Graded HyperRough Set and Linguistic HyperRough Set

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Abstract

Numerous mathematical frameworks have been developed to handle uncertainty, including Fuzzy Sets,[1] Intuitionistic Fuzzy Sets,[2] Hyperfuzzy Sets,[3] Picture Fuzzy Sets,[4] Hesitant Fuzzy Sets,[5,6] Neutrosophic Sets,[7] Plithogenic Sets,[8] and Soft Sets,[9] and research in this area continues to evolve rapidly. Rough set theory provides a foundational method for approximating subsets using lower and upper bounds based on equivalence relations, offering an effective approach to modeling uncertainty in classification and data analysis.[10,11] Building upon these foundations, extended models such as HyperRough Sets and SuperHyperRough Sets have been proposed.[12] In this paper, we present novel definitions that further generalize Graded Rough Sets and Linguistic Rough Sets—specifically, the Graded HyperRough Set and the Linguistic HyperRough Set. These new frameworks are expected to contribute to the advancement of research in fields such as decision-making, language theory, and artificial intelligence.

Keywords: Rough set; Hyperrough Set; Linguistic Rough Set; SuperHyperRough set; Graded Rough Set

1 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets under consideration are assumed to be finite.

1.1 Rough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership.[10]

Definition 1.1 (Universal Set)**.** A *universal set*, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.

Definition 1.2 (Rough Set Approximation)**.** [10] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

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1. *Lower Approximation*:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

2. *Upper Approximation*:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying
$$\underline{U} \subseteq U \subseteq \overline{U}.$$

1.2 Graded rough sets

In the graded rough set model the classical approximations are refined quantitatively by measuring the degree to which an equivalence class is included in a set X.[13,14]

Definition 1.3 (Graded Rough Set)**.** [15] Let $X \subseteq U$ and $B \subseteq A$. Define the *rough membership function* $\mu_B^X : U \to [0,1]$ by
$$\mu_B^X(x) = \frac{|[x]_B \cap X|}{|[x]_B|},$$
where $|\cdot|$ denotes the cardinality of a set. Fix a threshold $k \in (0.5, 1]$. Then the *graded rough approximations* of X with respect to the attribute set B are defined as follows:
$$\underline{R}_B^k(X) = \{x \in U \mid \mu_B^X(x) \geq k\}$$
and
$$\overline{R}_B^k(X) = \{x \in U \mid \mu_B^X(x) > 1 - k\}.$$
The pair
$$R_B^k(X) = \left(\underline{R}_B^k(X),\, \overline{R}_B^k(X)\right)$$
is called the *Graded Rough Set* of X at precision level k. If $\underline{R}_B^k(X) = \overline{R}_B^k(X)$, then X is said to be k-exact; otherwise, it is k-rough.

Example 1.4 (Graded Rough Set)**.** Consider the universe
$$U = \{1, 2, 3, 4, 5, 6\}.$$
Assume that the equivalence classes induced by R_B are:
$$[1]_B = \{1, 2, 3\}, \quad [4]_B = \{4, 5\}, \quad [6]_B = \{6\}.$$
Let
$$X = \{1, 2, 4, 6\}.$$
We compute the rough membership function for each x:

- For any $x \in \{1, 2, 3\}$:
$$\mu_B^X(x) = \frac{|\{1, 2, 3\} \cap \{1, 2, 4, 6\}|}{|\{1, 2, 3\}|} = \frac{2}{3} \approx 0.67.$$
- For any $x \in \{4, 5\}$:
$$\mu_B^X(x) = \frac{|\{4, 5\} \cap \{1, 2, 4, 6\}|}{|\{4, 5\}|} = \frac{1}{2} = 0.5.$$

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- For $x = 6$:

$$\mu_B^X(6) = \frac{|\{6\} \cap \{1,2,4,6\}|}{|\{6\}|} = 1.$$

Now, choose $k = 0.7$. Then:

$$\underline{R}_B^{0.7}(X) = \{x \in U \mid \mu_B^X(x) \geq 0.7\} = \{6\},$$

since only $\mu_B^X(6) = 1 \geq 0.7$. Moreover,

$$\overline{R}_B^{0.7}(X) = \{x \in U \mid \mu_B^X(x) > 1 - 0.7 = 0.3\} = U,$$

because $0.67 > 0.3$, $0.5 > 0.3$, and $1 > 0.3$. Thus, the graded rough set representation is

$$R_B^{0.7}(X) = \Big(\{6\}, U\Big),$$

so X is 0.7-rough.

1.3 Linguistic Rough Set

The Linguistic Rough Set model replaces numerical degrees of inclusion with qualitative linguistic labels, effectively capturing the inherent vagueness of human assessments.[16] Related concepts include Hesitant Fuzzy Linguistic Term Sets[17,18] and Linguistic Neutrosophic Sets,[19,20] which offer additional approaches for modeling uncertainty in linguistic information.

Definition 1.5 (Linguistic Label Set and Mapping). [16] Let $\mathcal{S}$ be a finite, totally ordered set of linguistic labels:

$$\mathcal{S} = \{s_{-2}, s_{-1}, s_0, s_1, s_2\},$$

with

$$s_{-2} \prec s_{-1} \prec s_0 \prec s_1 \prec s_2.$$

For example, one may define:

$$\begin{aligned} s_{-2} &= \text{“extremely low”}, \\ s_{-1} &= \text{“low”}, \\ s_0 &= \text{“medium”}, \\ s_1 &= \text{“high”}, \\ s_2 &= \text{“extremely high”}. \end{aligned}$$

Define a *linguistic mapping* $\ell : [0,1] \to \mathcal{S}$ by partitioning the interval $[0,1]$ with fixed thresholds $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < 1$. For example, one may set

$$\ell(t) = \begin{cases} s_{-1}, & 0 \leq t < \alpha_2, \\ s_0, & \alpha_2 \leq t < \alpha_3, \\ s_1, & \alpha_3 \leq t < \alpha_4, \\ s_2, & \alpha_4 \leq t \leq 1. \end{cases}$$

Definition 1.6 (Linguistic Rough Set). [16] Let $X \subseteq U$ and $B \subseteq A$. First, for each $x \in U$ the rough membership function is computed as

$$\mu_B^X(x) = \frac{|[x]_B \cap X|}{|[x]_B|}.$$

Then, the *linguistic evaluation* of x is given by

$$\lambda(x) = \ell\big(\mu_B^X(x)\big).$$

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The *Linguistic Rough Set* (LRS) of X is defined as

$$LRS(X) = \{\, ([x]_B, \lambda(x)) \mid x \in U \,\}.$$

Furthermore, by selecting two linguistic thresholds s_θ and s_φ from $\mathcal{S}$ (with s_θ typically representing a higher level of certainty), one can define the qualitative approximations as:

$$L_{LRS}(X) = \{\, [x]_B \mid \lambda(x) \succeq s_\theta \,\} \quad \text{(linguistic lower approximation)},$$

$$U_{LRS}(X) = \{\, [x]_B \mid \lambda(x) \succ s_\varphi \,\} \quad \text{(linguistic upper approximation)}.$$

Example 1.7 (Linguistic Rough Set)**.** Using the same universe and partition as in the previous example, let

$$U = \{1, 2, 3, 4, 5, 6\}, \quad U/R_B = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\},$$

and take

$$X = \{1, 2, 4, 6\}.$$

Previously, we computed

$$\mu_B^X(x) = \begin{cases} \frac{2}{3} \approx 0.67, & x \in \{1, 2, 3\}, \\ 0.5, & x \in \{4, 5\}, \\ 1, & x = 6. \end{cases}$$

Assume the linguistic mapping thresholds are set as

$$\alpha_2 = 0.6, \quad \alpha_3 = 0.8, \quad \alpha_4 = 0.9.$$

Then:

- For $x \in \{1, 2, 3\}$: $\mu_B^X(x) \approx 0.67$ so $\lambda(x) = \ell(0.67) = s_0$ ("medium").
- For $x \in \{4, 5\}$: $\mu_B^X(x) = 0.5$ so $\lambda(x) = \ell(0.5) = s_{-1}$ ("low").
- For $x = 6$: $\mu_B^X(6) = 1$ so $\lambda(6) = \ell(1) = s_2$ ("extremely high").

