Best Proximity Point Theorems in Neutrosophic Complete Metric Spaces

A. Sreelakshmi Unni[1], V. Pragadeeswarar[1,*], Manuel De La Sen[2]
[1]Department of Mathematics, Amrita School of Physical Sciences Coimbatore, Amrita Vishwa Vidyapeetham, India
[2]Institute of Research and Development of Processes IIDP, University of the Basque Country, Campus of Leioa, 48940 Leioa, Bizkaia, Spain
Emails: ua_sreelakshmi@cb.students.amrita.edu; v_pragadeeswarar@cb.amrita.edu; manuel.delasen@ehu.es

Abstract

In this work, we introduce the notion of best proximity point for a non-self map defined in a neutrosophic complete metric space. Moreover, we define the class of neutrosophic proximal contraction of first kind and second kind, and we prove theorems which ensures existence and uniqueness of best proximity point for such mappings in neutrosophic complete metric spaces. Additionally, a technique to identify an optimal approximation solution intended as a best proximity point is demonstrated.

Keywords: Best proximity point; Neutrosophic complete metric space; Fixed point

1 Introduction

Fuzzy sets introduced by Zadeh[1] in 1965, deals with uncertainities and vagueness, especially in situations where traditional binary logic fails to reach. Unlike classical sets where an element either belongs or does not belong to the set, fuzzy sets allow partial memebership, characterized by a membership function that assigns each element a degree of membership between 0 and 1. This concept is particularly useful in fields like artificial intelligence, control systems, and decision-making, where impreciseor subjective information is common. Fuzzy set theory provides a flexible and intuitive way to model and reason about such ambiguity.
A fuzzy metric space is a generalization of the classical metric space that incorporates the concept of fuzziness to handle uncertainity and imprecision in measuring distances between points. Subsequent to Zadeh's work, Kramosil and Michálek[2] extended the fuzzy notions to metric spaces and defined KM fuzzy metric space. Later on, George and Veeramani[3] generalized KM fuzzy metric space to GV fuzzy metric spaces. Many authors have described the fuzzy metric space in distinct ways.[4,5] Since fuzzy metric spaces are generalization of metric spaces, many existing results in fixed point theory are extended to the fuzzy settings for more details refer.[6–9]
Furthermore, Smarandache,[11] introduced the concepts of neutrosphic logic and neutrosophic sets, a novel version of the idea of classical sets. Neutrosophic sets comprise of degree of membership, degree of indeterminancy and degree of non-membership. Later, using the concepts of neutrosophic sets, Kiriski et al.[12] defined neutrosphic metric spaces (NMSs) and in[13] they proved fixed point results in a given complete NMS. Motivated by this, several researchers have considered generalised NMSs and have extended the existing fixed point results to the setting of NMSs, for more details, refer.[14–16] In particular, Saleem et al.,[14] have proved the fixed point result for multivalued maps, hence extended the concepts to fractals. In,[15] Ishtiaq et al. defined orthogonal NMS and established the existence of fixed point.

On the other hand, suppose M is a non-empty subset of a metric space (X, d). A map $\Gamma : M \to X$ is said to have a fixed point if $\Gamma m = m$, for some $m \in M$. Consider the scenario of finding the fixed point when the equation $\Gamma m = m$ does not have a solution that is, $d(m, \Gamma m) > 0, \forall m \in M$. In such case, we have to look for an element $m \in M$ such that $d(m, \Gamma m)$ is minimum. That is, let M, N be two non-empty subsets of a metric space (X, d) and $\Gamma : M \to N$. Then Γ is said to have a best proximity point in M, if we can find an element $m \in M$ such that $d(m, \Gamma m) = d(M, N) = \inf\{d(m, n) : m \in M \text{ and } n \in N\}$. Best proximity point theorems have been studied to find the necessary conditions such that the minimization problem $\min_{m\in M} d(m, \Gamma m)$ has at least one solution. Al-Thagafi and Shahzad in[17] has established the existence and convergence of best proximity point and hence it is an important application of optimization theory.

Thus, the importance of best proximity point theory emerges in the absence of fixed points. In the literature, for now, the concept of best proximity point for a non-self map in a neutrosophic complete metric space is undefined. Therefore, in this manuscript, we establish the existence and uniqueness of the best proximity point for a non-self map in a neutrosophic complete metric space, which is Archimedean. Thereby proving the fundamental result on existence of best proximity point in the Archimedean neutrosophic metric spaces simply known as neutrosophic metric spaces.

The structure of the paper is as follows. First we extend the notion of best proximity point and related concepts to the setting of neutrosophic metric spaces. Secondly, we define neutrosophic proximal contraction of first kind and second kind. Lastly, we find the sufficient conditions that the non-self map should satify to guarantee the existence and uniqueness of best proximity point for both non-self contractions we defined. Hence we proved a proximal version of Banach contraction principle in a neutrosophic complete metric space.

2 Preliminaries:

Definition 2.1. [10] An operation $* : [0, 1]^2 \to [0, 1]$ is a continuous t-norm (CTN) if $*$ is satisfying the following conditions:

(I) $a * 1 = a$,

(II) $*$ is continuous

(III) $*$ is commutative and associative

(IV) $a * b \le c * d$ whenever $a \le c$, $b \le d$, and $a, b, c, d \in [0, 1]$.

Definition 2.2. [10] An operation $\diamond : [0, 1]^2 \to [0, 1]$is a continuous t-conorm (CTCN) if $\diamond$ is satisfying the following conditions:

(I) $a \diamond 0 = a$

(II) $\diamond$ is continuous

(III) $\diamond$ is commutative and associative

(IV) $a \diamond b \le c \diamond d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0, 1]$.

Definition 2.3. [13] Suppose $R \ne \emptyset$. Given a six-tuple $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ where $*$ is a continuous t-norm (CTN), $\diamond$ is a continuous t-conorm (CTCN), and P_φ, E_φ, Z_φ are neutrosophic sets on $R \times R \times (0, \infty)$. If $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ satisfies the following conditions for all $\omega, \nu, z \in R$ and $\tau, s > 0$:

(I) $P_\varphi(\omega, \nu, \tau) + E_\varphi(\omega, \nu, \tau) + Z_\varphi(\omega, \nu, \tau) \le 3$

(II) $0 \le P_\varphi(\omega, \nu, \tau) \le 1$

(III) $P_\varphi(\omega, \nu, \tau) = 1 \iff \omega = \nu$

(IV) $P_\varphi(\omega, \nu, \tau) = P_\varphi(\nu, \omega, \tau)$

(V) $P_\varphi(\omega, z, \tau + s) \geq P_\varphi(\omega, \nu, \tau) * P_\varphi(\nu, z, s)$

(VI) $P_\varphi(\omega, \nu, \cdot) : [0, \infty) \to [0, 1]$ is continuous

(VII) $\lim_{\tau\to\infty} P_\varphi(\omega, \nu, \tau) = 1$

(VIII) $0 \leq E_\varphi(\omega, \nu, \tau) \leq 1$

(IX) $E_\varphi(\omega, \nu, \tau) = 0 \iff \omega = \nu$

(X) $E_\varphi(\omega, \nu, \tau) = E_\varphi(\nu, \omega, \tau)$

(XI) $E_\varphi(\omega, z, \tau + s) \leq E_\varphi(\omega, \nu, \tau) \diamond E_\varphi(\nu, z, s)$

