x_i} + \sum_{k=1}^{m} \lambda_k \left(\frac{\partial g_k}{\partial x_i}\right)\right]\right\} \; ; j = 1,2,\dots,n$$

$$x_1^{(2)} = 1 + 3.65\left\{\frac{1}{-20.24}\left[-4 + \left(\frac{14}{5}\right)(2)\right]\right\} = 0.71$$

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$$x_2^{(2)} = 1 + 3.65\left\{\frac{1}{-20.24}\left[-6 + \left(\frac{14}{5}\right)(1)\right]\right\} = 1.58$$

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.71, 1.58)$$

- We replace this point with the restriction:

$$2(0.71) + 1.58 = 3$$

The constraint is satisfied.

- We replace it with the objective function:

$$f\left(x^{(1)}\right) = (0.71 - 3)^2 + (1.58 - 4)^2 = 11.1$$

$$f\left(x^{(1)}\right) = 11.1$$

$$f\left(x^{(1)}\right) < f\left(x^{(0)}\right)$$

- Since $\lambda_{0\left(x^{(0)}\right)} \neq 0$, this solution is not optimal. We continue searching for the optimal solution using the new solution:

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.71, 1.58)$$

Stage second: We take $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) = (0.71, 1.58)$ and $s = 3.65$, we find:

$$\lambda = \frac{14}{5}$$

$$2\lambda_{0\left(x^{(1)}\right)} = -10.93$$

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.37, 2.26)$$

$$f\left(x^{(2)}\right) = 9.95$$

$$f\left(x^{(2)}\right) < f\left(x^{(1)}\right)$$

$$f\left(x^{(1)}\right) < f\left(x^{(0)}\right)$$

- Since $\lambda_{0\left(x^{(1)}\right)} \neq 0$, this solution is not optimal. We continue searching for the optimal solution using the new solution:

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.37, 2.26)$$

Stage third: We take $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.37, 2.26)$ and $s = 3.65$, we find:

$$\lambda = \frac{14}{5}$$

$$2\lambda_{0\left(x^{(2)}\right)} = -2.97$$

$$x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (-0.05, 3.1)$$

$$f\left(x^{(3)}\right) = 10.11$$

$$f\left(x^{(3)}\right) > f\left(x^{(2)}\right)$$

- We note that at this point, we have not obtained an improvement to the objective function in the third stage, because we are looking to minimize the objective function, and the value we obtained in this stage is greater than the value we obtained in the previous stage:

To continue searching for the optimal solution, we take the solution resulting from the second stage:

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.37, 2.26)$$

We consider this point a preliminary solution, reduce the step size, and continue working as before. The problem becomes

First attempt:

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Stage One:

Taking the step size $s = 1.5$ instead, and the initial rule is the point $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (0.37,2.26)$ we find:

$$\lambda = \frac{14}{5}$$

$$2\lambda_{0(x^{(2)})} = -2.97$$

$$x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.2,2.6)$$

$$f\left(x^{(3)}\right) = 9.8$$

$$f\left(x^{(3)}\right) < f\left(x^{(2)}\right)$$

- ☐ Since $\lambda_{0(x^{(2)})} \neq 0$, this solution is not optimal. We continue searching for the optimal solution using the new solution:

$$x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.2,2.6)$$

Stage Two:

Taking the step size $s = 1.5$ instead, and the initial rule is the point $x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}\right) = (0.2,2.6)$ we find:

$$\lambda = \frac{14}{5}$$

$$\lambda_{0(x^{(3)})} = 0$$

$$x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.2,2.6)$$

$$f\left(x^{(4)}\right) = 9.8$$

- ☐ **Since the value of $\lambda_{0(x^{(3)})} = 0$, this means that the base point $x^{(4)} = \left(x_1^{(4)}, x_2^{(4)}\right) = (0.2,2.6)$represents the optimal solution.**

$$f\left(x^{(4)}\right) = 9.8$$

We summarize the previous study in the following table:

starting point: $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$,and $s = 3.65$, we get: $f\left(x^{(0)}\right) = 13$

Table 2: Results of the previous study

Results stage	s	λ_k	$2\lambda_0$	$x_j^{(\delta+1)}$	$f\left(x_j^{(\delta+1)}\right)$
Stage One	3.65	$\frac{14}{5}$	-20.24	(0.71,1.58)	11.1
Stage Two	3.65	$\frac{14}{5}$	-10.93	(0.37,2.26)	9.95
Stage third	3.65	$\frac{14}{5}$	-2.97	$(-0.05,3.1)$	10.11
First attempt after changing the step size s					
Stage One	1.5	$\frac{14}{5}$	-2.97	(0.2,2.6)	9.8
Stage Two	1.5	$\frac{14}{5}$	0	(0.2,2.6)	9.8

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- **Section Three: Finding the optimal solution for nonlinear neutrosophic models constrained by equality constraints using the gradient projection method**.