If we choose $s_\theta = s_1$ (i.e., we require at least a "high" rating) for the lower approximation, then

$$L_{LRS}(X) = \{\, [x]_B \mid \lambda(x) \succeq s_1 \,\} = \{\, \{6\} \,\}.$$

Similarly, setting $s_\varphi = s_{-1}$ for the upper approximation (including all granules rated better than "low"), we obtain

$$U_{LRS}(X) = \{\, [x]_B \mid \lambda(x) \succ s_{-1} \,\} = \{\{1, 2, 3\}, \{6\}\}.$$

Thus, the complete linguistic rough set representation is

$$LRS(X) = \{\, (\{1, 2, 3\}, \text{"medium"}), (\{4, 5\}, \text{"low"}), (\{6\}, \text{"extremely high"}) \,\}.$$

1.4 HyperRough Set and Superhyperrough Set

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below.[12]

Definition 1.8 (HyperRough Set)**.** [12] Let X be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, with $[x]_R$ denoting the equivalence class of x. A *HyperRough Set* over X is a pair (F, J), where:

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- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \dots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 1.9 (A simple HyperRough Set over two attributes)**.** Let the finite universe be

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Define an equivalence relation R on X by the partition

$$[x_1]_R = \{x_1, x_2\}, \quad [x_3]_R = \{x_3\}, \quad [x_4]_R = \{x_4, x_5\}, \quad [x_6]_R = \{x_6\}.$$

Take two attributes:

$$T_1 = \text{Color}, \;\; J_1 = \{\text{red}, \text{blue}\}; \qquad T_2 = \text{Phase}, \;\; J_2 = \{\alpha, \beta\},$$

so that $J = J_1 \times J_2 = \{(\text{red}, \alpha), (\text{red}, \beta), (\text{blue}, \alpha), (\text{blue}, \beta)\}$. Define $F : J \to \mathcal{P}(X)$ by

$a \in J$	$F(a)$
(red, α)	$\{x_1, x_3, x_4\}$
(red, β)	$\{x_2, x_4\}$
(blue, α)	$\{x_5\}$
(blue, β)	$\{x_2, x_6\}$

For each $a \in J$, the lower/upper rough approximations are

$$\underline{F(a)} = \{\, x \in X : [x]_R \subseteq F(a) \,\}, \qquad \overline{F(a)} = \{\, x \in X : [x]_R \cap F(a) \neq \emptyset \,\}.$$

A direct check using the R-classes above gives:

a	$\underline{F(a)}$	$\overline{F(a)}$
(red, α)	$\{x_3\}$	$\{x_1, x_2, x_3, x_4, x_5\}$
(red, β)	$\varnothing$	$\{x_1, x_2, x_4, x_5\}$
(blue, α)	$\varnothing$	$\{x_4, x_5\}$
(blue, β)	$\{x_6\}$	$\{x_1, x_2, x_6\}$

Observe in each case $\underline{F(a)} \subseteq \overline{F(a)}$, and the extreme cases hold: if $F(a) = \varnothing$ (resp. $F(a) = X$) then $\underline{F(a)} = \overline{F(a)} = \varnothing$ (resp. X), in agreement with the definition.

An n-SuperHyperRough Set generalizes rough sets by using power sets of attribute values to produce nuanced approximations under uncertainty. The definition of n-SuperHyperRough Sets is described as follows.

Definition 1.10 (n-SuperHyperRough Set)**.** [12] Let X be a nonempty finite universe, and let $T_1, T_2, \dots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \dots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote its power set. Define the set of all possible attribute value combinations as

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X. An *n-SuperHyperRough Set* over X is a pair (F, J), where:

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- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$ (with $A_i \subseteq J_i$ for all i) a subset $F(A) \subseteq X$.
- For each $A \in J$, the lower and upper approximations are defined as

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Thus, $\underline{F(A)}$ consists of all elements whose equivalence classes are entirely contained in $F(A)$, and $\overline{F(A)}$ includes those elements whose equivalence classes intersect $F(A)$. The following properties hold for all $A \in J$:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$,

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 1.11 (Real-life Example of a 2-SuperHyperRough Set)**.** Let $X = \{S_1, S_2, S_3, S_4\}$ represent four online shoppers. We use two attributes:

$$T_1 = \text{Preferred OS}, \quad J_1 = \{\text{iOS}, \text{ Android}\},$$

$$T_2 = \text{Screen Size}, \quad J_2 = \{\text{Small}, \text{ Large}\}.$$

Then

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2)$$

has 16 combinations. Define an equivalence relation R by "same region," with classes

$$[S_1]_R = [S_2]_R = \{S_1, S_2\}, \quad [S_3]_R = [S_4]_R = \{S_3, S_4\}.$$

Specify the mapping $F\colon J \to \mathcal{P}(X)$ on key combinations:

$$\begin{aligned} F\big(\{\text{iOS}\}, \{\text{Large}\}\big) &= \{S_1, S_2\}, \\ F\big(\{\text{Android}\}, \{\text{Small}\}\big) &= \{S_3\}, \\ F\big(\{\text{Android}\}, \{\text{Large}\}\big) &= \{S_4\}, \\ F(A) &= \varnothing \text{ for all other } A \in J. \end{aligned}$$

1. For $A = (\{\text{iOS}\}, \{\text{Large}\})$, $F(A) = \{S_1, S_2\}$. Since each shopper in $\{S_1, S_2\}$ shares the same region class,

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq \{S_1, S_2\}\} = \{S_1, S_2\},$$

$$\overline{F(A)} = \{x \in X \mid [x]_R \cap \{S_1, S_2\} \neq \varnothing\} = \{S_1, S_2\}.$$

Thus this concept is crisply captured.

2. For $B = (\{\text{Android}\}, \{\text{Small}\})$, $F(B) = \{S_3\}$. But $[S_3]_R = \{S_3, S_4\} \not\subseteq \{S_3\}$, so

$$\underline{F(B)} = \varnothing,$$

while

$$\overline{F(B)} = \{x \in X \mid [x]_R \cap \{S_3\} \neq \varnothing\} = \{S_3, S_4\}.$$

This illustrates a non-crisp (rough) concept: only possibly S_4 belongs to the "Android + Small" group.

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Example 1.12 (Real-life Example of a 3-SuperHyperRough Set)**.** Let $X = \{U_1, U_2, U_3, U_4, U_5, U_6\}$ be six movie subscribers. We use three attributes:

$$T_1 = \text{Genres},\ \ J_1 = \{\text{Action}, \text{Drama}, \text{Comedy}\},$$

$$T_2 = \text{Languages},\ \ J_2 = \{\text{English}, \text{Spanish}\},$$

$$T_3 = \text{Time of Day},\ \ J_3 = \{\text{Morning}, \text{Evening}\},$$

so

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \mathcal{P}(J_3).$$

Define R by "same subscription tier," with classes $\{U_1, U_2, U_3\}$ and $\{U_4, U_5, U_6\}$. Specify F on two combinations:

$$F\big(\{\text{Action}, \text{Comedy}\}, \{\text{English}\}, \{\text{Evening}\}\big) = \{U_1, U_2, U_3\},$$
$$F\big(\{\text{Drama}\}, \{\text{Spanish}\}, \{\text{Morning}\}\big) = \{U_4\},$$
$$F(A) = \varnothing \text{ for all other } A \in J.$$

1. For $A = (\{\text{Action}, \text{Comedy}\}, \{\text{English}\}, \{\text{Evening}\})$,

$$F(A) = \{U_1, U_2, U_3\}, \quad [U_i]_R = \{U_1, U_2, U_3\}\ (i = 1, 2, 3).$$

Hence

$$\underline{F(A)} = \{U_1, U_2, U_3\}, \quad \overline{F(A)} = \{U_1, U_2, U_3\},$$

giving a crisp concept.

2. For $B = (\{\text{Drama}\}, \{\text{Spanish}\}, \{\text{Morning}\})$,

$$F(B) = \{U_4\}, \quad [U_4]_R = \{U_4, U_5, U_6\},$$

so

$$\underline{F(B)} = \varnothing, \quad \overline{F(B)} = \{U_4, U_5, U_6\}.$$

This shows a rough concept: U_5, U_6 are only possible members of the "Drama + Spanish + Morning" group.

1.5 (m, n)-SuperHyperRough Set

The results of this paper do not use it, but for reference, we provide the definition of the (m, n)-SuperHyperRough Set. Let X be a nonempty finite universe and let

$$R \subseteq X \times X$$

be an equivalence relation on X. We write $[x]_R = \{\, y \in X \mid (x, y) \in R\}$ for the R-equivalence class of x. For each $k \geq 0$, define the iterated power set

$$P^0(X) = X, \quad P^{k+1}(X) = \mathcal{P}\big(P^k(X)\big).$$

We will lift R to an equivalence R^k on $P^k(X)$ and then define (m, n)-SuperHyperRough Sets.