(XII) $E_\varphi(\omega, \nu, \cdot) : [0, \infty) \to [0, 1]$ is continuous

(XIII) $\lim_{\tau\to\infty} E_\varphi(\omega, \nu, \tau) = 0$

(XIV) $0 \leq Z_\varphi(\omega, \nu, \tau) \leq 1$

(XV) $Z_\varphi(\omega, \nu, \tau) = 0 \iff \omega = \nu$

(XVI) $Z_\varphi(\omega, \nu, \tau) = Z_\varphi(\nu, \omega, \tau)$

(XVII) $Z_\varphi(\omega, z, \tau + s) \leq Z_\varphi(\omega, \nu, \tau) \diamond Z_\varphi(\nu, z, s)$

(XVIII) $Z_\varphi(\omega, \nu, \cdot) : [0, \infty) \to [0, 1]$ is continuous

(XIX) $\lim_{\tau\to\infty} Z_\varphi(\omega, \nu, \tau) = 0$

(XX) If $\tau \leq 0$, then $P_\varphi(\omega, \nu, \tau) = 0$, $E_\varphi(\omega, \nu, \tau) = 1$, and $Z_\varphi(\omega, \nu, \tau) = 1$

Then, $(P_\varphi, E_\varphi, Z_\varphi)$ is a *neutrosophic metric* on R and $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ is called a **neutrosophic metric space (NMS)**. The functions $P_\varphi(\omega, \nu, \tau)$, $E_\varphi(\omega, \nu, \tau)$, and $Z_\varphi(\omega, \nu, \tau)$ represent the degree of nearness, indeterminacy, and non-nearness between ω and ν with respect to τ, respectively.

Remark 2.4. Let $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ be an NMS. Then for any $x, y \in R$, we have

(i) $P_\varphi(x, y, .)$ is non-decreasing

(ii) $E_\varphi(x, y, .)$ is non-increasing

(iii) $Z_\varphi(x, y, .)$ is non-increasing.

Lemma 2.5. *Let $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ be an NMS. Then P_φ, E_φ and Z_φ are continuous functions on $R \times R \times (0, \infty)$.*

Proof. Let $a, b \in R$ and $\tau > 0$. Consider a sequence $(a_n, b_n, \tau_n)_n$ in $R \times R \times (0, \infty)$ converging to (a, b, τ).
Claim I P_φ is continuous on $R \times R \times (0, \infty)$.
We have $P_\varphi(a_n, b_n, \tau_n)_n \in (0, 1)$ for $n \in \mathbb{N}$. Hence, we can find a convergent subsequence $(a'_n, b'_n, \tau'_n)_n$ of $(a_n, b_n, \tau_n)_n$ such that the sequence $P_\varphi(a'_n, b'_n, \tau'_n)_n$ converges in $[0, 1]$. Choose $\delta > 0$ such that $\delta < \frac{\tau}{2}$. Now there exists $n_0 \in \mathbb{N}$ such that $|\tau - \tau_n| < \delta$, $\forall n \geq n_0$. Thus, by the properties of NMS, we obtain

$$P_\varphi(a'_n, b'_n, \tau'_n) \geq P_\varphi(a'_n, a, \frac{\delta}{2}) * P_\varphi(a, b, \tau - 2\delta) * P_\varphi(b, b'_n, \frac{\delta}{2})$$

and

$$P_\varphi(a, b, \tau + 2\delta) \geq P_\varphi(a, a'_n, \frac{\delta}{2}) * P_\varphi(a'_n, b'_n, \tau_n) * P_\varphi(b'_n, b, \frac{\delta}{2})$$

As $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty} P_\varphi(a'_n, b'_n, \tau'_n) &\geq 1 * P_\varphi(a, b, \tau - 2\delta) * 1 \\ &= P_\varphi(a, b, \tau - 2\delta)\end{aligned}$$

and

$$P_\varphi(a,b,\tau+2\delta) \geq 1 * \lim_{n\to\infty} P_\varphi(a'_n,b'_n,\tau_n) * 1$$
$$= \lim_{n\to\infty} P_\varphi(a'_n,b'_n,\tau_n).$$

Since the function $\tau \to P_\varphi(a,b,\tau)$ is continous, $\lim_{n\to\infty} P_\varphi(a'_n,b'_n,\tau_n) = P_\varphi(a,b,\tau)$. Thus P_φ is continuous on $R \times R \times (0,\infty)$.

Claim II E_φ is continuous on $R \times R \times (0,\infty)$.

We have $E_\varphi(a_n,b_n,\tau_n)_n \in (0,1)$ for $n \in \mathbb{N}$. Hence, we can find a convergent subsequence $(a'_n,b'_n,\tau'_n)_n$ of $(a_n,b_n,\tau_n)_n$ such that the sequence $E_\varphi(a'_n,b'_n,\tau'_n)_n$ converges in $[0,1]$. Choose $\delta > 0$ such that $\delta < \frac{\tau}{2}$. Now there exists $n_0 \in \mathbb{N}$ such that $|\tau - \tau_n| < \delta$, $\forall n \geq n_0$. Thus, by the properties of NMS, we obtain

$$E_\varphi(a'_n,b'_n,\tau'_n) \leq E_\varphi(a'_n,a,\frac{\delta}{2}) \diamond E_\varphi(a,b,\tau-2\delta) \diamond E_\varphi(b,b'_n,\frac{\delta}{2})$$

and

$$E_\varphi(a,b,\tau+2\delta) \leq E_\varphi(a,a'_n,\frac{\delta}{2}) \diamond E_\varphi(a'_n,b'_n,\tau_n) \diamond E_\varphi(b'_n,b,\frac{\delta}{2})$$

As $n \to \infty$, we get

$$\lim_{n\to\infty} E_\varphi(a'_n,b'_n,\tau'_n) \leq 0 \diamond E_\varphi(a,b,\tau-2\delta) \diamond 0$$
$$= E_\varphi(a,b,\tau-2\delta)$$

and

$$E_\varphi(a,b,\tau+2\delta) \leq 0 \diamond \lim_{n\to\infty} E_\varphi(a'_n,b'_n,\tau_n) \diamond 0$$
$$= \lim_{n\to\infty} E_\varphi(a'_n,b'_n,\tau_n).$$

Since the function $\tau \to E_\varphi(a,b,\tau)$ is continuous, $\lim_{n\to\infty} E_\varphi(a'_n,b'_n,\tau_n) = E_\varphi(a,b,\tau)$. Thus E_φ is continuous on $R \times R \times (0,\infty)$.

Claim III Z_φ is continuous on $R \times R \times (0,\infty)$.

The proof is similar to Claim II. □

Example 2.6. [13] Let (R,Ω) be a metric space. Define the functions:

$$P_\varphi(\omega,\nu,\tau) = \frac{\tau}{\tau+\Omega(\omega,\nu)}, \quad E_\varphi(\omega,\nu,\tau) = \frac{\Omega(\omega,\nu)}{\tau+\Omega(\omega,\nu)}, \quad Z_\varphi(\omega,\nu,\tau) = \frac{\Omega(\omega,\nu)}{\tau}$$

for all $\omega,\nu \in R$ and $\tau > 0$. Then $(R,P_\varphi,E_\varphi,Z_\varphi,*,\diamond)$ is called the *standard NMS* induced by the metric Ω.

Definition 2.7. Let $(R,P_\varphi,E_\varphi,Z_\varphi,*,\diamond)$ be a NMS. The open ball with center $\omega \in R$, radius $0 < r < 1$, and parameter $\tau > 0$ is defined as:

$$B(\omega,r,\tau) = \{\nu \in R : P_\varphi(\omega,\nu,\tau) > 1-r,\ E_\varphi(\omega,\nu,\tau) < r,\ Z_\varphi(\omega,\nu,\tau) < r\}$$

Define the topology:

$$\tau(P_\varphi,E_\varphi) = \{A \subset R : \forall\omega \in A,\ \exists\tau > 0, r \in (0,1) \text{ such that } B(\omega,r,\tau) \subset A\}$$

Then $\tau(P_\varphi,E_\varphi)$ is the topology on R induced by the neutrosophic metric.

