



On the Formal Foundations of D-Off Numbers and Neutrosophic D-Numbers

Takaaki Fujita^{1,*}, Arif Mehmood², Arkan A. Ghaib³

¹Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

²Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan

³Department of Information Technology, Management Technical College, Southern Technical University, Basrah, 61004, Iraq

Emails: Takaaki.fujita060@gmail.com; mehdaniyal@gmail.com; arkan.ghaib@stu.edu.iq

Abstract

A variety of uncertainty-handling frameworks—such as Fuzzy Sets,¹ Hyperfuzzy Sets,² Bipolar Fuzzy Sets,³ Neutrosophic Sets,⁴ Vague Set,⁵ Hesitant Fuzzy Sets,^{6,7} Picture Fuzzy Sets,⁸ Soft Sets,^{9,10} Rough Sets,¹¹ and Plithogenic Sets^{12,13}—have been extensively studied for modeling and reasoning under vagueness and imprecision. A fuzzy set extends classical set theory by assigning each element a membership value in the unit interval $[0, 1]$, thereby capturing partial inclusion.¹ Neutrosophic Sets further generalize this idea by introducing three independent membership functions—truth, indeterminacy, and falsity—each mapping into $[0, 1]$. Many of these frameworks have been enriched by incorporating *offset* concepts, which permit membership degrees to take values beyond the unit interval. Similarly, D-numbers extend Dempster–Shafer belief functions by assigning to each subset $B \subseteq X$ a mass $D(B) \in [0, 1]$ with $\sum_B D(B) \leq 1$, thus accommodating incomplete uncertainty.¹⁴ In this work, we introduce and formally define four new constructs: *D-OffNumber*, *D-OverNumber*, *D-UnderNumber*, and *Neutrosophic D-Number*, and we investigate their mathematical foundations, structural properties, and interrelationships. The present study focuses exclusively on theoretical development, leaving potential applications—such as their integration into decision-making frameworks—for future research.

Keywords: Fuzzy Offset; Neutrosophic OffSet; Fuzzy Set; Neutrosophic Set; D-Number; Neutrosophic D-Number

1 Preliminaries

This section outlines the key concepts and definitions necessary for the discussions in this paper. The numbers considered in this paper are assumed to be finite.

1.1 Fuzzy Set and Neutrosophic Set

A fuzzy set extends classical set theory by assigning each element a membership value in the interval $[0, 1]$, reflecting degrees of partial or uncertain inclusion.^{1,15} A *Neutrosophic Set* further generalizes this by introducing three independent membership functions—truth, indeterminacy, and falsity—each valued in $[0, 1]$.¹⁶

Definition 1.1 (Fuzzy set). ¹ A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \rightarrow [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y , then δ is called a *fuzzy relation on τ* if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 1.2 (Fuzzy Set in Product Quality Evaluation). Let $Y = \{\text{Product A, Product B, Product C}\}$ be a set of products evaluated for overall quality. Define a fuzzy set τ on Y where $\tau(y)$ represents the degree of quality satisfaction of product y , measured on a scale from 0 (completely unsatisfactory) to 1 (perfect quality). Suppose the evaluations are:

$$\tau = \{(\text{Product A}, 0.85), (\text{Product B}, 0.65), (\text{Product C}, 0.40)\}.$$

Here, Product A has a high satisfaction degree (0.85), indicating good quality; Product B has a moderate satisfaction degree (0.65); Product C has a low satisfaction degree (0.40), indicating that it fails to meet several quality criteria.

Definition 1.3 (Neutrosophic Set). ¹⁷ Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \rightarrow [0, 1], \quad I_A : X \rightarrow [0, 1], \quad F_A : X \rightarrow [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.4 (Neutrosophic Set in Medical Diagnosis). Let $X = \{\text{Patient A, Patient B, Patient C}\}$ represent three patients being evaluated for a specific disease. Define a neutrosophic set A on X where:

$$\begin{aligned} T_A(x) &= \text{degree of evidence supporting the diagnosis,} \\ I_A(x) &= \text{degree of uncertainty in the diagnosis,} \\ F_A(x) &= \text{degree of evidence against the diagnosis.} \end{aligned}$$

Suppose the diagnostic results are:

$$A = \{(\text{Patient A}, 0.9, 0.05, 0.1), (\text{Patient B}, 0.6, 0.2, 0.4), (\text{Patient C}, 0.3, 0.5, 0.8)\}.$$

Here, Patient A has a high degree of truth (0.9) and very low indeterminacy (0.05), suggesting strong evidence for the disease. Patient B's diagnosis is moderately supported (0.6) with higher uncertainty (0.2) and some contradictory evidence (0.4). Patient C's diagnosis has low supporting evidence (0.3), high uncertainty (0.5), and strong evidence against the disease (0.8).