Definition 1.13 (Lifted Relation R^k)**.** Define recursively for each $k \geq 0$:

$$R^0 = R \subseteq X \times X,$$

and for $k \geq 1$,

$$R^k \ \subseteq\ P^k(X) \times P^k(X)$$

by declaring, for $A, B \in P^k(X)$,

$$A\, R^k\, B \quad \Longleftrightarrow \quad \big(\forall\, a \in A\ \exists\, b \in B:\ (a, b) \in R^{k-1}\big)\ \wedge\ \big(\forall\, b \in B\ \exists\, a \in A:\ (a, b) \in R^{k-1}\big).$$

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Definition 1.14 ((m,n)-SuperHyperRough Set)**.** Fix integers $m,n \geq 0$. An (m,n)-*SuperHyperRough Set* on (X,R) is a function

$$F:\ P^m(X) \longrightarrow P^n(X).$$

For each $A \in P^m(X)$, set $C = F(A) \in P^n(X)$. Its *lower* and *upper* approximations in $P^{n-1}(X)$ are

$$\underline{C} = \{\, B \in P^{n-1}(X) \mid [B]_{R^{n-1}} \subseteq C\}, \qquad \overline{C} = \{\, B \in P^{n-1}(X) \mid [B]_{R^{n-1}} \cap C \neq \emptyset\},$$

where $[B]_{R^{n-1}} = \{\, D \in P^{n-1}(X) \mid B\, R^{n-1}\, D\}$. Thus each A yields the rough pair $\big(\underline{F(A)},\ \overline{F(A)}\big)$.

Example 1.15 ($(m,n)=(1,2)$: element–to–family case with explicit R^1 computation)**.** Let $X = \{1,2,3,4\}$ and let R be the equivalence on X with two classes

$$C_1 = \{1,2\}, \qquad C_2 = \{3,4\}.$$

For $k \geq 1$ lift R to R^k as in the paper. In particular, for $k=1$ and any $A,B \subseteq X$,

$$A\, R^1\, B \iff (\forall a \in A\ \exists b \in B:\ aRb)\ \wedge\ (\forall b \in B\ \exists a \in A:\ aRb),$$

so $A\, R^1\, B$ holds iff A and B meet exactly the same R-classes (i.e. they have the same "class support"). Thus the R^1-classes in $\mathcal{P}(X)$ are:

$$\begin{array}{ll} \text{support } \varnothing: & \{\varnothing\}, \\ \text{support } \{C_1\}: & \big\{\{1\},\{2\},\{1,2\}\big\}, \\ \text{support } \{C_2\}: & \big\{\{3\},\{4\},\{3,4\}\big\}, \\ \text{support } \{C_1,C_2\}: & \big\{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\big\}. \end{array}$$

Fix $(m,n)=(1,2)$, so the domain is $P^1(X) = \mathcal{P}(X)$ and the codomain is $P^2(X) = \mathcal{P}(\mathcal{P}(X))$. Define a function $F: \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))$ by prescribing its value at

$$A_0 := \{1,3\} \quad \text{and setting} \quad F(A) = \varnothing \ \text{ for all } A \neq A_0.$$

Specifically, let

$$C := F(A_0) = \Big(\underbrace{\{S \subseteq X \mid \varnothing \neq S \subseteq \{1,2\}\}}_{\text{all subsets with support } \{C_1\}}\Big) \ \cup\ \{\{1,3\}\}.$$

Concretely,

$$C = \big\{\{1\},\{2\},\{1,2\},\{1,3\}\big\} \ \subseteq\ \mathcal{P}(X).$$

The lower/upper approximations of C live in $P^{n-1}(X) = P^1(X) = \mathcal{P}(X)$:

$$\underline{C} = \big\{B \in \mathcal{P}(X) \mid [B]_{R^1} \subseteq C\big\}, \qquad \overline{C} = \big\{B \in \mathcal{P}(X) \mid [B]_{R^1} \cap C \neq \varnothing\big\}.$$

Since an R^1-class is the collection of all subsets with a fixed support, we obtain:

1) For support $\{C_1\}$: the entire R^1-class $\{\{1\},\{2\},\{1,2\}\}$ is contained in C; hence

$$\{B : \mathrm{supp}(B) = \{C_1\}\} \ \subseteq\ \underline{C}.$$

2) For support $\{C_2\}$: no member $\{3\},\{4\},\{3,4\}$ lies in C; hence no B with support $\{C_2\}$ belongs to $\underline{C}$ or $\overline{C}$.

3) For support $\{C_1,C_2\}$: only $\{1,3\}$ from this R^1-class lies in C. Therefore the class is not fully contained in C (so it contributes nothing to $\underline{C}$), but it does intersect C (so it contributes everything to $\overline{C}$).

4) For support $\varnothing$: $\varnothing \notin C$, so $\varnothing \notin \underline{C}$ and $\varnothing \notin \overline{C}$.

Hence, explicitly,

$$\underline{C} = \big\{\{1\},\{2\},\{1,2\}\big\},$$

$$\overline{C} = \big\{\{1\},\{2\},\{1,2\}\big\} \ \cup\ \big\{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\big\}.$$

We have $\underline{C} \subsetneq \overline{C}$, so $F(A_0)$ is genuinely rough at level $n=2$.

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Example 1.16 ($(m,n)=(2,1)$: family–to–element case with explicit counts)**.** Let $X=\{a,b,c,d,e\}$ with R given by the partition

$$[a]_R=[b]_R=\{a,b\},\qquad [c]_R=[d]_R=\{c,d\},\qquad [e]_R=\{e\}.$$

Here $P^2(X)=\mathcal{P}(\mathcal{P}(X))$ and $P^1(X)=\mathcal{P}(X)$, so fix $(m,n)=(2,1)$ and define a function

$$F:\ P^2(X)\longrightarrow P^1(X)=\mathcal{P}(X),\qquad F(\mathcal{A}):=\bigcup_{S\in\mathcal{A}} S \quad \text{(set–union operator)}.$$

Choose the concrete input

$$\mathcal{A}_0=\big\{\{a\},\{c,e\}\big\}\ \in P^2(X).$$

Then

$$C:=F(\mathcal{A}_0)=\{a,c,e\}\ \subseteq X.$$

Since $n=1$, the lower/upper approximations lie in $P^{n-1}(X)=P^0(X)=X$ and use $R^0=R$:

$$\underline{C}=\big\{x\in X\mid [x]_R\subseteq C\big\},\qquad \overline{C}=\big\{x\in X\mid [x]_R\cap C\neq\varnothing\big\}.$$

Compute elementwise (writing the numerator/denominator sizes explicitly for clarity):

$$\begin{array}{lll}
x=a: & [a]_R=\{a,b\},\ \ |[a]_R\cap C|=|\{a,b\}\cap\{a,c,e\}|=1\ (>0), & [a]_R\nsubseteq C\\
x=b: & [b]_R=\{a,b\},\ \ |[b]_R\cap C|=1\ (>0), & [b]_R\nsubseteq C\\
x=c: & [c]_R=\{c,d\},\ \ |[c]_R\cap C|=1\ (>0), & [c]_R\nsubseteq C\\
x=d: & [d]_R=\{c,d\},\ \ |[d]_R\cap C|=1\ (>0), & [d]_R\nsubseteq C\\
x=e: & [e]_R=\{e\},\ \ \ |[e]_R\cap C|=1\ (>0), & [e]_R\subseteq C.
\end{array}$$

Therefore

$$\underline{C}=\{e\},\qquad \overline{C}=\{a,b,c,d,e\}=X.$$

Thus with $(m,n)=(2,1)$ the output $F(\mathcal{A}_0)$ produces a classical rough pair on X (lower/upper at level $n-1=0$), and the boundary region is $X\setminus\{e\}$.

2 Results of This Paper

This section presents the results obtained in this paper.

2.1 Graded HyperRough Set and Graded SuperHyperRough Set

The Graded HyperRough Set and the Graded SuperHyperRough Set are defined as follows.

Definition 2.1 (Graded HyperRough Set)**.** Let X be a nonempty universe with equivalence relation R, and let $T_1,\ldots,T_n$ be n attributes with domains $J_1,\ldots,J_n$. Define

$$J=J_1\times J_2\times\cdots\times J_n.$$

Let $F:J\to\mathcal{P}(X)$ be a mapping. For each $a\in J$, consider the subset $F(a)\subseteq X$. Define the *rough membership function* with respect to $F(a)$ by

$$\mu_R^{F(a)}(x)=\frac{|[x]_R\cap F(a)|}{|[x]_R|},\quad \text{for all } x\in X.$$

For a fixed threshold $k\in(0.5,1]$ the *graded hyperrough approximations* of $F(a)$ are defined as

$$\underline{F}^k(a)=\{x\in X\mid \mu_R^{F(a)}(x)\geq k\}$$

and

$$\overline{F}^k(a)=\{x\in X\mid \mu_R^{F(a)}(x)>1-k\}.$$

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Then the ordered pair

$$R_F^k(a) = \left(\underline{F}^k(a), \overline{F}^k(a)\right)$$

represents the graded hyperrough set approximation of $F(a)$. The collection

$$\{ R_F^k(a) \mid a \in J \}$$

is called the *Graded HyperRough Set* of F at precision level k.