Definition 2.8. [13] Let $(R,P_\varphi,E_\varphi,Z_\varphi,*,\diamond)$ be a NMS.

- a sequence $\{\omega_n\}$ in R is converging to a point $\omega \in R$ if for each $\tau > 0$, $\lim_{n\to\infty} P_\varphi(\omega_n, \omega, \tau) = 1; \lim_{n\to\infty} E_\varphi(\omega_n, \omega, \tau) = 0; \lim_{n\to\infty} Z_\varphi(\omega_n, \omega, \tau) = 0$.
- a sequence $\{\omega_n\}$ in R is said to be Cauchy if for each $\epsilon > 0$ and $\tau > 0$ there exist $n_0 \in N$ such that $P_\varphi(\omega_n, \omega_m, \tau) > 1 - \epsilon; E_\varphi(\omega_n, \omega_m, \tau) < \epsilon; Z_\varphi(\omega_n, \omega_m, \tau) < \epsilon$ for all $n, m \leq n_0$.
- The space is said to be *complete* if every Cauchy sequence in R converges.
- The space is said to be *compact* if every sequence $\{\omega_n\}$ in R has a convergent subsequence $\{\omega_{n_k}\}$.

The notion of best proximity point for non-self mappings is follows.

Definition 2.9. Let (C, D) be a pair of two non-empty subsets of a neutrosophic metric space $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$. An element $x \in C$ is said to be a best proximity point of the non-self mappings $\Gamma : C \to D$ if it satisfies the conditions that $P_\varphi(x, \Gamma x, \tau) = P_\varphi(C, D, \tau)$, $E_\varphi(x, \Gamma x, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(x, \Gamma x, \tau) = Z_\varphi(C, D, \tau)$ for all $\tau > 0$.

Given non-empty subsets C and D of a neutrosophic metric space $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$, the following notions are used subsequently:

$$P_\varphi(C, D, \tau) := \sup\{P_\varphi(x, y, \tau) : x \in C \text{ and } y \in D\}$$
$$E_\varphi(C, D, \tau) := \inf\{E_\varphi(x, y, \tau) : x \in C \text{ and } y \in D\}$$
$$Z_\varphi(C, D, \tau) := \inf\{Z_\varphi(x, y, \tau) : x \in C \text{ and } y \in D\}$$
$$C_0(\tau) = \left\{ x \in C \;\middle|\; \begin{array}{l} P_\varphi(x, y, \tau) = P_\varphi(C, D, \tau), E_\varphi(x, y, \tau) = E_\varphi(C, D, \tau), \text{ and} \\ Z_\varphi(x, y, \tau) = Z_\varphi(C, D, \tau) \quad \text{for some } y \in D \end{array} \right\}$$
$$D_0(\tau) = \left\{ y \in D \;\middle|\; \begin{array}{l} P_\varphi(x, y, \tau) = P_\varphi(C, D, \tau), E_\varphi(x, y, \tau) = E_\varphi(C, D, \tau), \text{ and} \\ Z_\varphi(x, y, \tau) = Z_\varphi(C, D, \tau) \quad \text{for some } x \in C \end{array} \right\}$$

for all $\tau > 0$.

Let us define the notion of a neutrosophic proximal contraction maps of first and second kind as follows.

Definition 2.10. A mapping $\Gamma : C \to D$ is said to be a neutrosophic proximal contraction of first kind if there exists a non-negative number $k \in [0, 1)$ such that, for all u_1, u_2, x_1, x_2 in C,

$$\left.\begin{array}{l} P_\varphi(u_1, \Gamma x_1, \tau) = P_\varphi(C, D, \tau) \\ P_\varphi(u_2, \Gamma x_2, \tau) = P_\varphi(C, D, \tau) \end{array}\right\} \implies P_\varphi(u_1, u_2, k\tau) \geq P_\varphi(x_1, x_2, \tau)$$

$$\left.\begin{array}{l} E_\varphi(u_1, \Gamma x_1, \tau) = E_\varphi(C, D, \tau) \\ E_\varphi(u_2, \Gamma x_2, \tau) = E_\varphi(C, D, \tau) \end{array}\right\} \implies E_\varphi(u_1, u_2, k\tau) \leq E_\varphi(x_1, x_2, \tau)$$

$$\left.\begin{array}{l} Z_\varphi(u_1, \Gamma x_1, \tau) = Z_\varphi(C, D, \tau) \\ Z_\varphi(u_2, \Gamma x_2, \tau) = Z_\varphi(C, D, \tau) \end{array}\right\} \implies Z_\varphi(u_1, u_2, k\tau) \leq Z_\varphi(x_1, x_2, \tau)$$

for all $\tau > 0$.

Definition 2.11. A mapping $\Gamma : C \to D$ is said to be a neutrosophic proximal contraction of second kind if there exists a non-negative number $k \in [0, 1)$ such that, for all u_1, u_2, x_1, x_2 in C,

$$\left.\begin{array}{l} P_\varphi(u_1, \Gamma x_1, \tau) = P_\varphi(C, D, \tau) \\ P_\varphi(u_2, \Gamma x_2, \tau) = P_\varphi(C, D, \tau) \end{array}\right\} \implies P_\varphi(\Gamma u_1, \Gamma u_2, k\tau) \geq P_\varphi(\Gamma x_1, \Gamma x_2, \tau)$$

$$\left.\begin{array}{l} E_\varphi(u_1, \Gamma x_1, \tau) = E_\varphi(C, D, \tau) \\ E_\varphi(u_2, \Gamma x_2, \tau) = E_\varphi(C, D, \tau) \end{array}\right\} \implies E_\varphi(\Gamma u_1, \Gamma u_2, k\tau) \leq E_\varphi(\Gamma x_1, \Gamma x_2, \tau)$$

$$\left.\begin{array}{l} Z_\varphi(u_1, \Gamma x_1, \tau) = Z_\varphi(C, D, \tau) \\ Z_\varphi(u_2, \Gamma x_2, \tau) = Z_\varphi(C, D, \tau) \end{array}\right\} \implies Z_\varphi(\Gamma u_1, \Gamma u_2, k\tau) \leq Z_\varphi(\Gamma x_1, \Gamma x_2, \tau)$$

for all $\tau > 0$.

3 Best Proximity Point Theorems

Now, let us state our main result.

Theorem 3.1. *Let (C, D) be a pair of two non-empty closed subsets of a neutrosophic complete metric space $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$ such that $C_0(\tau)$ is non-empty. Let $\Gamma : C \to D$ be a map satisfying the following conditions:*
(i) Γ is a continuous and neutrosophic proximal contraction of first kind,
(ii) $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$.
Then, there exists a unique best proximity point $x \in C$ for Γ. That is, $P_\varphi(x, \Gamma x, \tau) = P_\varphi(C, D, \tau), E_\varphi(x, \Gamma x, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(x, \Gamma x, \tau) = Z_\varphi(C, D, \tau)$. Further, for any fixed element $u_0 \in C_0(\tau)$, the sequence $\{u_n\}$, defined by $P_\varphi(u_{n+1}, \Gamma u_n, \tau) = P_\varphi(C, D, \tau), E_\varphi(u_{n+1}, \Gamma u_n, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(u_{n+1}, \Gamma u_n, \tau) = Z_\varphi(C, D, \tau)$ converges to the element x.