1.2 Fuzzy Offset and Single-Valued Neutrosophic Offset

The definitions of Fuzzy Offset and Single-Valued Neutrosophic Offset are provided below.¹⁸ While this paper primarily focuses on the definition of the Offset, any concept defined using the Offset can also be defined using the Overset or Underset. For example, a Fuzzy Overset or Fuzzy Underset can define a Fuzzy Offset.

Definition 1.5 (Fuzzy Offset). (cf.¹⁹) Let X be a universe of discourse. A *Fuzzy Offset* \tilde{A} in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\},$$

where $\Omega > 1$ and $\Psi < 0$. There exist elements $x, y \in X$ such that $\mu_{\tilde{A}}(x) > 1$ and $\mu_{\tilde{A}}(y) < 0$.

Example 1.6 (Fuzzy Offset for Machine Reliability). Let $X = \{M_1, M_2\}$ represent two machines in a production line. Choose $\Psi = -0.1$ and $\Omega = 1.2$. Define the fuzzy offset

$$\tilde{A} = \{(M_1, \mu_{\tilde{A}}(M_1)), (M_2, \mu_{\tilde{A}}(M_2))\},$$

with

$$\mu_{\tilde{A}}(M_1) = 1.1, \quad \mu_{\tilde{A}}(M_2) = -0.05,$$

and $\mu_{\tilde{A}}(x) = 0$ for any other $x \in X$. Here $\mu_{\tilde{A}}(M_1) > 1$ and $\mu_{\tilde{A}}(M_2) < 0$, while all values lie in $[\Psi, \Omega] = [-0.1, 1.2]$. Thus \tilde{A} is a valid Fuzzy Offset modeling over-confidence in M_1 and under-confidence in M_2 .

Definition 1.7 (Single-Valued Neutrosophic OffSet). (cf.¹⁹) A *Single-Valued Neutrosophic OffSet*, denoted $A_{\text{off}} \subseteq U_{\text{off}}$, is a set within a universe of discourse U_{off} in which certain elements may possess neutrosophic degrees—truth, indeterminacy, or falsity—that extend beyond the standard limits, either above 1 or below 0. It is formally defined as:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x) \rangle) \mid x \in U_{\text{off}}, \exists (T(x) > 1 \text{ or } F(x) < 0)\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of each $x \in U_{\text{off}}$.
- $T(x), I(x), F(x) \in [\Psi, \Omega]$, where $\Omega > 1$ (termed the *OverLimit*) and $\Psi < 0$ (termed the *UnderLimit*), allow the possibility for $T(x)$, $I(x)$, or $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.

Example 1.8 (Single-Valued Neutrosophic OffSet for Machine Reliability). Let $U_{\text{off}} = \{M_1, M_2\}$ denote two machines in a production line. Choose UnderLimit $\Psi = -0.1$ and OverLimit $\Omega = 1.2$. Define the Single-Valued Neutrosophic OffSet

$$A_{\text{off}} = \{(M_1, \langle 1.1, 0.05, 0 \rangle), (M_2, \langle 0.8, 0.10, -0.05 \rangle)\},$$

with all other $x \in U_{\text{off}}$ omitted. Here:

$$T(M_1) = 1.1 > 1, \quad I(M_1) = 0.05, \quad F(M_1) = 0,$$

$$T(M_2) = 0.8, \quad I(M_2) = 0.10, \quad F(M_2) = -0.05 < 0.$$

All degrees lie within $[\Psi, \Omega] = [-0.1, 1.2]$, and M_1 and M_2 satisfy the OffSet condition ($T > 1$ or $F < 0$). Hence A_{off} is a valid Single-Valued Neutrosophic OffSet modeling over-confidence in M_1 and under-falsity for M_2 .

1.3 D-Number

D-number is a set-based generalization of Dempster-Shafer belief functions that relaxes exclusivity and completeness: it assigns belief masses $D(B)$ in $[0, 1]$ to subsets B , summing ≤ 1 , capturing incomplete uncertainty.^{14,20,21}

Definition 1.9 (D number).²² Let X be a finite nonempty set (the *frame of discernment*). A *D number* on X is a mapping

$$D : 2^X \longrightarrow [0, 1]$$

such that

1. $D(\emptyset) = 0$,
2. $\sum_{B \subseteq X} D(B) \leq 1$.

In particular, one may write

$$D = \{(b_1, v_1), (b_2, v_2), \dots, (b_n, v_n)\}, \quad b_i \subseteq X, \quad v_i = D(b_i) > 0,$$

with $\sum_{i=1}^n v_i \leq 1$, and $D(B) = 0$ for any other $B \subseteq X$.

Example 1.10 (D number for weather prediction). Weather prediction is the process of forecasting atmospheric conditions like temperature, precipitation, and wind using scientific models and observational data (cf.²³). Let the possible weather outcomes be

$$X = \{\text{Rain, Sunny, Snow}\}.$$

Define a D number D by

$$D(\{\text{Rain}\}) = 0.6, \quad D(\{\text{Sunny}\}) = 0.2, \quad D(\{\text{Rain,Sunny}\}) = 0.1,$$

and $D(B) = 0$ for all other $B \subseteq X$. Then

$$D(\emptyset) = 0, \quad D(\{\text{Rain}\}) + D(\{\text{Sunny}\}) + D(\{\text{Rain,Sunny}\}) = 0.9 \leq 1,$$

so D is a valid D number. This models, for instance, 60% belief in rain, 20% in sunshine, 10% in either rain or sunshine (uncertain), and 10% unassigned.