Theorem 2.2 (Generalization of HyperRough Set and Graded Rough Set)**.** *Let the notation be as in the previous definition.*

1. *If $k = 1$, then for each $a \in J$ and for every $x \in X$ we have*

$$\mu_R^{F(a)}(x) \geq 1 \iff [x]_R \subseteq F(a).$$

Consequently,

$$\underline{F}^1(a) = \{x \in X \mid [x]_R \subseteq F(a)\} \quad \textit{and} \quad \overline{F}^1(a) = \{x \in X \mid [x]_R \cap F(a) \neq \varnothing\}.$$

That is, setting $k = 1$ in the graded hyperrough set definition recovers the classical hyperrough set.

2. *If the attribute space J is a singleton (or if F is constant over J), then the mapping F defines a unique subset $A \subseteq X$ and the graded hyperrough set reduces to the graded rough set approximation of A.*

Proof. (i) For any $x \in X$, note that by definition $\mu_R^{F(a)}(x) = \frac{|[x]_R \cap F(a)|}{|[x]_R|} \leq 1$. Therefore, $\mu_R^{F(a)}(x) \geq 1$ if and only if

$$|[x]_R \cap F(a)| = |[x]_R|,$$

which holds if and only if $[x]_R \subseteq F(a)$. In addition, since $1 - k = 0$ when $k = 1$, we have $\mu_R^{F(a)}(x) > 0$ if and only if $|[x]_R \cap F(a)| > 0$; that is,

$$\overline{F}^1(a) = \{x \in X \mid [x]_R \cap F(a) \neq \varnothing\}.$$

Thus, the graded approximations with $k = 1$ exactly coincide with the classical hyperrough approximations.
(ii) If J is a singleton (say, $J = \{a_0\}$), then F defines a single subset $F(a_0) = A$ of X. Hence the definitions reduce to

$$\underline{F}^k(a_0) = \{x \in X \mid \mu_R^A(x) \geq k\} \quad \text{and} \quad \overline{F}^k(a_0) = \{x \in X \mid \mu_R^A(x) > 1 - k\},$$

which is precisely the graded rough set of A. □

Theorem 2.3 (Inclusion Property)**.** *For every $a \in J$ and threshold $k \in (0.5, 1]$,*

$$\underline{F}^k(a) \subseteq \overline{F}^k(a).$$

Proof. Fix $a \in J$. By definition,

$$\underline{F}^k(a) = \{x \in X \mid \mu_R^{F(a)}(x) \geq k\}, \quad \overline{F}^k(a) = \{x \in X \mid \mu_R^{F(a)}(x) > 1 - k\}.$$

Since $k > 0.5$ we have $k > 1 - k$. Thus if $\mu_R^{F(a)}(x) \geq k$, then in particular $\mu_R^{F(a)}(x) > 1 - k$, so $x \in \overline{F}^k(a)$. Hence $\underline{F}^k(a) \subseteq \overline{F}^k(a)$. □

Theorem 2.4 (Monotonicity in k)**.** *Let $0.5 < k_1 \leq k_2 \leq 1$. Then for every $a \in J$:*

$$\underline{F}^{k_2}(a) \subseteq \underline{F}^{k_1}(a), \quad \overline{F}^{k_1}(a) \subseteq \overline{F}^{k_2}(a).$$

Proof. Suppose $x \in \underline{F}^{k_2}(a)$. Then $\mu_R^{F(a)}(x) \geq k_2$. Since $k_2 \geq k_1$, it follows $\mu_R^{F(a)}(x) \geq k_1$, so $x \in \underline{F}^{k_1}(a)$. Thus $\underline{F}^{k_2}(a) \subseteq \underline{F}^{k_1}(a)$.

Next, assume $x \in \overline{F}^{k_1}(a)$. Then $\mu_R^{F(a)}(x) > 1 - k_1$. Because $k_2 \geq k_1$, we have $1 - k_2 \leq 1 - k_1$, so $\mu_R^{F(a)}(x) > 1 - k_2$. Hence $x \in \overline{F}^{k_2}(a)$ and $\overline{F}^{k_1}(a) \subseteq \overline{F}^{k_2}(a)$. □

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Theorem 2.5 (Monotonicity in Concept)**.** *If $a, b \in J$ satisfy $F(a) \subseteq F(b)$ then, for any $k \in (0.5, 1]$,*

$$\underline{F}^k(a) \subseteq \underline{F}^k(b), \quad \overline{F}^k(a) \subseteq \overline{F}^k(b).$$

Proof. Fix $x \in X$. Since $[x]_R \cap F(a) \subseteq [x]_R \cap F(b)$, we have

$$\mu_R^{F(a)}(x) = \frac{|[x]_R \cap F(a)|}{|[x]_R|} \leq \frac{|[x]_R \cap F(b)|}{|[x]_R|} = \mu_R^{F(b)}(x).$$

Hence:

- If $x \in \underline{F}^k(a)$, then $\mu_R^{F(a)}(x) \geq k$ and so $\mu_R^{F(b)}(x) \geq k$; thus $x \in \underline{F}^k(b)$.
- If $x \in \overline{F}^k(a)$, then $\mu_R^{F(a)}(x) > 1 - k$ and so $\mu_R^{F(b)}(x) > 1 - k$; hence $x \in \overline{F}^k(b)$.

This yields the desired inclusions. □

Theorem 2.6 (Boundary Region and k-Exactness)**.** *Define the boundary region*

$$\mathrm{BN}_F^k(a) = \overline{F}^k(a) \setminus \underline{F}^k(a).$$

Then for each $a \in J$:

$$\mathrm{BN}_F^k(a) = \varnothing \iff \underline{F}^k(a) = \overline{F}^k(a),$$

*i.e. $F(a)$ is k-*exact *if and only if its boundary region is empty.*

Proof. By definition $\mathrm{BN}_F^k(a) = \varnothing$ means $\overline{F}^k(a) \subseteq \underline{F}^k(a)$. Combined with the Inclusion Property ($\underline{F}^k(a) \subseteq \overline{F}^k(a)$), we obtain $\underline{F}^k(a) = \overline{F}^k(a)$. Conversely, if $\underline{F}^k(a) = \overline{F}^k(a)$, then certainly no element lies strictly in the upper without being in the lower, so $\mathrm{BN}_F^k(a) = \varnothing$. □

Example 2.7 (Graded HyperRough Set)**.** Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose the equivalence relation R partitions X into

$$[1]_R = \{1, 2, 3\}, \quad [4]_R = \{4, 5\}, \quad [6]_R = \{6\}.$$

Assume two attributes with domains $J_1 = \{a, b\}$ and $J_2 = \{0, 1\}$. Then the Cartesian product is

$$J = J_1 \times J_2.$$

Suppose the mapping F is given by:

$$F(a, 0) = \{1, 2, 4\}, \quad F(a, 1) = \{1, 2, 3, 6\}, \quad F(b, 0) = \{4, 5\}, \quad F(b, 1) = \{6\}.$$

Choose $k = 0.8$. Then, for example, for $a = (a, 1)$ we compute for any $x \in [1]_R = \{1, 2, 3\}$:

$$\mu_R^{F(a,1)}(x) = \frac{|\{1, 2, 3\} \cap \{1, 2, 3, 6\}|}{3} = \frac{3}{3} = 1.$$

Thus, $\{1, 2, 3\} \subseteq \underline{F}^{0.8}(a, 1)$. Similar calculations can be performed for the other attribute combinations.

When the attribute domain is expanded to include all possible subsets, we have the notion of a SuperHyperRough Set. We now define its graded version.

Definition 2.8 (Graded SuperHyperRough Set)**.** Let X be a nonempty universe with equivalence relation R and let $T_1, \ldots, T_n$ be attributes with respective domains $J_1, \ldots, J_n$. Define the domain

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n),$$

where $\mathcal{P}(J_i)$ is the power set of J_i. Let

$$F : J \to \mathcal{P}(X)$$

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be a mapping. For each $A = (A_1, A_2, \ldots, A_n) \in J$, define the rough membership function with respect to $F(A)$ by

$$\mu_R^{F(A)}(x) = \frac{|[x]_R \cap F(A)|}{|[x]_R|}, \quad \text{for all } x \in X.$$

Then, for a fixed threshold $k \in (0.5, 1]$, define the *graded superhyperrough approximations* by

$$\underline{F}^k(A) = \{x \in X \mid \mu_R^{F(A)}(x) \geq k\}$$

and

$$\overline{F}^k(A) = \{x \in X \mid \mu_R^{F(A)}(x) > 1 - k\}.$$

The pair

$$R_F^k(A) = \Big(\underline{F}^k(A), \, \overline{F}^k(A)\Big)$$

is the graded superhyperrough approximation of $F(A)$, and the collection

$$\{ \, R_F^k(A) \mid A \in J \, \}$$

is called the *Graded SuperHyperRough Set* of F at precision level k.