Proof. Choose $u_0 \in C_0(\tau)$. Since $\Gamma u_0 \in \Gamma(C_0(\tau)) \subseteq D_0(\tau)$, there exists $u_1 \in C_0(\tau)$ such that $P_\varphi(u_1, \Gamma u_0, \tau) = P_\varphi(C, D, \tau), E_\varphi(u_1, \Gamma u_0, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(u_1, \Gamma u_0, \tau) = Z_\varphi(C, D, \tau)$. Since $\Gamma u_1 \in \Gamma(C_0(\tau)) \subseteq D_0(\tau)$, there exists $u_2 \in C_0(\tau)$ such that $P_\varphi(u_2, \Gamma u_1, \tau) = P_\varphi(C, D, \tau), E_\varphi(u_2, \Gamma u_1, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(u_2, \Gamma u_1, \tau) = Z_\varphi(C, D, \tau)$. Continuing this process, we can find a sequence $\{u_n\}$ in $C_0(\tau)$ such that

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, \Gamma u_n, \tau) = P_\varphi(C, D, \tau), \\ &E_\varphi(u_{n+1}, \Gamma u_n, \tau) = E_\varphi(C, D, \tau), \\ &Z_\varphi(u_{n+1}, \Gamma u_n, \tau) = Z_\varphi(C, D, \tau), \text{ for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{1}$$

If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $P_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = P_\varphi(C, D, \tau), E_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = Z_\varphi(C, D, \tau)$. This means that u_{n_0} is a best proximity point of Γ and the proof is finished. Thus, we can suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since Γ is a neutrosophic proximal contraction of first kind, it follows that

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, u_n, k\tau) \geq P_\varphi(u_n, u_{n-1}, \tau), \\ &E_\varphi(u_{n+1}, u_n, k\tau) \leq E_\varphi(u_n, u_{n-1}, \tau), \\ &Z_\varphi(u_{n+1}, u_n, k\tau) \leq Z_\varphi(u_n, u_{n-1}, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{2}$$

By Mathematical induction, we obtain

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, u_n, \tau) \geq P_\varphi(u_0, u_1, \tfrac{\tau}{k^{n-1}}), \\ &E_\varphi(u_{n+1}, u_n, \tau) \leq E_\varphi(u_0, u_1, \tfrac{\tau}{k^{n-1}}), \\ &Z_\varphi(u_{n+1}, u_n, \tau) \leq Z_\varphi(u_0, u_1, \tfrac{\tau}{k^{n-1}}), \quad \forall n \in \mathbb{N} \text{ and } k \in (0,1). \end{aligned}\right\} \tag{3}$$

Thus for any positive integer p, we have

$$\left.\begin{aligned} P_\varphi(u_{n+p}, u_n, \tau) &\geq P_\varphi(u_n, u_{n+1}, \tfrac{\tau}{p}) * \cdots \text{(p-times)} \cdots * P_\varphi(u_{n+p-1}, u_{n+p}, \tfrac{\tau}{p}) \\ &\geq P_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n-1}}) * \cdots \text{(p-times)} \cdots * P_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n+p-2}}), \\ E_\varphi(u_{n+p}, u_n, \tau) &\leq E_\varphi(u_n, u_{n+1}, \tfrac{\tau}{p}) \diamond \cdots \text{(p-times)} \cdots \diamond E_\varphi(u_{n+p-1}, u_{n+p}, \tfrac{\tau}{p}) \\ &\leq E_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n-1}}) \diamond \cdots \text{(p-times)} \cdots \diamond E_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n+p-2}}), \\ Z_\varphi(u_{n+p}, u_n, \tau) &\leq Z_\varphi(u_n, u_{n+1}, \tfrac{\tau}{p}) \diamond \cdots \text{(p-times)} \cdots \diamond Z_\varphi(u_{n+p-1}, u_{n+p}, \tfrac{\tau}{p}) \\ &\leq Z_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n-1}}) \diamond \cdots \text{(p-times)} \cdots \diamond Z_\varphi(u_0, u_1, \tfrac{\tau}{pk^{n+p-2}}). \end{aligned}\right\} \tag{4}$$

Now by (4) and the definition of NMS conditions, we get

$$\begin{aligned} &\lim_{n\to\infty} P_\varphi(u_{n+p}, u_n, \tau) \geq 1 * \cdots \text{ (p-times) } \cdots * 1 = 1, \\ &\lim_{n\to\infty} E_\varphi(u_{n+p}, u_n, \tau) \leq 0 \diamond \cdots \text{ (p-times) } \cdots \diamond 0 = 0, \\ &\lim_{n\to\infty} Z_\varphi(u_{n+p}, u_n, \tau) \leq 0 \diamond \cdots \text{ (p-times) } \cdots \diamond 0 = 0. \end{aligned}$$

Thus, $\{u_n\}$ is a Cauchy sequence in C and hence converges to some element $x \in C$. Since Γ is continuous, we get $\Gamma u_n \to \Gamma x$ as $n \to \infty$. Now, using $u_n \to x$, $\Gamma u_n \to \Gamma x$, and the definition of neutrosophic proximity in (1), we obtain:

$$P_\varphi(x, \Gamma x, \tau) = P_\varphi(C, D, \tau),$$
$$E_\varphi(x, \Gamma x, \tau) = E_\varphi(C, D, \tau),$$
$$Z_\varphi(x, \Gamma x, \tau) = Z_\varphi(C, D, \tau).$$

Hence, x is the best proximity point for Γ.

Let us now prove the uniqueness of the best proximity point. Suppose there exists another element $y \in C$ such that:

$$P_\varphi(y, \Gamma y, \tau) = P_\varphi(C, D, \tau),$$
$$E_\varphi(y, \Gamma y, \tau) = E_\varphi(C, D, \tau),$$
$$Z_\varphi(y, \Gamma y, \tau) = Z_\varphi(C, D, \tau).$$

Since Γ is a neutrosophic proximal contraction of first kind, for every $\tau \in (0, \infty)$ and fixed $k \in (0, 1)$, we have:

$$P_\varphi(x, y, k\tau) \geq P_\varphi(x, y, \tau),$$
$$E_\varphi(x, y, k\tau) \leq E_\varphi(x, y, \tau),$$
$$Z_\varphi(x, y, k\tau) \leq Z_\varphi(x, y, \tau).$$

Now, from the above, we get:

$$\begin{aligned}
1 &\geq P_\varphi(x, y, \tau) \geq P_\varphi\left(x, y, \frac{\tau}{k}\right) \geq P_\varphi\left(x, y, \frac{\tau}{k^2}\right) \\
&\geq \cdots \geq P_\varphi\left(x, y, \frac{\tau}{k^n}\right) \to 1 \quad \text{as } n \to \infty, \\
0 &\leq E_\varphi(x, y, \tau) \leq E_\varphi\left(x, y, \frac{\tau}{k}\right) \leq E_\varphi\left(x, y, \frac{\tau}{k^2}\right) \\
&\leq \cdots \leq E_\varphi\left(x, y, \frac{\tau}{k^n}\right) \to 0 \quad \text{as } n \to \infty, \\
0 &\leq Z_\varphi(x, y, \tau) \leq Z_\varphi\left(x, y, \frac{\tau}{k}\right) \leq Z_\varphi\left(x, y, \frac{\tau}{k^2}\right) \\
&\leq \cdots \leq Z_\varphi\left(x, y, \frac{\tau}{k^n}\right) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, from the definition of a neutrosophic metric space (NMS), we get $x = y$. Therefore, Γ has a **unique best proximity point**. □