2 Result: D-OffNumber

We newly define the concept of the D-OffNumber. It extends the structure of the Fuzzy Offset into the framework of D-Numbers.

Definition 2.1 (D-Offnumber). Let $\Psi < 0 < 1 < \Omega$ be real constants and let X be a finite nonempty set (the *frame of discernment*). A *D-Offnumber* on X is a function

$$D_{\text{off}} : 2^X \longrightarrow [\Psi, \Omega]$$

such that

1. $D_{\text{off}}(\emptyset) = 0$.
2. $\sum_{B \subseteq X} D_{\text{off}}(B) \leq 1$.
3. There exists at least one subset $B \subseteq X$ with $D_{\text{off}}(B) > 1$ or $D_{\text{off}}(B) < 0$.

If we denote by

$$\mathcal{F} = \{B \subseteq X \mid D_{\text{off}}(B) \neq 0\}$$

the set of *focal sets*, then we can write

$$D_{\text{off}} = \{(B, v_B) \mid B \in \mathcal{F}, v_B = D_{\text{off}}(B)\},$$

with $\sum_{B \in \mathcal{F}} v_B \leq 1$ and $D_{\text{off}}(B) = 0$ for all $B \notin \mathcal{F}$.

Example 2.2 (D-Offnumber for Weather Prediction). Let $X = \{\text{Rain, Sunny}\}$. Choose offset bounds $\Psi = -0.2, \Omega = 1.2$. The focal-set family is

$$\mathcal{F} = \{\{\text{Rain}\}, \{\text{Sunny}\}, \{\text{Rain, Sunny}\}\},$$

and we define

$$D_{\text{off}}(\{\text{Rain}\}) = 1.2, \quad D_{\text{off}}(\{\text{Sunny}\}) = -0.1, \quad D_{\text{off}}(\{\text{Rain, Sunny}\}) = -0.1,$$

with $D_{\text{off}}(B) = 0$ for all $B \notin \mathcal{F}$. Then

$$D_{\text{off}}(\emptyset) = 0, \quad \sum_{B \in \mathcal{F}} D_{\text{off}}(B) = 1.2 + (-0.1) + (-0.1) = 1.0 \leq 1,$$

and since $D_{\text{off}}(\{\text{Rain}\}) > 1$ and $D_{\text{off}}(\{\text{Sunny}\}) < 0$, this is a valid D-Offnumber. It models a 120% belief in rain, a -10% “delay” on sunshine, and -10% residual uncertainty between the two, with no remaining mass.

Example 2.3 (D-Offnumber for Medical Test Interpretation). Let $X = \{\text{Flu}, \text{COVID-19}\}$. Choose offset bounds $\Psi = -0.2, \Omega = 1.3$. The focal-set family is

$$\mathcal{F} = \{\{\text{Flu}\}, \{\text{COVID-19}\}, \{\text{Flu}, \text{COVID-19}\}\},$$

and define

$$D_{\text{off}}(\{\text{Flu}\}) = 1.3, \quad D_{\text{off}}(\{\text{COVID-19}\}) = -0.15, \quad D_{\text{off}}(\{\text{Flu}, \text{COVID-19}\}) = -0.15,$$

with $D_{\text{off}}(B) = 0$ for all $B \notin \mathcal{F}$. Then

$$D_{\text{off}}(\emptyset) = 0, \quad \sum_{B \in \mathcal{F}} D_{\text{off}}(B) = 1.3 + (-0.15) + (-0.15) = 1.0 \leq 1,$$

and since $D_{\text{off}}(\{\text{Flu}\}) > 1$ and $D_{\text{off}}(\{\text{COVID-19}\}) < 0$, this is a valid D-Offnumber. It models a 130% belief in influenza, a -15% deficit belief in COVID-19, and -15% joint uncertainty, with no remaining mass.

Theorem 2.4 (Reduction to D-number under $\Psi = 0, \Omega = 1$). Let X be a finite nonempty set. Suppose

$$D_{\text{off}} : 2^X \rightarrow [0, 1]$$

satisfies

$$D_{\text{off}}(\emptyset) = 0, \quad \sum_{B \subseteq X} D_{\text{off}}(B) \leq 1.$$

Then D_{off} is exactly a D-number on X .

Proof. We verify each of the three defining properties of a D-number in turn.

(i) Codomain. By hypothesis, $D_{\text{off}}(B) \in [0, 1]$ for every subset $B \subseteq X$. This matches the requirement that a D-number m satisfies $m(B) \in [0, 1]$ for all B .