Theorem 2.9 (Inclusion Property). *For every $A \in J$ and $k \in (0.5, 1]$,*

$$\underline{F}^k(A) \; \subseteq \; \overline{F}^k(A).$$

Proof. By definition,

$$\underline{F}^k(A) = \{x \in X \mid \mu_R^{F(A)}(x) \geq k\}, \quad \overline{F}^k(A) = \{x \in X \mid \mu_R^{F(A)}(x) > 1 - k\}.$$

Since $k > 1 - k$ whenever $k > 0.5$, any x satisfying $\mu_R^{F(A)}(x) \geq k$ also satisfies $\mu_R^{F(A)}(x) > 1 - k$. Hence $\underline{F}^k(A) \subseteq \overline{F}^k(A)$. □

Theorem 2.10 (Monotonicity in k). *If $0.5 < k_1 \leq k_2 \leq 1$, then for every $A \in J$,*

$$\underline{F}^{k_2}(A) \; \subseteq \; \underline{F}^{k_1}(A), \quad \overline{F}^{k_1}(A) \; \subseteq \; \overline{F}^{k_2}(A).$$

Proof. Assume $x \in \underline{F}^{k_2}(A)$. Then $\mu_R^{F(A)}(x) \geq k_2 \geq k_1$, so $x \in \underline{F}^{k_1}(A)$. Thus $\underline{F}^{k_2}(A) \subseteq \underline{F}^{k_1}(A)$.

Next, assume $x \in \overline{F}^{k_1}(A)$. Then $\mu_R^{F(A)}(x) > 1 - k_1$. Since $k_2 \geq k_1$ implies $1 - k_2 \leq 1 - k_1$, we have $\mu_R^{F(A)}(x) > 1 - k_2$, so $x \in \overline{F}^{k_2}(A)$. Hence $\overline{F}^{k_1}(A) \subseteq \overline{F}^{k_2}(A)$. □

Theorem 2.11 (Monotonicity in Concept). *If $A, B \in J$ satisfy $F(A) \subseteq F(B)$, then for any $k \in (0.5, 1]$,*

$$\underline{F}^k(A) \; \subseteq \; \underline{F}^k(B), \quad \overline{F}^k(A) \; \subseteq \; \overline{F}^k(B).$$

Proof. For each $x \in X$,

$$[x]_R \cap F(A) \; \subseteq \; [x]_R \cap F(B),$$

so

$$\mu_R^{F(A)}(x) = \frac{|[x]_R \cap F(A)|}{|[x]_R|} \leq \frac{|[x]_R \cap F(B)|}{|[x]_R|} = \mu_R^{F(B)}(x).$$

Hence:

- If $x \in \underline{F}^k(A)$, then $\mu_R^{F(A)}(x) \geq k$ implies $\mu_R^{F(B)}(x) \geq k$, so $x \in \underline{F}^k(B)$.
- If $x \in \overline{F}^k(A)$, then $\mu_R^{F(A)}(x) > 1 - k$ implies $\mu_R^{F(B)}(x) > 1 - k$, so $x \in \overline{F}^k(B)$.

Thus both inclusions hold. □

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Theorem 2.12 (Boundary Region and k-Exactness)**.** *Define the boundary region*

$$\mathrm{BN}_F^k(A) = \overline{F}^k(A) \setminus \underline{F}^k(A).$$

Then for each $A \in J$,

$$\mathrm{BN}_F^k(A) = \varnothing \quad \Longleftrightarrow \quad \underline{F}^k(A) = \overline{F}^k(A),$$

i.e. $F(A)$ is k-exact *if and only if its boundary region is empty.*

Proof. ($\Rightarrow$) If $\mathrm{BN}_F^k(A) = \varnothing$, then $\overline{F}^k(A) \subseteq \underline{F}^k(A)$. Together with the Inclusion Property, we get $\underline{F}^k(A) = \overline{F}^k(A)$.

($\Leftarrow$) Conversely, if $\underline{F}^k(A) = \overline{F}^k(A)$, then no element lies exclusively in the upper approximation, so the boundary is empty. □

Theorem 2.13 (Generalization)**.** *Under the notation of Definition:*

1. *Setting $k = 1$ recovers the classical n-SuperHyperRough Set approximations:*

$$\underline{F}^1(A) = \{x \mid [x]_R \subseteq F(A)\}, \quad \overline{F}^1(A) = \{x \mid [x]_R \cap F(A) \neq \varnothing\}.$$

2. *Restricting J to $J' = J_1 \times \cdots \times J_n$ yields exactly the Graded HyperRough Set on J'.*

Proof. We prove each part in detail.

(1) Recovery of the classical n-SuperHyperRough Set when $k = 1$.

Fix $A \in J$. Recall

$$\mu_R^{F(A)}(x) = \frac{\big|[x]_R \cap F(A)\big|}{\big|[x]_R\big|},$$

and by construction $0 \leq \mu_R^{F(A)}(x) \leq 1$.

(a) Lower approximation.

$$\underline{F}^1(A) = \{\, x \in X \mid \mu_R^{F(A)}(x) \geq 1\}.$$

Since $\mu_R^{F(A)}(x) \leq 1$, the inequality $\mu_R^{F(A)}(x) \geq 1$ holds precisely when $\mu_R^{F(A)}(x) = 1$. By definition of the ratio,

$$\mu_R^{F(A)}(x) = 1 \iff |[x]_R \cap F(A)| = |[x]_R| \iff [x]_R \subseteq F(A).$$

Hence

$$\underline{F}^1(A) = \{\, x \in X \mid [x]_R \subseteq F(A)\},$$

which is exactly the lower approximation in the classical n-SuperHyperRough Set.

(b) Upper approximation.

$$\overline{F}^1(A) = \{\, x \in X \mid \mu_R^{F(A)}(x) > 1 - 1\} = \{\, x \in X \mid \mu_R^{F(A)}(x) > 0\}.$$

Since $\mu_R^{F(A)}(x) > 0$ if and only if $|[x]_R \cap F(A)| > 0$, we obtain

$$\overline{F}^1(A) = \{\, x \in X \mid [x]_R \cap F(A) \neq \varnothing\},$$

matching the classical upper approximation.

Thus, for $k = 1$, the graded superhyperrough approximations coincide with the classical ones.

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(2) Reduction to the Graded HyperRough Set on $J' = J_1 \times \cdots \times J_n$.

Suppose we restrict our domain from

$$J = \mathcal{P}(J_1) \times \cdots \times \mathcal{P}(J_n)$$

to its Cartesian subproduct

$$J' = J_1 \times \cdots \times J_n.$$

Then each $A \in J'$ has the form

$$A = (\{a_1\}, \{a_2\}, \ldots, \{a_n\})$$

with $a_i \in J_i$. In this case the mapping

$$F\colon J \to \mathcal{P}(X)$$

when restricted to J' coincides with the mapping used in the definition of the *Graded HyperRough Set*:

$$F'(a_1, \ldots, a_n) := F\big(\{a_1\}, \{a_2\}, \ldots, \{a_n\}\big).$$

But then for any $x \in X$,

$$\mu_R^{F(A)}(x) = \frac{\big|[x]_R \cap F(A)\big|}{\big|[x]_R\big|} = \frac{\big|[x]_R \cap F'(a_1, \ldots, a_n)\big|}{\big|[x]_R\big|},$$

which is exactly the rough membership function for the graded hyperrough approximation of $F'(a_1, \ldots, a_n)$. Consequently, for any $k \in (0.5, 1]$,

$$\underline{F}^k(A) = \{x \mid \mu_R^{F'(a_1,\ldots,a_n)}(x) \geq k\}, \quad \overline{F}^k(A) = \{x \mid \mu_R^{F'(a_1,\ldots,a_n)}(x) > 1 - k\},$$

and these are precisely the $\underline{R}^k$ and $\overline{R}^k$ sets in the Graded HyperRough framework.