Example 3.2. Consider $\mathbb{R}^2$ with metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Choose $C = \{0\} \times \mathbb{R}$ and $D = \{1\} \times \mathbb{R}$. Here, $C_0(t) = C$ and $D_0(t) = D$. Define $\Gamma : C \to D$ as $\Gamma(0, x) = (1, \frac{x}{2})$. Choose the CTN $*$ as $a * b = ab$ and the CCTN $\diamond$ as $a \diamond b = \max\{a, b\}$. The functions $P_\varphi, E_\varphi, Z_\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty)$ are defined as follows:

$$P_\varphi(z_i, z_{i+1}, t) = \frac{t}{t + d(z_i, z_{i+1})},$$
$$E_\varphi(z_i, z_{i+1}, t) = \frac{d(z_i, z_{i+1})}{t + d(z_i, z_{i+1})},$$
$$Z_\varphi(z_i, z_{i+1}, t) = \frac{d(z_i, z_{i+1})}{t}$$

where $z_i = (x_i, y_i)$ and $z_{i+1} = (x_{i+1}, y_{i+1})$, for $x_i, y_i, x_{i+1}, y_{i+1} \in \mathbb{R}$. And we have

$$P_\varphi(C, D, t) = \frac{t}{t + 1},$$
$$E_\varphi(C, D, t) = \frac{1}{t + 1},$$
$$Z_\varphi(C, D, t) = \frac{1}{t},\ t > 0.$$

Claim: Γ is a neutrosophic proximal contraction of first kind.
Step 1: Suppose we have $(0,u_1),(0,u_2),(0,x_1),(0,x_2)\in C$ such that

$$P_\varphi((0,u_1),\Gamma(0,x_1),t)=P_\varphi(C,D,t)=\frac{t}{t+1}=P_\varphi((0,u_2),\Gamma(0,x_2),t). \tag{5}$$

we have to verify that $P_\varphi((0,u_1),(0,u_2),kt)\geq P_\varphi((0,x_1),(0,x_2),t)$, for some $k\in[0,1)$.
Then by (5), we obtain $u_1=\frac{x_1}{2}$ and $u_2=\frac{x_2}{2}$, $\forall(0,x_1),(0,x_2)\in C$. Now, consider,

$$P_\varphi((0,u_1),(0,u_2),kt)=\frac{kt}{kt+|u_1-u_2|}=\frac{t}{t+\frac{|u_1-u_2|}{k}}$$

and

$$P_\varphi((0,x_1),(0,x_2),t)=\frac{t}{t+|x_1-x_2|}=\frac{t}{t+2|u_1-u_2|}.$$

Hence, for any $k\in[\frac{1}{2},1)$, we have $P_\varphi((0,u_1),(0,u_2),kt)\geq P_\varphi((0,x_1),(0,x_2),t)$.

Step 2: Suppose we have $(0,u_1),(0,u_2),(0,x_1),(0,x_2)\in C$ such that

$$E_\varphi((0,u_1),\Gamma(0,x_1),t)=E_\varphi(C,D,t)=\frac{1}{t+1}=E_\varphi((0,u_2),\Gamma(0,x_1),t). \tag{6}$$

we have to verify that $E_\varphi((0,u_1),(0,u_2),kt)\leq E_\varphi((0,x_1),(0,x_2),t)$, for some $k\in[0,1)$.
Then by (6), we obtain $u_1=\frac{x_1}{2}$ and $u_2=\frac{x_2}{2}$, $\forall(0,x_1),(0,x_2)\in C$. Now, consider,

$$E_\varphi((0,u_1),(0,u_2),kt)=\frac{|u_1-u_2|}{kt+|u_1-u_2|}$$

and

$$E_\varphi((0,x_1),(0,x_2),t)=\frac{|x_1-x_2|}{t+|x_1-x_2|}=\frac{2|u_1-u_2|}{t+2|u_1-u_2|}.$$

Hence for any $k\in[\frac{1}{2},1)$, we have $E_\varphi((0,u_1),(0,u_2),kt)\leq E_\varphi((0,x_1),(0,x_2),t)$.

Step 3: Suppose we have $(0,u_1),(0,u_2),(0,x_1),(0,x_2)\in C$ such that

$$Z_\varphi((0,u_1),\Gamma(0,x_1),t)=Z_\varphi(C,D,t)=\frac{1}{t}=Z_\varphi((0,u_2),\Gamma(0,x_2),t). \tag{7}$$

we have to verify that $Z_\varphi((0,u_1),(0,u_2),kt)\leq Z_\varphi((0,x_1),(0,x_2),t)$, for some $k\in[0,1)$.
Then by (7), we obtain $u_1=\frac{x_1}{2}$ and $u_2=\frac{x_2}{2}$, $\forall(0,x_1),(0,x_2)\in C$. Now, consider,

$$Z_\varphi((0,u_1),(0,u_2),kt)=\frac{|u_1-u_2|}{kt}$$

and

$$Z_\varphi((0,x_1),(0,x_2),t)=\frac{|x_1-x_2|}{t}=\frac{2|u_1-u_2|}{t}.$$

Hence for any $k\in[\frac{1}{2},1)$, we have $Z_\varphi((0,u_1),(0,u_2),kt)\leq Z_\varphi((0,x_1),(0,x_2),t)$.
Thus Γ is a neutrosophic proximal contraction of first kind. Now by Theorem 3.1, Γ has a unique best proximity point $(0,0)$ in C.

Theorem 3.3. *Assume that $C_0(\tau)$ is closed instead of assuming Γ is continuous in Theorem 3.1.*

Proof. Following the proof of Theorem 3.1, there exists a sequence $\{u_n\}$ in C satisfying the following conditions:

$$\left.\begin{aligned}P_\varphi(u_{n+1},\Gamma u_n,\tau)&=P_\varphi(C,D,\tau),\\ E_\varphi(u_{n+1},\Gamma u_n,\tau)&=E_\varphi(C,D,\tau),\\ Z_\varphi(u_{n+1},\Gamma u_n,\tau)&=Z_\varphi(C,D,\tau),\quad\text{for all } n\in\mathbb{N}.\end{aligned}\right\} \tag{8}$$

Moreover, $u_n \to x$ in C. Note that $\{u_n\} \subseteq C_0(\tau)$ and $C_0(\tau)$ is closed, therefore $x \in C_0(\tau)$. Since $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$, we get $\Gamma x \in D_0(\tau)$.

As $\Gamma x \in D_0(\tau)$, there exists $z \in C_0(\tau)$ such that

$$\left.\begin{aligned} P_\varphi(z, \Gamma x, \tau) = P_\varphi(C, D, \tau), \\ E_\varphi(z, \Gamma x, \tau) = E_\varphi(C, D, \tau), \\ Z_\varphi(z, \Gamma x, \tau) = Z_\varphi(C, D, \tau). \end{aligned}\right\} \tag{9}$$

From (8), (9), and since Γ is a neutrosophic proximal contraction of first kind, we have

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, z, k\tau) \geq P_\varphi(u_n, x, \tau), \\ &E_\varphi(u_{n+1}, z, k\tau) \leq E_\varphi(u_n, x, \tau), \\ &Z_\varphi(u_{n+1}, z, k\tau) \leq Z_\varphi(u_n, x, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{10}$$

By the definition of NMS and using $u_n \to x$ in (10), as $n \to \infty$, we get

$$\left.\begin{aligned} 1 \geq P_\varphi(x, z, k\tau) \geq P_\varphi(x, x, \tau) = 1, \\ 0 \leq E_\varphi(x, z, k\tau) \leq E_\varphi(x, x, \tau) = 0, \\ 0 \leq Z_\varphi(x, z, k\tau) \leq Z_\varphi(x, x, \tau) = 0. \end{aligned}\right\} \tag{11}$$

Hence, we get $x = z$. From (9), it follows that x is a best proximity point for Γ. The uniqueness can be proved similarly to Theorem 3.1. □

Theorem 3.4. *Let (C, D) be a pair of non-empty closed subsets of a neutrosophic complete metric space $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$. Let $\Gamma : C \to D$ be a map satisfying the following conditions:*

(i) Γ *is a neutrosophic proximal contraction of second kind,*

(ii) $\Gamma(C_0(\tau))$ *is non-empty and closed,*

(iii) $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$.