The Neutrosophic Mathematical Model: [1]

In an example problem where the objective and constraints are in the form of neutrosophic mathematical functions, the neutrosophic mathematical model is written in the following form:

$$f_N = f_N(x_1, x_2, --, x_n) \rightarrow (Max)or\ (Min)$$

Subject to the following constraints:

$$g_{iN}(x_1, x_2, --, x_n)(\leq \geq =)b_{iN} \quad ; i = 1,2, ---, m$$

$$x_1, x_2, --, x_n \geq 0$$

In this model, the variables in the objective function and constraints, as well as the second side of the relationships representing the constraints (at least one of which), are neutrosophic values. This model is a nonlinear model if any component of the objective function or constraints is a nonlinear expression, and nonlinear expressions may be present in both.

We demonstrate how to use the gradient projection method through the following example:

Example 3:

Find the minimum value of the function:

$$f(x) = (x_1 - 3)^2 + (x_2 - 4)^2$$

Taking into account the constraint:

$$g(x) = 2x_1 + x_2 = [2,4]$$

Stage first:

1. We take the step size as a neutrosophic value:

$$s_N \in [1.5,3.65]$$

2. We take an acceptable initial solution:

$$x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}\right) = (1,1)$$

3. We calculate the value of the objective function and find:

$$f\left(x^{(0)}\right) = 13$$

4. We calculate the partial derivatives of the objective function and the constraint function at the initial point as follows:

$$\left(\frac{\partial f}{\partial x_1}\right)_{x^{(0)}} = -4 \quad , \quad \left(\frac{\partial f}{\partial x_2}\right)_{x^{(0)}} = -6$$

$$\left(\frac{\partial g}{\partial x_1}\right)_{x^{(0)}} = 2 \quad , \quad \left(\frac{\partial g}{\partial x_2}\right)_{x^{(0)}} = 1$$

5. We calculate λ_k from the equation(7):

$$2(-4) + 1(-6) = -\{(2)\lambda(2) + (1)\lambda(1)\}$$

$$\lambda_1 = \frac{14}{5}$$

6. We calculate λ_0 from the equation (8):

$$4\lambda_0^2 = \left\{(-4)^2 + (-4)\left(\frac{14}{5}\right)(2)\right\}^2 + \left\{(-6)^2 + (-6)\left(\frac{14}{5}\right)(1)\right\}^2 = 409.6$$

$$2\lambda_{0(x^{(0)})} = \pm 20.24$$

Since we are looking for the minimum value of the objective function, we take the negative component, i.e.:

$$2\lambda_{0(x^{(0)})} = -20.24$$

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7. We calculate the coordinates of the new point from the relation (9):

$$x_{1N}^{(1)} = 1 + [2,3.65]\left\{\frac{1}{-20.24}\left[-4 + \left(\frac{14}{5}\right)(2)\right]\right\} \in [0.71,0.84]$$

$$x_{2N}^{(1)} = 1 + [2,3.65]\left\{\frac{1}{-20.24}\left[-6 + \left(\frac{14}{5}\right)(1)\right]\right\} \in [1.31,1.58]$$

$$x_N^{(1)} \in ([0.71,0.84],[1.31,1.58])$$

8. Substitute into the constraint:

$$2([0.71,0.84]) + [1.31,1.58] = [2.73,3.26] \subseteq [2,4]$$

The constraint is satisfied.

9. Substitute into the objective function:

$$f_N\left(x_N^{(1)}\right) = ([0.71,0.84] - 3)^2 + ([1.31,1.58] - 4)^2 \in [10.52,12.48]$$

$$f\left(x_N^{(1)}\right) \in [10.52,12.48]$$

10. Compare the current value of the objective function with the previous value of the objective function:

$$[10.52,12.48] \subseteq [13,13]$$

$$f\left(x_N^{(1)}\right) < f\left(x^{(0)}\right)$$

Stage second:

We notice that we have improved the value of the objective function. We continue the solution by taking the new point as an acceptable solution and repeating the previous steps. We get:

$$\lambda_{2N} \in \left[\frac{3.12}{5},\frac{14.54}{5}\right]$$

$$2\lambda_{0N\left(x^{(1)}\right)} \in (-[9.22,14.45])$$

$$x_N^{(1)} = ([0.54,0.53],[1.71,2.20])$$

$$f\left(x_N^{(2)}\right) \in [9.34,11.3]$$

$$f\left(x_N^{(2)}\right) < f\left(x_N^{(1)}\right)$$

Stage third:

We notice that we have improved the value of the objective function. We continue the solution by taking the new point as an acceptable solution and repeating the previous steps. We get:

$$\lambda_{2N} \in \left[\frac{3.44}{5},\frac{14.46}{5}\right]$$

$$2\lambda_{0N\left(x^{(2)}\right)} \in (-[3.28,8.78])$$

$$x_N^{(3)} = ([0.26,0.18],[2.28,2.90])$$

$$f\left(x_N^{(3)}\right) \in [9.16,10.47]$$

$$f\left(x_N^{(3)}\right) < f\left(x_N^{(2)}\right)$$

Stage Four:

We notice that we have improved the value of the objective function. We continue the solution by taking the new point as an acceptable solution and repeating the previous steps. We get:

$$\lambda_{3N} \in \left[\frac{13.16}{5},\frac{14.72}{5}\right]$$

$$2\lambda_{0N\left(x^{(3)}\right)} \in (-[1.51,2.21])$$

$$x_N^{(4)} = ([-0.23,0.55],[1.71,3.72])$$

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We substitute the constraint:

$$2([-0.23,0.55]) + [1.71,3.72] = [1.25,2.62] \nsubseteq [2,4]$$

The constraint is not satisfied.

To continue searching for the optimal solution, we reduce the step size and substitute into the relation:

Since $2\lambda_{0N(x^{(3)})} = -[1.51,2.21]$, i.e., λ_{0N} is a small value (close to zero), this means that the previous solution is very close to the optimal solution, so we take a small step size, for example, $s_N \in [0.01,0.5]$

We calculate the coordinates of the new point and find:

$$x_N^{(4)} = ([0.26,0.33],[2.28,3.01])$$

Substitute the constraint:

$$2([0.26,0.33]) + [2.28,3.01] = [2.8,3.67] \subseteq [2,4]$$

The constraint is satisfied:

Substitute the objective function:

$$f\left(x_N^{(4)}\right) = ([0.26,0.33] - 3)^2 + ([2.28,3.01] - 4)^2 \in [7.9,10.47]$$

$$f\left(x_N^{(4)}\right) \in [7.9,10.47]$$

$$f\left(x_N^{(4)}\right) < f\left(x_N^{(3)}\right)$$

We note that we have improved the value of the objective function because:

$$[7.9,10.47] \subseteq [9.16,10.47]$$

Since the upper bound of the two intervals is the same, we can consider the points we obtained as an optimal solution point, and the required minimum value is:

$$f\left(x_N^{(4)}\right) \in [7.9,10.47]$$

At the point:

$$x_N^{(4)} = ([0.26,0.33],[2.28,3.01])$$

We summarize the previous study in the following table:

Table 3: Results of the previous study

Results stage	s_N	λ_{kN}	$2\lambda_{0N}$	$x_N^{(\delta+1)}$	$f\left(x_N^{(\delta+1)}\right)$
Stage One	$[1.5,3.65]$	$\frac{14}{5}$	-20.24	$([0.71,0.84],[1.31,1.58])$	$[10.52,12.48]$
Stage Two	$[1.5,3.65]$	$\left[\frac{3.12}{5},\frac{14.54}{5}\right]$	$-[9.22,14.45]$	$([0.54,0.53],[1.71,2.20])$	$[9.34,11.3]$
Stage third	$[1.5,3.65]$	$\left[\frac{3.44}{5},\frac{14.46}{5}\right]$	$-[3.28,8.78]$	$([0.26,0.18],[2.28,2.90])$	$[9.16,10.47]$
Stage Four	$[1.5,3.65]$	$\left[\frac{13.16}{5},\frac{14.72}{5}\right]$	$-[1.51,2.21]$	$([-0.23,0.55],[1.71,3.72])$	$---$
Attempts to reduce step size					
First attempt	$[0.01,0.5]$	$\left[\frac{13.16}{5},\frac{14.72}{5}\right]$	$-[1.51,2.21]$	$([0.26,0.33],[2.28,3.01])$	$[7.9,10.47]$
Second attempt	$[0.01,0.5]$	$\left[\frac{12.94}{5},\frac{14,12}{5}\right]$	$[2.06,2.68]$	$([0.26,0.27],[2.28,3.12])$	$[8.23,10.47]$

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We notice that we did not get an improvement in the value of the objective function in the second attempt (after reducing the step size). If we wanted to continue, this would require reducing the step size again. Since the step size we used in the last calculations is small enough and the upper limit for the two periods is the same, we can consider the points we obtained in the first improvement as an optimal solution point, and the minimum value required is:

$$f\left(x_N^{(4)}\right) \in [7.9,10.47]$$

At the point:

$$x_N^{(4)} = ([0.26,0.27],[2.28,3.12])$$

3. Conclusion and Results

In this research, we presented a study of the gradient projection method used to find the optimal solution for nonlinear models constrained by equal constraints. What is new in this study is the proposed improvement of this method. By comparing the results shown in the first and second tables, we find that we obtained the optimal solution through the proposed method with a number of steps less than the number we did in the method before the improvement, through which we did not obtain the optimal solution because the matter requires a large number of attempts. As for the neutrosophic study presented in the third section, we obtained more comprehensive results through it. In addition, the minimum range $[7.9,10.47]$, which expresses the values of the objective function at the optimal solution point, is smaller than the optimal value (9.8) for the objective function that we obtained in the method after the improvement.

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