Thus, restricting J to J' recovers the Graded HyperRough Set on the product $J_1 \times \cdots \times J_n$. □

Example 2.14 (Real-life Example of a Graded 2-SuperHyperRough Set)**.** Let $X = \{1, 2, 3, 4, 5, 6\}$ be six online shoppers. Define an equivalence relation R by "same postal region," with classes

$$[1]_R = [2]_R = [3]_R = \{1, 2, 3\}, \quad [4]_R = [5]_R = \{4, 5\}, \quad [6]_R = \{6\}.$$

We use two attributes:

$$T_1 = \text{Loyalty Tier},\ J_1 = \{\text{Silver}, \text{Gold}\}, \qquad T_2 = \text{Purchase Channel},\ J_2 = \{\text{Online}, \text{InStore}\}.$$

Then

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Select one combination

$$A = (\{\text{Gold}\}, \{\text{Online}\}),$$

and define

$$F(A) = \{1, 2, 4\}.$$

For each $x \in X$, the rough membership degree is

$$\mu_R^{F(A)}(x) = \frac{\big|[x]_R \cap F(A)\big|}{\big|[x]_R\big|}.$$

Compute:

$$\mu_R^{F(A)}(1) = \frac{|\{1,2,3\} \cap \{1,2,4\}|}{3} = \frac{2}{3} \approx 0.67,$$
$$\mu_R^{F(A)}(4) = \frac{|\{4,5\} \cap \{1,2,4\}|}{2} = \frac{1}{2} = 0.5,$$
$$\mu_R^{F(A)}(6) = \frac{|\{6\} \cap \{1,2,4\}|}{1} = 0.$$

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Fix threshold $k = 0.7$. Then

$$\underline{F}^{0.7}(A) = \{x \in X \mid \mu_R^{F(A)}(x) \geq 0.7\} = \varnothing,$$

since no class attains degree ≥ 0.7, and

$$\overline{F}^{0.7}(A) = \{x \in X \mid \mu_R^{F(A)}(x) > 0.3\} = \{1, 2, 3, 4, 5\},$$

because $0.67 > 0.3$ and $0.5 > 0.3$. Thus the graded 2-SuperHyperRough approximation at $k = 0.7$ is

$$R_F^{0.7}(A) = \big(\varnothing,\ \{1, 2, 3, 4, 5\}\big).$$

Example 2.15 (Real-life Example of a Graded 3-SuperHyperRough Set)**.** Let $X = \{1, 2, 3, 4, 5, 6\}$ and equivalence classes as above. Use three attributes:

$$T_1 = \text{Loyalty Tier},\ J_1 = \{\text{Bronze}, \text{Silver}, \text{Gold}\},$$

$$T_2 = \text{Channel},\ J_2 = \{\text{Online}, \text{InStore}\},$$

$$T_3 = \text{Feedback},\ J_3 = \{\text{Positive}, \text{Negative}\}.$$

Form $\mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \mathcal{P}(J_3)$. Choose

$$B = (\{\text{Silver}, \text{Gold}\}, \{\text{Online}\}, \{\text{Positive}\}),$$

and let

$$F(B) = \{1, 2, 3, 4\}.$$

Then for each $x \in X$,

$$\mu_R^{F(B)}(x) = \frac{\big|[x]_R \cap \{1, 2, 3, 4\}\big|}{|[x]_R|}.$$

Compute:

$$\mu_R^{F(B)}(1) = \frac{|\{1, 2, 3\} \cap \{1, 2, 3, 4\}|}{3} = 1, \quad \mu_R^{F(B)}(4) = \frac{|\{4, 5\} \cap \{1, 2, 3, 4\}|}{2} = \frac{1}{2} = 0.5, \quad \mu_R^{F(B)}(6) = 0.$$

Set $k = 0.8$. Then

$$\underline{F}^{0.8}(B) = \{x \mid \mu_R^{F(B)}(x) \geq 0.8\} = \{1, 2, 3\},$$

since only the first class has degree $1 \geq 0.8$, and

$$\overline{F}^{0.8}(B) = \{x \mid \mu_R^{F(B)}(x) > 0.2\} = \{1, 2, 3, 4, 5\},$$

because $0.5 > 0.2$. Therefore, the graded 3-SuperHyperRough approximation is

$$R_F^{0.8}(B) = \big(\{1, 2, 3\},\ \{1, 2, 3, 4, 5\}\big).$$

This illustrates how graded superhyperrough sets capture nuanced membership levels across combinations of three attribute subsets.

2.2 Linguistic HyperRough Set and Linguistic SuperHyperRough Set

The Linguistic HyperRough Set and the Linguistic SuperHyperRough Set are defined as follows.

Definition 2.16 (Linguistic HyperRough Set)**.** Let X be a universe with equivalence relation R and let $T_1, T_2, \ldots, T_n$ be n attributes with domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $F : J \to \mathcal{P}(X)$ be a mapping that assigns to each $a \in J$ a subset $F(a) \subseteq X$.

Assume that S is a finite, totally ordered set of linguistic labels and that $\ell : [0, 1] \to S$ is a given linguistic mapping.

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For each $a \in J$ and each $x \in X$, define the rough membership function with respect to $F(a)$ by

$$\mu_R^{F(a)}(x) = \frac{|[x]_R \cap F(a)|}{|[x]_R|}.$$

Then the *linguistic evaluation* of x is given by

$$\lambda^{F(a)}(x) = \ell\Big(\mu_R^{F(a)}(x)\Big).$$

Finally, define the linguistic approximation of $F(a)$ by

$$LHS(F(a)) = \{\, ([x]_R,\, \lambda^{F(a)}(x)) \mid x \in X \,\},$$

and the *Linguistic HyperRough Set* associated with F by

$$LHRS(F) = \{(a,\, LHS(F(a))) \mid a \in J\}.$$

Definition 2.17 (Linguistic Approximations)**.** Let $s_\theta, s_\varphi \in S$ be two linguistic thresholds with $s_\theta \succeq s_\varphi$. For each $a \in J$, define

$$L^\theta_{LHRS}(F(a)) = \big\{[x]_R \mid \lambda^{F(a)}(x) \succeq s_\theta\big\}, \qquad U^\varphi_{LHRS}(F(a)) = \big\{[x]_R \mid \lambda^{F(a)}(x) \succ s_\varphi\big\}.$$

These are called the *linguistic lower* and *upper approximations* of $F(a)$.

Theorem 2.18 (Inclusion Property)**.** *For any $a \in J$ and thresholds $s_\theta \succeq s_\varphi$,*

$$L^\theta_{LHRS}(F(a)) \;\subseteq\; U^\varphi_{LHRS}(F(a)).$$

Proof. Let $[x]_R \in L^\theta_{LHRS}(F(a))$. Then $\lambda^{F(a)}(x) \succeq s_\theta$. Since $s_\theta \succeq s_\varphi$, we have $\lambda^{F(a)}(x) \succ s_\varphi$. Thus $[x]_R \in U^\varphi_{LHRS}(F(a))$, proving the inclusion. □

Theorem 2.19 (Monotonicity in Thresholds)**.** *If $s_{\theta_1} \succeq s_{\theta_2}$ and $s_{\varphi_1} \succeq s_{\varphi_2}$, then for any $a \in J$:*

$$L^{\theta_1}_{LHRS}(F(a)) \;\subseteq\; L^{\theta_2}_{LHRS}(F(a)), \quad U^{\varphi_1}_{LHRS}(F(a)) \;\subseteq\; U^{\varphi_2}_{LHRS}(F(a)).$$

Proof. Suppose $[x]_R \in L^{\theta_1}_{LHRS}(F(a))$. Then $\lambda^{F(a)}(x) \succeq s_{\theta_1} \succeq s_{\theta_2}$, so $[x]_R \in L^{\theta_2}_{LHRS}(F(a))$. The argument for the upper approximations is analogous: $\lambda^{F(a)}(x) \succ s_{\varphi_1} \succeq s_{\varphi_2}$ implies $[x]_R \in U^{\varphi_2}_{LHRS}(F(a))$. □

Theorem 2.20 (Generalization)**.** *The Linguistic HyperRough Set specializes to:*

1. *the classical HyperRough Set when ℓ is the indicator mapping $\mathbf{1}_{t>0}$;*
2. *the Linguistic Rough Set when J is a singleton so that $F(a) = A \subseteq X$.*

Proof. We show each specialization in turn, with all intermediate steps made explicit.

(1) Recovery of the classical HyperRough Set via the indicator mapping.

Suppose we choose the linguistic mapping $\ell : [0,1] \to \{0,1\}$ by

$$\ell(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0. \end{cases}$$

Then for any $a \in J$ and any $x \in X$, by definition

$$\mu_R^{F(a)}(x) = \frac{\big|[x]_R \cap F(a)\big|}{\big|[x]_R\big|}\,.$$

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Hence:

$$\lambda^{F(a)}(x) = \ell\big(\mu_R^{F(a)}(x)\big) = \begin{cases} 1, & \text{if } \mu_R^{F(a)}(x) > 0, \\ 0, & \text{if } \mu_R^{F(a)}(x) = 0. \end{cases}$$

But $\mu_R^{F(a)}(x) > 0$ exactly when $[x]_R \cap F(a) \neq \varnothing$. Therefore:

$$\lambda^{F(a)}(x) = 1 \iff [x]_R \cap F(a) \neq \varnothing,$$

and

$$\lambda^{F(a)}(x) = 0 \iff [x]_R \cap F(a) = \varnothing.$$

Recalling the definition

$$LHS(F(a)) = \big\{\big([x]_R, \lambda^{F(a)}(x)\big) \mid x \in X\big\},$$

we see that $LHS(F(a))$ partitions the granules $[x]_R$ into two groups:

$$\{\,([x]_R, 1) \mid [x]_R \cap F(a) \neq \varnothing\} \quad \text{and} \quad \{\,([x]_R, 0) \mid [x]_R \cap F(a) = \varnothing\}.$$

This is exactly the information contained in the classical HyperRough Set approximations:

$$\overline{F(a)} = \{x \mid [x]_R \cap F(a) \neq \varnothing\}, \quad \underline{F(a)} = \{x \mid [x]_R \subseteq F(a)\}.$$

In particular, those granules labeled "1" correspond to the upper approximation, and those labeled "0" to its complement. Thus the linguistic description $LHS(F(a))$ reproduces the classical HyperRough Set.