(ii) Empty-set mass. It is given that

$$D_{\text{off}}(\emptyset) = 0,$$

which coincides exactly with the D-number axiom $m(\emptyset) = 0$.

(iii) Total mass. The assumption

$$\sum_{B \subseteq X} D_{\text{off}}(B) \leq 1$$

is precisely the normalization condition for a D-number, namely $\sum_{B \subseteq X} m(B) \leq 1$.

Since all three conditions—values in $[0, 1]$, zero mass on the empty set, and total mass at most one—are satisfied, we conclude that D_{off} fulfills the definition of a D-number on X . \square

Theorem 2.5 (Normalization to a D-number). Let

$$D_{\text{off}} = \{(B_i, v_i)\}_{i=1}^n$$

be any D-Offnumber on X whose focal values are all nonnegative (i.e. $v_i = D_{\text{off}}(B_i) \geq 0$) and satisfy $\sum_{i=1}^n v_i \leq 1$. Define the residual mass

$$r := 1 - \sum_{i=1}^n v_i.$$

Then $r \in [0, 1]$, and the mapping

$$\tilde{D}(B) := \begin{cases} v_i, & B = B_i, \\ r, & B = X, \\ 0, & \text{otherwise,} \end{cases}$$

defines a normalized D-number on X .

Proof. Since $\sum_{i=1}^n v_i \leq 1$ and each $v_i \geq 0$, it follows that $0 \leq r \leq 1$. Moreover,

$$\tilde{D}(\emptyset) = 0, \quad \sum_{B \subseteq X} \tilde{D}(B) = \sum_{i=1}^n v_i + r = 1,$$

and every $\tilde{D}(B)$ lies in $[0, 1]$. Therefore \tilde{D} meets all the requirements of a D-number. □

Theorem 2.6 (Singleton D-Offnumber yields a Fuzzy Offset). *Let $\Psi < 0 < 1 < \Omega$ be real constants and let $X = \{x_1, \dots, x_n\}$. Suppose*

$$D_{\text{off}} : 2^X \longrightarrow [\Psi, \Omega]$$

is a D-Offnumber whose only nonzero values occur on the singletons:

$$D_{\text{off}}(\{x_i\}) = v_i \in [\Psi, \Omega], \quad i = 1, \dots, n, \quad D_{\text{off}}(B) = 0 \text{ for } B \notin \{\{x_i\}\}.$$

Define

$$\mu : X \longrightarrow [\Psi, \Omega], \quad \mu(x_i) := v_i, \quad i = 1, \dots, n.$$

Then (X, μ) is a Fuzzy Offset, i.e. μ takes values in $[\Psi, \Omega]$ and there exists at least one index j such that $\mu(x_j) > 1$ or $\mu(x_j) < 0$.

Proof. (Codomain.) By construction, each $v_i = D_{\text{off}}(\{x_i\}) \in [\Psi, \Omega]$. Hence $\mu(x_i) = v_i$ lies in $[\Psi, \Omega]$ for every i , so μ maps X into $[\Psi, \Omega]$.

(Offset property.) Axiom (iii) of the D-Offnumber definition ensures there is at least one focal set $B \subseteq X$ with

$$D_{\text{off}}(B) > 1 \quad \text{or} \quad D_{\text{off}}(B) < 0.$$

Since the only focal sets are the singletons $\{x_i\}$, this set must be $\{x_j\}$ for some j . Therefore

$$\mu(x_j) = D_{\text{off}}(\{x_j\}) > 1 \quad \text{or} \quad \mu(x_j) = D_{\text{off}}(\{x_j\}) < 0,$$

showing that μ attains an “offset” value outside $[0, 1]$.

Together, these two facts establish that (X, μ) satisfies the definition of a Fuzzy Offset. □

D-OverNumber and D-UnderNumber are subclasses of the D-OffNumber. Their definitions and concrete examples are provided below.

Definition 2.7 (D-Overnumber). Let $\Omega > 1$ and X be a finite nonempty set (the frame of discernment). A *D-Overnumber* on X is a mapping

$$D_{\text{over}} : 2^X \longrightarrow [0, \Omega]$$

such that

1. $D_{\text{over}}(\emptyset) = 0$.
2. $\sum_{B \subseteq X} D_{\text{over}}(B) \leq \Omega$.
3. There exists at least one $B \subseteq X$ with $D_{\text{over}}(B) > 1$.

If we denote the focal-set family by

$$\mathcal{F} = \{ B \subseteq X \mid D_{\text{over}}(B) > 0 \},$$

then we may write

$$D_{\text{over}} = \{ (B, v_B) \mid B \in \mathcal{F}, v_B = D_{\text{over}}(B) \},$$

with $\sum_{B \in \mathcal{F}} v_B \leq \Omega$ and $D_{\text{over}}(B) = 0$ for all $B \notin \mathcal{F}$.