Then, there exists a best proximity point $z \in C$ for Γ. That is, $P_\varphi(z, \Gamma z, \tau) = P_\varphi(C, D, \tau)$, $E_\varphi(z, \Gamma z, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(z, \Gamma z, \tau) = Z_\varphi(C, D, \tau)$. Further, if z^* is another best proximity point of Γ, then $\Gamma z = \Gamma z^*$, hence Γ is a constant on the set of all best proximity point of Γ.

Proof. Following the proof of Theorem 3.1, there exists a sequence $\{u_n\}$ in C satisfying the following conditions:

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, \Gamma u_n, \tau) = P_\varphi(C, D, \tau), \\ &E_\varphi(u_{n+1}, \Gamma u_n, \tau) = E_\varphi(C, D, \tau), \\ &Z_\varphi(u_{n+1}, \Gamma u_n, \tau) = Z_\varphi(C, D, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{12}$$

If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $P_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = P_\varphi(C, D, \tau), E_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = E_\varphi(C, D, \tau)$, and $Z_\varphi(u_{n_0}, \Gamma u_{n_0}, \tau) = Z_\varphi(C, D, \tau)$. This means that u_{n_0} is a best proximity point of Γ and the proof is finished. Thus, we can suppose that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since Γ is a neutrosophic proximal contraction of second kind, it follows that

$$\left.\begin{aligned} &P_\varphi(\Gamma u_{n+1}, \Gamma u_n, k\tau) \geq P_\varphi(\Gamma u_n, \Gamma u_{n-1}, \tau), \\ &E_\varphi(\Gamma u_{n+1}, \Gamma u_n, k\tau) \leq E_\varphi(\Gamma u_n, \Gamma u_{n-1}, \tau), \\ &Z_\varphi(\Gamma u_{n+1}, \Gamma u_n, k\tau) \leq Z_\varphi(\Gamma u_n, \Gamma u_{n-1}, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{13}$$

By Mathematical induction, we obtain

$$\left.\begin{aligned} &P_\varphi(\Gamma u_{n+1}, \Gamma u_n, \tau) \geq P_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{k^{n-1}}),\\ &E_\varphi(\Gamma u_{n+1}, \Gamma u_n, \tau) \leq E_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{k^{n-1}}),\\ &Z_\varphi(\Gamma u_{n+1}, \Gamma u_n, \tau) \leq Z_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{k^{n-1}}), \quad \forall n \in \mathbb{N} \text{ and } k \in (0,1). \end{aligned}\right\} \tag{14}$$

Thus for any positive integer p, we have

$$\left.\begin{aligned} P_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) &\geq P_\varphi(\Gamma u_n, \Gamma u_{n+1}, \tfrac{\tau}{p}) * \cdots \text{(p-times)} \cdots * P_\varphi(\Gamma u_{n+p-1}, \Gamma u_{n+p}, \tfrac{\tau}{p})\\ &\geq P_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n-1}}) * \cdots \text{(p-times)} \cdots * P_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n+p-2}}),\\ E_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) &\leq E_\varphi(\Gamma u_n, \Gamma u_{n+1}, \tfrac{\tau}{p}) \diamond \cdots \text{(p-times)} \cdots \diamond E_\varphi(\Gamma u_{n+p-1}, \Gamma u_{n+p}, \tfrac{\tau}{p})\\ &\leq E_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n-1}}) \diamond \cdots \text{(p-times)} \cdots \diamond E_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n+p-2}}),\\ Z_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) &\leq Z_\varphi(\Gamma u_n, \Gamma u_{n+1}, \tfrac{\tau}{p}) \diamond \cdots \text{(p-times)} \cdots \diamond Z_\varphi(\Gamma u_{n+p-1}, \Gamma u_{n+p}, \tfrac{\tau}{p})\\ &\leq Z_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n-1}}) \diamond \cdots \text{(p-times)} \cdots \diamond Z_\varphi(\Gamma u_0, \Gamma u_1, \tfrac{\tau}{pk^{n+p-2}}). \end{aligned}\right\} \tag{15}$$

Now by (15) and the definition of NMS conditions, we get

$$\begin{aligned} &\lim_{n\to\infty} P_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) \geq 1 * \cdots \text{ (p-times) } \cdots * 1 = 1,\\ &\lim_{n\to\infty} E_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) \leq 0 \diamond \cdots \text{ (p-times) } \cdots \diamond 0 = 0,\\ &\lim_{n\to\infty} Z_\varphi(\Gamma u_{n+p}, \Gamma u_n, \tau) \leq 0 \diamond \cdots \text{ (p-times) } \cdots \diamond 0 = 0. \end{aligned}$$

Thus, $\{\Gamma u_n\}$ is a Cauchy sequence. Since $\Gamma(C_0(\tau))$ is closed subset of a complete NMS $(R, P_\varphi, E_\varphi, Z_\varphi, *, \diamond)$, there exists $x \in C_0(\tau)$ such that $\Gamma u_n \to \Gamma x \in \Gamma(C_0(\tau)) \subseteq D_0(\tau)$. Moreover, there exists $z \in C_0(\tau)$ such that

$$\left.\begin{aligned} P_\varphi(z, \Gamma x, \tau) = P_\varphi(C, D, \tau),\\ E_\varphi(z, \Gamma x, \tau) = E_\varphi(C, D, \tau),\\ Z_\varphi(z, \Gamma x, \tau) = Z_\varphi(C, D, \tau). \end{aligned}\right\} \tag{16}$$

From (12), (16), and since Γ is a neutrosophic proximal contraction of second kind, we have

$$\left.\begin{aligned} &P_\varphi(\Gamma u_{n+1}, \Gamma z, k\tau) \geq P_\varphi(\Gamma u_n, \Gamma x, \tau),\\ &E_\varphi(\Gamma u_{n+1}, \Gamma z, k\tau) \leq E_\varphi(\Gamma u_n, \Gamma x, \tau),\\ &Z_\varphi(\Gamma u_{n+1}, \Gamma z, k\tau) \leq Z_\varphi(\Gamma u_n, \Gamma x, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{17}$$

By the definition of NMS and using $\Gamma u_n \to \Gamma x$ in (17), we get

$$\left.\begin{aligned} 1 \geq P_\varphi(\Gamma x, \Gamma z, k\tau) \geq P_\varphi(\Gamma x, \Gamma x, \tau) = 1,\\ 0 \leq E_\varphi(\Gamma x, \Gamma z, k\tau) \leq E_\varphi(\Gamma x, \Gamma x, \tau) = 0,\\ 0 \leq Z_\varphi(\Gamma x, \Gamma z, k\tau) \leq Z_\varphi(\Gamma x, \Gamma x, \tau) = 0. \end{aligned}\right\} \tag{18}$$

Hence, we get $\Gamma x = \Gamma z$. From (16), it follows that z is a best proximity point for Γ.