(2) Reduction to the Linguistic Rough Set when J is a singleton.

Now assume $J = \{a_0\}$ consists of a single element, so $F(a_0) = A \subseteq X$. Then

$$LHRS(F) = \{\,\big(a_0, LHS(F(a_0))\big)\} = \{\,\big(a_0, LHS(A)\big)\}.$$

By Definition, the standard *Linguistic Rough Set* of A is

$$LRS(A) = \{\,\big([x]_R, \ell\big(\mu_R^A(x)\big)\big) \mid x \in X\big\},$$

which precisely coincides with $LHS(A)$. Hence the single pair $(a_0, LHS(A))$ in $LHRS(F)$ is exactly the Linguistic Rough Set of A.

Therefore, in these two limiting cases the Linguistic HyperRough Set framework specializes exactly to the classical HyperRough Set and to the standard Linguistic Rough Set, respectively. □

Example 2.21 (Linguistic HyperRough Set Example)**.** Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose the equivalence relation R partitions X as follows:

$$[1]_R = \{1, 2, 3\}, \quad [4]_R = \{4, 5\}, \quad [6]_R = \{6\}.$$

Consider two attributes with domains $J_1 = \{a, b\}$ and $J_2 = \{0, 1\}$ so that

$$J = J_1 \times J_2.$$

Define the mapping $F : J \to \mathcal{P}(X)$ by

$$F(a, 0) = \{1, 2, 4\}, \quad F(a, 1) = \{1, 2, 3, 6\}, \quad F(b, 0) = \{4, 5\}, \quad F(b, 1) = \{6\}.$$

Assume the linguistic label set is

$$S = \{\text{low, medium, high}\}$$

with the ordering low $<$ medium $<$ high and the linguistic mapping defined by

$$\ell(t) = \begin{cases} \text{low}, & 0 \leq t < 0.5, \\ \text{medium}, & 0.5 \leq t < 0.8, \\ \text{high}, & 0.8 \leq t \leq 1. \end{cases}$$

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For $a = (a, 1)$ we have $F(a, 1) = \{1, 2, 3, 6\}$. Then for any $x \in [1]_R = \{1, 2, 3\}$,

$$\mu_R^{F(a,1)}(x) = \frac{|\{1, 2, 3\} \cap \{1, 2, 3, 6\}|}{3} = 1,$$

so that $\lambda^{F(a,1)}(x) = \ell(1) =$ high. For $x \in [4]_R = \{4, 5\}$,

$$\mu_R^{F(a,1)}(x) = \frac{|\{4, 5\} \cap \{1, 2, 3, 6\}|}{2} = 0,$$

giving $\lambda^{F(a,1)}(x) = \ell(0) =$ low. Finally, for $x \in [6]_R = \{6\}$ we have

$$\mu_R^{F(a,1)}(6) = \frac{|\{6\} \cap \{1, 2, 3, 6\}|}{1} = 1,$$

so that $\lambda^{F(a,1)}(6) =$ high. Thus, one obtains

$$LHS(F(a, 1)) = \{(\{1, 2, 3\}, \text{high}),\ (\{4, 5\}, \text{low}),\ (\{6\}, \text{high})\}.$$

Therefore, the Linguistic HyperRough Set is

$$LHRS(F) = \{(a, 0, LHS(F(a, 0)),\ (a, 1, LHS(F(a, 1)),\ (b, 0, LHS(F(b, 0)),\ (b, 1, LHS(F(b, 1))\}.$$

Definition 2.22 (Linguistic SuperHyperRough Set)**.** Let X be a universe with equivalence relation R and let $T_1, T_2, \ldots, T_n$ be n attributes with respective domains $J_1, J_2, \ldots, J_n$. Define the expanded attribute domain

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n),$$

where $\mathcal{P}(J_i)$ denotes the power set of J_i. Let $F : J \to \mathcal{P}(X)$ be a mapping that assigns to each attribute set combination $A = (A_1, A_2, \ldots, A_n) \in J$ a subset $F(A) \subseteq X$.

Assume a fixed linguistic mapping $\ell : [0, 1] \to S$, where S is a finite totally ordered set of linguistic labels.

For each $A \in J$ and for each $x \in X$ define the rough membership function

$$\mu_R^{F(A)}(x) = \frac{|[x]_R \cap F(A)|}{|[x]_R|},$$

and then the corresponding linguistic evaluation

$$\lambda^{F(A)}(x) = \ell\Big(\mu_R^{F(A)}(x)\Big).$$

Define the linguistic approximation of $F(A)$ by

$$LSHS(F(A)) = \{\, ([x]_R,\ \lambda^{F(A)}(x)) \mid x \in X \,\}.$$

Finally, the *Linguistic SuperHyperRough Set* associated with F is defined as

$$LSHRS(F) = \{\, (A,\ LSHS(F(A))) \mid A \in J \,\}.$$

Definition 2.23 (Linguistic Lower and Upper Approximations)**.** Fix two linguistic thresholds $s_\theta, s_\varphi \in S$ with $s_\theta \succeq s_\varphi$. For each $A \in J$, define

$$L^\theta_{LSHRS}(F(A)) = \big\{[x]_R \mid \lambda^{F(A)}(x) \succeq s_\theta\big\}, \qquad U^\varphi_{LSHRS}(F(A)) = \big\{[x]_R \mid \lambda^{F(A)}(x) \succ s_\varphi\big\}.$$

These are called the *linguistic lower* and *linguistic upper* approximations of $F(A)$ in the SuperHyperRough setting.

Theorem 2.24 (Inclusion Property)**.** *For any $A \in J$ and thresholds $s_\theta \succeq s_\varphi$,*

$$L^\theta_{LSHRS}(F(A)) \ \subseteq\ U^\varphi_{LSHRS}(F(A)).$$

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Proof. Take an arbitrary granule $[x]_R \in L^{\theta}_{LSHRS}(F(A))$. By definition,

$$\lambda^{F(A)}(x) \succeq s_\theta.$$

Since $s_\theta \succeq s_\varphi$ in the total order on S, it follows that $\lambda^{F(A)}(x) \succ s_\varphi$. Hence $[x]_R$ belongs to

$$U^{\varphi}_{LSHRS}(F(A)) = \{[y]_R \mid \lambda^{F(A)}(y) \succ s_\varphi\},$$

and the inclusion is proved. □

Theorem 2.25 (Monotonicity in Thresholds)**.** *Let $s_{\theta_1} \succeq s_{\theta_2}$ and $s_{\varphi_1} \succeq s_{\varphi_2}$. Then for every $A \in J$:*

$$L^{\theta_1}_{LSHRS}(F(A)) \subseteq L^{\theta_2}_{LSHRS}(F(A)), \quad U^{\varphi_1}_{LSHRS}(F(A)) \subseteq U^{\varphi_2}_{LSHRS}(F(A)).$$

Proof. (*Lower approximations*) Suppose $[x]_R \in L^{\theta_1}_{LSHRS}(F(A))$. Then $\lambda^{F(A)}(x) \succeq s_{\theta_1}$. Since $s_{\theta_1} \succeq s_{\theta_2}$, we get $\lambda^{F(A)}(x) \succeq s_{\theta_2}$, so $[x]_R \in L^{\theta_2}_{LSHRS}(F(A))$.

(*Upper approximations*) Suppose $[x]_R \in U^{\varphi_1}_{LSHRS}(F(A))$. Then $\lambda^{F(A)}(x) \succ s_{\varphi_1}$. Because $s_{\varphi_1} \succeq s_{\varphi_2}$, it follows $\lambda^{F(A)}(x) \succ s_{\varphi_2}$, hence $[x]_R \in U^{\varphi_2}_{LSHRS}(F(A))$. □

Theorem 2.26 (Generalization)**.** *The Linguistic SuperHyperRough Set specializes to:*

1. *the classical SuperHyperRough Set when ℓ is the indicator $\ell(t) = \mathbf{1}_{t>0}$;*
2. *the Linguistic HyperRough Set when J is restricted to $J' = J_1 \times \cdots \times J_n$.*

Proof. We prove each specialization by unpacking all definitions and verifying term-by-term equivalence.

(1) Recovery of the classical SuperHyperRough Set via the indicator mapping.