Example 2.8 (D-Overnumber for Rain Prediction). Let $X = \{\text{Rain}, \text{Sunny}\}$ and choose $\Omega = 1.2$. The focal-set family is

$$\mathcal{F} = \{\{\text{Rain}\}\},$$

and we define

$$D_{\text{over}}(\{\text{Rain}\}) = 1.1, \quad D_{\text{over}}(B) = 0 \text{ for all } B \neq \{\text{Rain}\}.$$

Then

$$D_{\text{over}}(\emptyset) = 0, \quad \sum_{B \subseteq X} D_{\text{over}}(B) = 1.1 \leq 1.2,$$

and since $D_{\text{over}}(\{\text{Rain}\}) > 1$, this is a valid D-Overnumber. It represents a 110% confidence in rain, with no belief assigned elsewhere.

Example 2.9 (D-Overnumber for Customer Preference Survey). Let

$$X = \{\text{Product A}, \text{Product B}, \text{Product C}\}$$

and choose $\Omega = 1.5$. The focal-set family is

$$\mathcal{F} = \{\{\text{Product A}\}, \{\text{Product B}\}, \{\text{Product C}\}\},$$

and we define

$$D_{\text{over}}(\{\text{Product A}\}) = 1.2, \quad D_{\text{over}}(\{\text{Product B}\}) = 0.2, \quad D_{\text{over}}(\{\text{Product C}\}) = 0.1,$$

with $D_{\text{over}}(B) = 0$ for all $B \notin \mathcal{F}$. Then

$$D_{\text{over}}(\emptyset) = 0, \quad \sum_{B \subseteq X} D_{\text{over}}(B) = 1.2 + 0.2 + 0.1 = 1.5 \leq 1.5,$$

and since $D_{\text{over}}(\{\text{Product A}\}) > 1$, this is a valid D-Overnumber. It models a survey in which respondents exhibit a 120% preference strength for Product A, 20% for B, and 10% for C, indicating an overconfident favoring of A.

Definition 2.10 (D-Undernumber). Let $\Psi < 0$ be a real constant and let X be a finite nonempty set (the frame of discernment). A *D-Undernumber* on X is a mapping

$$D_{\text{under}} : 2^X \longrightarrow [\Psi, 1]$$

such that

1. $D_{\text{under}}(\emptyset) = 0$.
2. $\sum_{B \subseteq X} D_{\text{under}}(B) \leq 1$.
3. There exists at least one $B \subseteq X$ with $D_{\text{under}}(B) < 0$.

If we denote the focal-set family by

$$\mathcal{F} = \{B \subseteq X \mid D_{\text{under}}(B) \neq 0\},$$

then we may write

$$D_{\text{under}} = \{(B, v_B) \mid B \in \mathcal{F}, v_B = D_{\text{under}}(B)\},$$

with $\sum_{B \in \mathcal{F}} v_B \leq 1$ and $D_{\text{under}}(B) = 0$ for all $B \notin \mathcal{F}$.

Example 2.11 (D-Undernumber for Fault Diagnosis). Consider the binary fault diagnosis frame

$$X = \{\text{Fault}, \text{NoFault}\}, \quad \Psi = -0.2.$$

The focal-set family is

$$\mathcal{F} = \{\{\text{Fault}\}, \{\text{NoFault}\}\},$$

and we define

$$D_{\text{under}}(\{\text{Fault}\}) = -0.1, \quad D_{\text{under}}(\{\text{NoFault}\}) = 0.8,$$

with $D_{\text{under}}(B) = 0$ for all $B \notin \mathcal{F}$. Then

$$D_{\text{under}}(\emptyset) = 0, \quad \sum_{B \subseteq X} D_{\text{under}}(B) = -0.1 + 0.8 = 0.7 \leq 1,$$

and since $D_{\text{under}}(\{\text{Fault}\}) < 0$, this is a valid D-Undernumber. It models an 80% confidence in no-fault, with a -10% underestimation of fault likelihood and 20% residual uncertainty.

Example 2.12 (D-Undernumber for Credit Risk Assessment). Let

$$X = \{\text{Default}, \text{NoDefault}\}, \quad \Psi = -0.2.$$

The focal-set family is

$$\mathcal{F} = \{\{\text{Default}\}, \{\text{NoDefault}\}\},$$

and define

$$D_{\text{under}}(\{\text{Default}\}) = -0.1, \quad D_{\text{under}}(\{\text{NoDefault}\}) = 0.9,$$

with $D_{\text{under}}(B) = 0$ for all other $B \subseteq X$. Then

$$D_{\text{under}}(\emptyset) = 0, \quad \sum_{B \subseteq X} D_{\text{under}}(B) = -0.1 + 0.9 = 0.8 \leq 1,$$

and since $D_{\text{under}}(\{\text{Default}\}) < 0$, this is a valid D-Undernumber. It represents a 90% confidence in non-default, a -10% under-assessment of default risk, and 20% unassigned mass.

3 Result: Neutrosophic D-Number

We now introduce the concept of a Neutrosophic D-Number, which merges the ideas of D-numbers and neutrosophic sets.