Suppose there exists another element $z^* \in C_0(\tau)$ such that:

$$\begin{aligned} P_\varphi(z^*, \Gamma z^*, \tau) = P_\varphi(C, D, \tau),\\ E_\varphi(z^*, \Gamma z^*, \tau) = E_\varphi(C, D, \tau),\\ Z_\varphi(z^*, \Gamma z^*, \tau) = Z_\varphi(C, D, \tau). \end{aligned}$$

Since Γ is a neutrosophic proximal contraction of second kind, for every $\tau \in (0, \infty)$ and fixed $k \in (0,1)$, we have:

$$\begin{aligned} P_\varphi(\Gamma z, \Gamma z^*, k\tau) \geq P_\varphi(\Gamma z, \Gamma z^*, \tau),\\ E_\varphi(\Gamma z, \Gamma z^*, k\tau) \leq E_\varphi(\Gamma z, \Gamma z^*, \tau),\\ Z_\varphi(\Gamma z, \Gamma z^*, k\tau) \leq Z_\varphi(\Gamma z, \Gamma z^*, \tau). \end{aligned}$$

Now, from the above, we get:

$$\begin{aligned}1 &\geq P_\varphi(\Gamma z, \Gamma z^*, \tau) \geq P_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k}\right) \geq P_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^2}\right)\\ &\geq \cdots \geq P_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^n}\right) \to 1 \quad \text{as } n \to \infty,\\ 0 &\leq E_\varphi(\Gamma z, \Gamma z^*, \tau) \leq E_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k}\right) \leq E_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^2}\right)\\ &\leq \cdots \leq E_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^n}\right) \to 0 \quad \text{as } n \to \infty,\\ 0 &\leq Z_\varphi(\Gamma z, \Gamma z^*, \tau) \leq Z_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k}\right) \leq Z_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^2}\right)\\ &\leq \cdots \leq Z_\varphi\left(\Gamma z, \Gamma z^*, \frac{\tau}{k^n}\right) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, from the definition of a NMS, we get $\Gamma z = \Gamma z^*$. □

Example 3.5. Example 3.2 satisfies Theorem 3.4 as well. Here, in addition, we need to verify only the contraction condition.

Claim: Γ is a neutrosophic proximal contraction of second kind.
Step 1: Suppose we have $(0, u_1), (0, u_2), (0, x_1), (0, x_2) \in C$ such that

$$P_\varphi((0, u_1), \Gamma(0, x_1), t) = P_\varphi(C, D, t) = \frac{t}{t+1} = P_\varphi((0, u_2), \Gamma(0, x_2), t). \tag{19}$$

we have to verify that $P_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) \geq P_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t)$, for some $k \in [0, 1)$.
Then by (19), we obtain $u_1 = \frac{x_1}{2}$ and $u_2 = \frac{x_2}{2}$, $\forall (0, x_1), (0, x_2) \in C$. Now, consider,

$$P_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) = \frac{kt}{kt + |\frac{u_1}{2} - \frac{u_2}{2}|} = \frac{t}{t + \frac{|\frac{u_1}{2} - \frac{u_2}{2}|}{k}}$$

and

$$P_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t) = \frac{t}{t + |\frac{x_1}{2} - \frac{x_2}{2}|} = \frac{t}{t + 2|\frac{u_1}{2} - \frac{u_2}{2}|}.$$

Hence for any $k \in [\frac{1}{2}, 1)$, we have $P_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) \geq P_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t)$.

Step 2: Suppose we have $(0, u_1), (0, u_2), (0, x_1), (0, x_2) \in C$ such that

$$E_\varphi((0, u_1), \Gamma(0, x_1), t) = E_\varphi(C, D, t) = \frac{1}{t+1} = E_\varphi((0, u_2), \Gamma(0, x_1), t). \tag{20}$$

we have to verify that $E_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) \leq E_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t)$, for some $k \in [0, 1)$.
Then by (20), we obtain $u_1 = \frac{x_1}{2}$ and $u_2 = \frac{x_2}{2}$, $\forall (0, x_1), (0, x_2) \in C$. Now, consider,

$$E_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) = \frac{|\frac{u_1}{2} - \frac{u_2}{2}|}{kt + |\frac{u_1}{2} - \frac{u_2}{2}|}$$

and

$$E_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t) = \frac{|\frac{x_1}{2} - \frac{x_2}{2}|}{t + |\frac{x_1}{2} - \frac{x_2}{2}|} = \frac{2|\frac{u_1}{2} - \frac{u_2}{2}|}{t + 2|\frac{u_1}{2} - \frac{u_2}{2}|}.$$

Hence for any $k \in [\frac{1}{2}, 1)$, $E_\varphi(\Gamma(0, u_1), \Gamma(0, u_2), kt) \leq E_\varphi(\Gamma(0, x_1), \Gamma(0, x_2), t)$.

Step 3: Suppose we have $(0, u_1), (0, u_2), (0, x_1), (0, x_2) \in C$ such that

$$Z_\varphi((0, u_1), \Gamma(0, x_1), t) = Z_\varphi(C, D, t) = \frac{1}{t} = Z_\varphi((0, u_2), \Gamma(0, x_2), t). \tag{21}$$

we have to verify that $Z_\varphi(\Gamma(0,u_1),\Gamma(0,u_2),kt) \leq Z_\varphi(\Gamma(0,x_1),\Gamma(0,x_2),t)$, for some $k \in [0,1)$.
Then by (21), we obtain $u_1 = \frac{x_1}{2}$ and $u_2 = \frac{x_2}{2}$, $\forall (0,x_1),(0,x_2) \in C$. Now, consider,

$$Z_\varphi(\Gamma(0,u_1),\Gamma(0,u_2),kt) = \frac{|\frac{u_1}{2} - \frac{u_2}{2}|}{kt}$$

and

$$Z_\varphi(\Gamma(0,x_1),\Gamma(0,x_2),t) = \frac{|\frac{x_1}{2} - \frac{x_2}{2}|}{t} = 2.\frac{|\frac{u_1}{2} - \frac{u_2}{2}|}{t}.$$

Hence for any $k \in [\frac{1}{2},1)$, $Z_\varphi(\Gamma(0,u_1),\Gamma(0,u_2),kt) \leq Z_\varphi(\Gamma(0,x_1),\Gamma(0,x_2),t)$.
Thus Γ is a neutrosophic proximal contraction of second kind. Now by Theorem 3.4, Γ has a unique best proximity point $(0,0)$ in C.

Theorem 3.6. *Let (C,D) be a pair of two non-empty closed subsets of a neutrosophic complete metric space $(R,P_\varphi,E_\varphi,Z_\varphi,*,\diamond)$. Let $\Gamma : C \to D$ be a map satisfying the following conditions:*

(i) Γ *is an injective neutrosophic proximal contraction of second kind,*

(ii) $\Gamma(C_0(\tau))$ *is non-empty and closed,*

(iii) $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$.

Then, there exists a unique best proximity point $z \in C$ for Γ. That is, $P_\varphi(z,\Gamma z,\tau) = P_\varphi(C,D,\tau)$, $E_\varphi(z,\Gamma z,\tau) = E_\varphi(C,D,\tau)$, and $Z_\varphi(z,\Gamma z,\tau) = Z_\varphi(C,D,\tau)$.