Let $\ell : [0,1] \to \{0,1\}$ be defined by

$$\ell(t) \;=\; \begin{cases} 1, & t > 0, \\ 0, & t = 0. \end{cases}$$

Fix any $A \in J$ and $x \in X$. By definition,

$$\mu_R^{F(A)}(x) = \frac{|[x]_R \cap F(A)|}{|[x]_R|} \;\in [0,1].$$

Thus

$$\lambda^{F(A)}(x) = \ell\big(\mu_R^{F(A)}(x)\big) = \begin{cases} 1, & \text{if } |[x]_R \cap F(A)| > 0, \\ 0, & \text{if } |[x]_R \cap F(A)| = 0. \end{cases}$$

Since $|[x]_R \cap F(A)| > 0$ exactly when the granule $[x]_R$ intersects $F(A)$, we have:

$$\lambda^{F(A)}(x) = 1 \quad \Longleftrightarrow \quad [x]_R \cap F(A) \neq \varnothing,$$

$$\lambda^{F(A)}(x) = 0 \quad \Longleftrightarrow \quad [x]_R \cap F(A) = \varnothing.$$

Therefore the set

$$LSHS(F(A)) = \big\{\big([x]_R, \lambda^{F(A)}(x)\big) \mid x \in X\big\}$$

decomposes into two parts:

$$\{([x]_R, 1) \mid [x]_R \cap F(A) \neq \varnothing\} \quad \cup \quad \{([x]_R, 0) \mid [x]_R \cap F(A) = \varnothing\}.$$

But this exactly encodes the classical SuperHyperRough lower and upper approximations:

$$\underline{F(A)} = \{x \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \mid [x]_R \cap F(A) \neq \varnothing\}.$$

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In particular, granules labeled “1” correspond to the upper approximation, and those labeled “0” cover all others. Hence $LSHS(F(A))$ recovers the SuperHyperRough Set of $F(A)$.

(2) Reduction to the Linguistic HyperRough Set upon restricting J.

Now let $J' = J_1 \times \cdots \times J_n \subseteq J$. Then each element of J' is a tuple of single attribute values rather than subsets. Denote by F' the restriction $F|_{J'} : J' \to \mathcal{P}(X)$. For any $\alpha = (a_1, \ldots, a_n) \in J'$ and $x \in X$, the rough membership

$$\mu_R^{F'(\alpha)}(x) = \frac{|[x]_R \cap F'(\alpha)|}{|[x]_R|}$$

agrees with $\mu_R^{F(A)}(x)$ when $A = (\{a_1\}, \ldots, \{a_n\})$. Consequently, the linguistic evaluation

$$\lambda^{F'(\alpha)}(x) = \ell\big(\mu_R^{F'(\alpha)}(x)\big)$$

and the set of labeled granules

$$LHS(F'(\alpha)) = \{([x]_R, \lambda^{F'(\alpha)}(x)) \mid x \in X\}$$

are identical to the constructions in the Linguistic HyperRough Set. Hence, the full collection

$$\{(\alpha, LHS(F'(\alpha))) \mid \alpha \in J'\}$$

is precisely the Linguistic HyperRough Set on J'. This shows $LSHRS(F)$ restricts to the Linguistic HyperRough Set when J is specialized to J'. □

Example 2.27 (Real-life Example of a Linguistic 2-SuperHyperRough Set)**.** Let $X = \{U_1, U_2, U_3, U_4\}$ be four smartphone users. Define an equivalence relation R by “same support region” with classes

$$[U_1]_R = [U_2]_R = \{U_1, U_2\}, \quad [U_3]_R = [U_4]_R = \{U_3, U_4\}.$$

We consider two attributes:

$$T_1 = \text{Preferred OS},\ J_1 = \{\text{iOS}, \text{Android}\}, \qquad T_2 = \text{Subscription Plan},\ J_2 = \{\text{Basic}, \text{Premium}\}.$$

Then

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2).$$

Select the attribute-set combination

$$A = (\{\text{iOS}\}, \{\text{Premium}\}) \in J,$$

and let

$$F(A) = \{U_1\}.$$

Define the linguistic label set and mapping by

$$S = \{\text{low, medium, high}\}, \quad \ell(t) = \begin{cases} \text{low}, & 0 \le t < 0.5, \\ \text{medium}, & 0.5 \le t < 1, \\ \text{high}, & t = 1. \end{cases}$$

For each $x \in X$, the rough membership is

$$\mu_R^{F(A)}(x) = \frac{\big|[x]_R \cap \{U_1\}\big|}{\big|[x]_R\big|}.$$

Hence:

$$\mu_R^{F(A)}(U_1) = \frac{|\{U_1, U_2\} \cap \{U_1\}|}{2} = \tfrac{1}{2} \quad \Longrightarrow \quad \lambda^{F(A)}(U_1) = \text{medium},$$

$$\mu_R^{F(A)}(U_2) = \frac{|\{U_1, U_2\} \cap \{U_1\}|}{2} = \tfrac{1}{2} \quad \Longrightarrow \quad \lambda^{F(A)}(U_2) = \text{medium},$$

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$$\mu_R^{F(A)}(U_3) = \frac{|\{U_3, U_4\} \cap \{U_1\}|}{2} = 0 \quad \Longrightarrow \quad \lambda^{F(A)}(U_3) = \text{low},$$
$$\mu_R^{F(A)}(U_4) = \frac{|\{U_3, U_4\} \cap \{U_1\}|}{2} = 0 \quad \Longrightarrow \quad \lambda^{F(A)}(U_4) = \text{low}.$$

Therefore the linguistic superhyperrough description is

$$LSHS(F(A)) = \big\{(\{U_1, U_2\}, \text{medium}),\ (\{U_3, U_4\}, \text{low})\big\},$$

and the full set

$$LSHRS(F) = \{(A, LSHS(F(A))) \mid A \in J\}.$$

Example 2.28 (Real-life Example of a Linguistic 3-SuperHyperRough Set)**.** Let

$$X = \{R_1, R_2, R_3, R_4, R_5, R_6\}$$

be six restaurants. Define R by "same star-rating" with equivalence classes

$$[R_1]_R = [R_2]_R = [R_3]_R = \{R_1, R_2, R_3\}, \quad [R_4]_R = [R_5]_R = \{R_4, R_5\}, \quad [R_6]_R = \{R_6\}.$$

We use three attributes:

$$T_1 = \text{Price Tier},\ J_1 = \{\text{Cheap}, \text{Moderate}, \text{Expensive}\},$$
$$T_2 = \text{Cuisine},\ J_2 = \{\text{Italian}, \text{Japanese}\},$$
$$T_3 = \text{Location},\ J_3 = \{\text{CityCenter}, \text{Suburb}\}.$$

Then

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \mathcal{P}(J_3).$$

Choose

$$B = \big(\{\text{Cheap}, \text{Moderate}\}, \{\text{Italian}\}, \{\text{CityCenter}, \text{Suburb}\}\big),$$

and set

$$F(B) = \{R_1, R_3, R_4\}.$$

Use the same label set and mapping as above:

$$\ell(t) = \begin{cases} \text{low}, & 0 \le t < 0.5, \\ \text{medium}, & 0.5 \le t < 0.8, \\ \text{high}, & 0.8 \le t \le 1. \end{cases}$$

Compute for each granule:

$$\mu_R^{F(B)}(R_i) = \frac{\big|[R_i]_R \cap \{R_1, R_3, R_4\}\big|}{\big|[R_i]_R\big|}.$$

Thus,

$$\mu_R^{F(B)}(R_1) = \frac{|\{R_1, R_2, R_3\} \cap \{R_1, R_3, R_4\}|}{3} = \tfrac{2}{3} \approx 0.67 \quad \Longrightarrow \quad \lambda = \text{medium},$$
$$\mu_R^{F(B)}(R_4) = \frac{|\{R_4, R_5\} \cap \{R_1, R_3, R_4\}|}{2} = \tfrac{1}{2} = 0.5 \quad \Longrightarrow \quad \lambda = \text{medium},$$
$$\mu_R^{F(B)}(R_6) = \frac{|\{R_6\} \cap \{R_1, R_3, R_4\}|}{1} = 0 \quad \Longrightarrow \quad \lambda = \text{low}.$$

Therefore

$$LSHS(F(B)) = \big\{(\{R_1, R_2, R_3\}, \text{medium}),\ (\{R_4, R_5\}, \text{medium}),\ (\{R_6\}, \text{low})\big\},$$

and

$$LSHRS(F) = \{(B, LSHS(F(B))) \mid B \in J\}.$$

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3 Conclusion and Future Work

In this paper, we introduced two new frameworks that extend the classical rough-set paradigm: the *Graded HyperRough Set* and the *Linguistic HyperRough Set*. We then further generalized these ideas by defining the *Graded SuperHyperRough Set* and the *Linguistic SuperHyperRough Set*, and we investigated their internal structure and key properties.

As for future directions, we plan to explore connections between our superhyperrough models and related uncertainty-handling formalisms such as Soft Sets,[21] Hypersoft Sets,[22,23] and Hyperfuzzy Sets.[24–27] We will also investigate efficient algorithms for computing these approximations and examine potential applications in data mining, machine learning, and other areas of computer science. It may also be possible to explore extensions using HyperGraphs[28,29] and SuperHyperGraphs.[30]

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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