Definition 3.1 (Neutrosophic D-Number). Let X be a finite nonempty set (the frame). A Neutrosophic D-Number on X is a function

$$D_N : 2^X \rightarrow [0, 1]^3, \quad B \mapsto D_N(B) = (T_B, I_B, F_B),$$

such that

1. $D_N(\emptyset) = (0, 0, 0)$.
2. For each focal set $B \subseteq X$, $(T_B, I_B, F_B) \in [0, 1]^3$.
3. $\sum_{B \subseteq X} T_B \leq 1$, $\sum_{B \subseteq X} I_B \leq 1$, $\sum_{B \subseteq X} F_B \leq 1$.

If $\mathcal{F} = \{B \subseteq X \mid D_N(B) \neq (0, 0, 0)\}$ denotes the focal-set family, we may write

$$D_N = \{(B, (T_B, I_B, F_B)) \mid B \in \mathcal{F}\}.$$

Example 3.2 (Medical Diagnosis under Uncertain and Conflicting Evidence). A clinician must decide whether a patient has *Flu* or *Cold*, but the symptoms and test results are inconclusive. We model this with the frame

$$X = \{\text{Flu}, \text{Cold}\}.$$

Based on clinical observations, laboratory tests, and patient history, the doctor assigns the following neutrosophic masses:

$$\begin{aligned} D_N(\{\text{Flu}\}) &= (T_{\{\text{Flu}\}}, I_{\{\text{Flu}\}}, F_{\{\text{Flu}\}}) = (0.5, 0.2, 0.3), \\ D_N(\{\text{Cold}\}) &= (T_{\{\text{Cold}\}}, I_{\{\text{Cold}\}}, F_{\{\text{Cold}\}}) = (0.3, 0.4, 0.3), \\ D_N(\{\text{Flu}, \text{Cold}\}) &= (T_{\{\text{Flu}, \text{Cold}\}}, I_{\{\text{Flu}, \text{Cold}\}}, F_{\{\text{Flu}, \text{Cold}\}}) = (0.1, 0.1, 0.2), \end{aligned}$$

and $D_N(B) = (0, 0, 0)$ for all other $B \subseteq X$. Here:

- T_B is the degree of support (“truth”) for hypothesis B .
- I_B is the degree of indeterminacy (ambiguity) about B .
- F_B is the degree of contradiction (“falsity”) against B .

Step-by-step verification:

1. *Truth-sum:*

$$\sum_{B \subseteq X} T_B = T_{\{\text{Flu}\}} + T_{\{\text{Cold}\}} + T_{\{F,C\}} = 0.5 + 0.3 + 0.1 = 0.9 \leq 1.$$

2. *Indeterminacy-sum:*

$$\sum_{B \subseteq X} I_B = 0.2 + 0.4 + 0.1 = 0.7 \leq 1.$$

3. *Falsity-sum:*

$$\sum_{B \subseteq X} F_B = 0.3 + 0.3 + 0.2 = 0.8 \leq 1.$$

4. *Range check:* Each (T_B, I_B, F_B) lies in $[0, 1]^3$, and $D_N(\emptyset) = (0, 0, 0)$ by convention.

Since all three component-sums do not exceed 1, and every triple is within $[0, 1]$, the above assignment is a valid Neutrosophic D-Number on X . This example concretely captures how the clinician’s evidence can simultaneously support, contradict, and remain uncertain about each diagnosis.

Example 3.3 (Neutrosophic D-Number for Credit Risk Level Assessment). A financial institution assesses a borrower’s credit risk as *Low*, *Medium*, or *High*. Let

$$X = \{\text{Low}, \text{Medium}, \text{High}\}.$$

Based on credit history, income verification, and market analysis, the institution assigns the following neutrosophic masses:

$$D_N(\{\text{Low}\}) = (0.3, 0.1, 0.2), \quad D_N(\{\text{Medium}\}) = (0.2, 0.2, 0.1),$$

$$D_N(\{\text{High}\}) = (0.1, 0.1, 0.3), \quad D_N(\{\text{Low}, \text{Medium}\}) = (0.2, 0.1, 0.1),$$

and $D_N(B) = (0, 0, 0)$ for all other $B \subseteq X$.

Verification:

$$\sum_{B \subseteq X} T_B = 0.3 + 0.2 + 0.1 + 0.2 = 0.8 \leq 1,$$

$$\sum_{B \subseteq X} I_B = 0.1 + 0.2 + 0.1 + 0.1 = 0.5 \leq 1,$$

$$\sum_{B \subseteq X} F_B = 0.2 + 0.1 + 0.3 + 0.1 = 0.7 \leq 1.$$

Each $(T_B, I_B, F_B) \in [0, 1]^3$ and $D_N(\emptyset) = (0, 0, 0)$. Therefore D_N is a valid Neutrosophic D-Number on X , capturing the institution’s partial support, indeterminacy, and potential contradiction for each risk level.