Proof. The existence of a best proximity point can be proved by following the same argument as in the proof of Theorem 3.4. The uniqueness follows from the fact that Γ is injective. □

Theorem 3.7. *Let (C,D) be a pair of two non-empty closed subsets of a neutrosophic complete metric space $(R,P_\varphi,E_\varphi,Z_\varphi,*,\diamond)$ such that $C_0(\tau)$ is non-empty. Let $\Gamma : C \to D$ be a map satisfy the following conditions:*
(i) Γ is a neutrosophic proximal contraction of first kind as well as neutrosophic proximal contraction of second kind,
(ii) $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$,
Then, there exists a unique best proximity point $x \in C$ for Γ. That is, $P_\varphi(x,\Gamma x,\tau) = P_\varphi(C,D,\tau)$, $E_\varphi(x,\Gamma x,\tau) = E_\varphi(C,D,\tau)$, and $Z_\varphi(x,\Gamma x,\tau) = Z_\varphi(C,D,\tau)$. Further, for any fixed element $u_0 \in C_0(\tau)$, the sequence $\{u_n\}$, defined by $P_\varphi(u_{n+1},\Gamma u_n,\tau) = P_\varphi(C,D,\tau)$, $E_\varphi(u_{n+1},\Gamma u_n,\tau) = E_\varphi(C,D,\tau)$, and $Z_\varphi(u_{n+1},\Gamma u_n,\tau) = Z_\varphi(C,D,\tau)$ converges to the element x.

Proof. Following the proof of Theorem 3.1, there exists a sequence $\{u_n\}$ in C satisfying the following conditions:

$$\left.\begin{aligned} P_\varphi(u_{n+1},\Gamma u_n,\tau) &= P_\varphi(C,D,\tau),\\ E_\varphi(u_{n+1},\Gamma u_n,\tau) &= E_\varphi(C,D,\tau),\\ Z_\varphi(u_{n+1},\Gamma u_n,\tau) &= Z_\varphi(C,D,\tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{22}$$

As in Theorem 3.1, we have $\{u_n\}$ is a Cauchy sequence and converges to some $x \in C$. Further, as in Theorem 3.4, we have we have $\{\Gamma u_n\}$ is a Cauchy sequence and converges to some $y \in D$. Therefore, it follows that

$$\left.\begin{aligned} P_\varphi(x,y,\tau) &= \lim_{n\to\infty} P_\varphi(u_{n+1},\Gamma u_n,\tau) = P_\varphi(C,D,\tau),\\ E_\varphi(x,y,\tau) &= \lim_{n\to\infty} E_\varphi(u_{n+1},\Gamma u_n,\tau) = E_\varphi(C,D,\tau),\\ Z_\varphi(x,y,\tau) &= \lim_{n\to\infty} Z_\varphi(u_{n+1},\Gamma u_n,\tau) = Z_\varphi(C,D,\tau). \end{aligned}\right\} \tag{23}$$

That is $x \in C_0(\tau)$. Since $\Gamma(C_0(\tau)) \subseteq D_0(\tau)$, we get $\Gamma x \in D_0(\tau)$.

As $\Gamma x \in D_0(\tau)$, there exists $z \in C_0(\tau)$ such that

$$\left.\begin{aligned} P_\varphi(z,\Gamma x,\tau) &= P_\varphi(C,D,\tau),\\ E_\varphi(z,\Gamma x,\tau) &= E_\varphi(C,D,\tau),\\ Z_\varphi(z,\Gamma x,\tau) &= Z_\varphi(C,D,\tau). \end{aligned}\right\} \tag{24}$$

From (22), (24), and since Γ is a neutrosophic proximal contraction of first kind, we have

$$\left.\begin{aligned} &P_\varphi(u_{n+1}, z, k\tau) \geq P_\varphi(u_n, x, \tau), \\ &E_\varphi(u_{n+1}, z, k\tau) \leq E_\varphi(u_n, x, \tau), \\ &Z_\varphi(u_{n+1}, z, k\tau) \leq Z_\varphi(u_n, x, \tau), \quad \text{for all } n \in \mathbb{N}. \end{aligned}\right\} \tag{25}$$

By the definition of neutrosophic and using $u_n \to x$ in (25), we get

$$\left.\begin{aligned} &1 \geq P_\varphi(x, z, k\tau) \geq P_\varphi(x, x, \tau) = 1, \\ &0 \leq E_\varphi(x, z, k\tau) \leq E_\varphi(x, x, \tau) = 0, \\ &0 \leq Z_\varphi(x, z, k\tau) \leq Z_\varphi(x, x, \tau) = 0. \end{aligned}\right\} \tag{26}$$

Hence, we get $x = z$. From (24), it follows that x is a best proximity point for Γ. The uniqueness can be proved similarly to Theorem 3.1. □

4 Conclusion

In this work, we have introduced the basic theory of the existence of best proximity point for a non-self map in the framework of neutrosophic metric spaces. Here, we proved the results on existence and uniqueness of best proximity point for neutrosophic proximal contraction of first kind and second kind as well. And, we have provided examples to support our main results.
Motivated by our results, our findings can be broadened in a future study by extending them to weaker contractions, more generalized spaces, and even by altering the type of non-self mappings considered. Furthermore, our findings will aid in developing the theory for use in the domains of optimization, game theory, differential equations, non-linear analysis, and even fractal theory.

Declarations

Funding

This work has been funded by the Basque Government through Grant IT1555-22.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] L. A. Zadeh, Fuzzy sets, Information and control, vol.8, pp. 338–353, 1965.

[2] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, Kybernetika, vol.11, pp. 336–344, 1975.

[3] Z. Deng, Fuzzy pseudo metric spaces, Journal of Mathematical Analysis and Applications, vol.86, pp. 74–95, 1982.

[4] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy sets and systems, vol.12, pp. 215–229, 1984.

[5] B. Ali, M. Abbas, Fixed point theorems for multivalued contractive mappings in fuzzy metric spaces, American Journal of Applied Mathematics, vol.3, pp. 41–45, 2015.

[6] T. Dośenovic, D. Rakic, B. Caric, S. Radenovic, Multivalued generalizations of fixed point results in fuzzy metric spaces, Nonlinear Analysis: Modelling and Control, vol.21, pp. 211–222, 2016.

[7] F. Kiany, A. Amini-Harandi, Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces, Fixed Point Theory and Applications, vol.2011, pp. 1–9, 2011.

[8] S. U. Rehman, H. Aydi, G. X. Chen, S. Jabeen, S. U. Khan, Some set-valued and multi-valued contraction results in fuzzy cone metric spaces, Journal of Inequalities and Applications, vol.2021, pp. 110, 2021.

[9] C. Vetro, P. Salimi, Best proximity point results in non-Archimedean fuzzy metric spaces, Fuzzy Information and Engineering, vol.5, pp. 417–429, 2013.

[10] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy sets and systems, vol.64, pp. 395–399, 1994.

[11] F. Smarandache, Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis, Rehoboth, NM: American Research Press, 1998.

[12] M. Kirisci, N. Simsek, Neutrosophic metric spaces, Mathematical Sciences, vol.14, pp. 241–248, 2020.

[13] S. Sowndrarajan, M. Jeyaraman, F. Smarandache, Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems, vol.36, pp. 1, 2020.

[14] U. Ishtiaq, D. A. Kattan, K. Ahmad, T. A. Lazár, V. L. Lazár, L. Guran, On intuitionistic fuzzy N b metric space and related fixed point results with application to nonlinear fractional differential equations. Fractal and Fractional, vol.7, pp. 529, 2023.

[15] W. Shatanawi, K. Abodayeh, A. Mukheimer, Some fixed point theorems in extended b-metric spaces, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, vol.80, pp. 71–78, 2018.

[16] N. Alamgir, Q. Kiran, H. Aydi, A. Mukheimer, A Mizoguchi–Takahashi type fixed point theorem in complete extended b-metric spaces, Mathematics, vol.7, PP. 478, 2019.

[17] M. A. Al-Thagafi, Naseer Shahzad, Convergence and existence results for best proximity points, Nonlinear Analysis: Theory, Methods & Applications, vol.70, pp. 3665–3671, 2009.