Theorem 3.4 (Reduction to D-number and Neutrosophic Set). *Let*

$$D_N = \{(B_i, (T_i, I_i, F_i)) \mid i = 1, \dots, n\}$$

be a Neutrosophic D-Number on X . Then:

1. If $I_i = F_i = 0$ for all i , the function

$$D : 2^X \rightarrow [0, 1], \quad D(B_i) = T_i, \quad D(B) = 0 \text{ for } B \notin \{B_1, \dots, B_n\},$$

satisfies $D(\emptyset) = 0$ and $\sum_{B \subseteq X} D(B) \leq 1$, so D is a D-number on X .

2. If additionally $n = |X|$ and each $B_i = \{x_i\}$ is a singleton, then setting

$$T_A(x_i) = T_i, \quad I_A(x_i) = I_i, \quad F_A(x_i) = F_i, \quad i = 1, \dots, n,$$

and $T_A(x) = I_A(x) = F_A(x) = 0$ for $x \notin \{x_1, \dots, x_n\}$, defines a neutrosophic set $A = \{(x, \langle T_A(x), I_A(x), F_A(x) \rangle) \mid x \in X\}$.

Proof. (1) Reduction to D-number. Since $I_i = F_i = 0$, each $T_i \in [0, 1]$ and by the Neutrosophic D-number axiom $\sum_i T_i \leq 1$. Also $D(\emptyset) = T(\emptyset) = 0$ and $D(B) = 0$ for all other B . Thus D meets the D-number conditions.

(2) Reduction to Neutrosophic Set. Under the singleton assumption $B_i = \{x_i\}$, define T_A, I_A, F_A on X as above. Each value lies in $[0, 1]$, and

$$0 \leq T_A(x) + I_A(x) + F_A(x) = T_i + I_i + F_i \leq 3$$

for $x = x_i$, with trivial equality for other x . Hence A is a neutrosophic set on X . □

Example 3.5 (Reduction to a D-number). Let $X = \{\text{Rain, Sunny, Snow}\}$. Consider the Neutrosophic D-Number

$$D_N = \{(\{\text{Rain, Sunny}\}, (0.4, 0, 0)), (\{\text{Snow}\}, (0.5, 0, 0))\},$$

with $D_N(B) = (0, 0, 0)$ for all other $B \subseteq X$. Here the truth-components are $T_1 = 0.4$, $T_2 = 0.5$, and each indeterminacy and falsity component is zero. Since

$$\sum_{i=1}^2 T_i = 0.4 + 0.5 = 0.9 \leq 1, \quad \sum_{i=1}^2 I_i = 0, \quad \sum_{i=1}^2 F_i = 0,$$

Theorem (1) allows us to define

$$D(\{\text{Rain, Sunny}\}) = 0.4, \quad D(\{\text{Snow}\}) = 0.5, \quad D(B) = 0 \text{ otherwise.}$$

Then $D(\emptyset) = 0$ and $\sum_{B \subseteq X} D(B) = 0.9 \leq 1$, so D is a valid D-number on X .

Example 3.6 (Reduction to a Neutrosophic Set). Let $X = \{\text{Flu, Cold}\}$. Consider the Neutrosophic D-Number concentrated on singletons:

$$D_N = \{(\{\text{Flu}\}, (0.6, 0.2, 0.1)), (\{\text{Cold}\}, (0.3, 0.1, 0.4))\},$$

and $D_N(B) = (0, 0, 0)$ for all other $B \subseteq X$. We verify

$$\sum T_i = 0.6 + 0.3 = 0.9 \leq 1, \quad \sum I_i = 0.2 + 0.1 = 0.3 \leq 1, \quad \sum F_i = 0.1 + 0.4 = 0.5 \leq 1.$$

By Theorem (2), define

$$T_A(\text{Flu}) = 0.6, \quad I_A(\text{Flu}) = 0.2, \quad F_A(\text{Flu}) = 0.1, \quad T_A(\text{Cold}) = 0.3, \quad I_A(\text{Cold}) = 0.1, \quad F_A(\text{Cold}) = 0.4.$$

Then for each $x \in X$,

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3,$$

so

$$A = \{(\text{Flu}, \langle 0.6, 0.2, 0.1 \rangle), (\text{Cold}, \langle 0.3, 0.1, 0.4 \rangle)\}$$

is a valid Neutrosophic Set on X .

Theorem 3.7 (Marginalization to a Neutrosophic Set). *Let X be a finite nonempty set and let*

$$D_N = \{(B_i, (T_i, I_i, F_i)) \mid i = 1, \dots, n, B_i \subseteq X\}$$

be a Neutrosophic D-Number on X , with focal-set family $\mathcal{F} = \{B_1, \dots, B_n\}$ and

$$(T_i, I_i, F_i) \in [0, 1]^3, \quad \sum_{i=1}^n T_i \leq 1, \quad \sum_{i=1}^n I_i \leq 1, \quad \sum_{i=1}^n F_i \leq 1.$$

Define functions $T^, I^*, F^* : X \rightarrow [0, 1]$ by*

$$T^*(x) = \sum_{i:x \in B_i} T_i, \quad I^*(x) = \sum_{i:x \in B_i} I_i, \quad F^*(x) = \sum_{i:x \in B_i} F_i.$$

Then

$$A = \{(x, \langle T^*(x), I^*(x), F^*(x) \rangle) \mid x \in X\}$$

is a Neutrosophic Set on X .

Proof. We verify the neutrosophic set conditions for each $x \in X$.

(i) **Range bound.** Since each $T_i, I_i, F_i \in [0, 1]$ and $\sum_{i=1}^n T_i \leq 1$, we have

$$0 \leq T^*(x) = \sum_{i:x \in B_i} T_i \leq \sum_{i=1}^n T_i \leq 1.$$

Similarly $0 \leq I^*(x) \leq 1$ and $0 \leq F^*(x) \leq 1$.

(ii) **Sum-condition.** Using the component sums,

$$T^*(x) + I^*(x) + F^*(x) = \sum_{i:x \in B_i} (T_i + I_i + F_i) \leq \sum_{i=1}^n (T_i + I_i + F_i) \leq 1 + 1 + 1 = 3.$$

Thus $\langle T^*(x), I^*(x), F^*(x) \rangle \in [0, 1]^3$ with total at most 3, which are exactly the axioms of a neutrosophic set. \square

Example 3.8 (Marginalization of a Weather-Forecast Neutrosophic D-Number). Let

$$X = \{\text{Rain, Sunny, Snow}\},$$

and suppose

$$D_N = \left\{ (\{\text{Rain}\}, (0.4, 0.1, 0.2)), (\{\text{Sunny}\}, (0.3, 0.2, 0.1)), \right. \\ \left. (\{\text{Rain, Sunny}\}, (0.2, 0.3, 0.2)), (\{\text{Snow}\}, (0.1, 0.2, 0.3)) \right\},$$

with all other subsets assigned $(0, 0, 0)$. One checks

$$\sum_i T_i = 0.4 + 0.3 + 0.2 + 0.1 = 1.0 \leq 1, \\ \sum_i I_i = 0.1 + 0.2 + 0.3 + 0.2 = 0.8 \leq 1, \\ \sum_i F_i = 0.2 + 0.1 + 0.2 + 0.3 = 0.8 \leq 1.$$

By Theorem 3.7, for each $x \in X$ define

$$T^*(\text{Rain}) = T_1 + T_3 = 0.4 + 0.2 = 0.6, \quad T^*(\text{Sunny}) = T_2 + T_3 = 0.3 + 0.2 = 0.5, \quad T^*(\text{Snow}) = T_4 = 0.1, \\ I^*(\text{Rain}) = I_1 + I_3 = 0.1 + 0.3 = 0.4, \quad I^*(\text{Sunny}) = I_2 + I_3 = 0.2 + 0.3 = 0.5, \quad I^*(\text{Snow}) = I_4 = 0.2, \\ F^*(\text{Rain}) = F_1 + F_3 = 0.2 + 0.2 = 0.4, \quad F^*(\text{Sunny}) = F_2 + F_3 = 0.1 + 0.2 = 0.3, \quad F^*(\text{Snow}) = F_4 = 0.3.$$

Hence the marginal neutrosophic set is

$$A = \left\{ (\text{Rain}, \langle 0.6, 0.4, 0.4 \rangle), (\text{Sunny}, \langle 0.5, 0.5, 0.3 \rangle), (\text{Snow}, \langle 0.1, 0.2, 0.3 \rangle) \right\},$$

and one verifies $0 \leq T^*(x), I^*(x), F^*(x) \leq 1$ and $T^*(x) + I^*(x) + F^*(x) \leq 3$ for each x . Thus A is a valid neutrosophic set on X .

4 Conclusion and Future Work

In this paper, we have formally introduced and rigorously defined four novel mathematical constructs—*D-OffNumber*, *D-OverNumber*, *D-UnderNumber*, and the *Neutrosophic D-Number*. We examined their foundational properties and clarified the relationships among them within a unified theoretical framework. These constructs provide a fresh perspective for representing and analyzing indeterminate, excessive, and deficient information across diverse contexts.

Looking ahead, we intend to pursue several avenues of further research. First, we will design and conduct computational experiments to validate our theoretical models and assess their practical performance. Second, we plan to explore applications in multi-criteria decision-making and integrate these constructs into more advanced analytical frameworks. Third, we will analyze the algorithmic complexity—both in time and space—associated with these new number systems. Finally, we believe there are promising interdisciplinary opportunities, particularly in the social sciences and systems that handle uncertainty.

Moreover, we aim to investigate whether the *D-OffNumber* and *D-Number* concepts can be extended to existing combinatorial structures such as graphs,²⁴ hypergraphs,^{25–27} and superhypergraphs.^{28,29}

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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