On Convex Combinations of Starlike and Convex Functions Associated with the Epicycloid Domain

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Abstract

This paper introduces the class $\mathcal{M}_{\varepsilon,4L}$, defined through a convex combination of starlike and convex functions associated with a four-cusped epicycloid domain, where the parameter satisfies $0 \leq \varepsilon \leq 1$. Unlike earlier studies that focused on circular or conic domains, this work extends the geometric framework to epicycloidal domains. Within this framework, sharp estimates for the first coefficients are obtained, together with the Fekete-Szegö inequality and the second Hankel determinant evaluations. These findings extend several classical results for starlike and convex functions and offer new perspectives on analytic function theory related to epicycloidal domains.

Keywords: ε-convex functions; epicycloid domain; univalent functions; coefficient bounds; Fekete–Szegö inequality; Hankel determinant

1 Introduction

Consider the family $\mathcal{H}(\Delta)$ consisting of analytical functions within an open unit disk $\Delta = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Let $\mathcal{A}$ consist of functions $g \in \mathcal{H}(\Delta)$, normalize by conditions $g^{'}(0) = 1$ and $g(0) = 0$, which having the following series form

$$g(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad \xi \in \Delta. \tag{1}$$

We define $\mathcal{S}$ the subclass of $\mathcal{A}$ that are injective in Δ. Examples of well-known types within this class include starlike and convex functions, as discussed in [3].

Let P represent the set of Carathéodory functions comprising analytic functions $p(\xi)$ which holds under the condition $\mathrm{Re} p(\xi) > 0$ where $\mathrm{Re}\, p(\xi)$ denotes the real part of $p(\xi)$ and analytic in Δ. For every $p \in P$ be a function represented in the form

$$p(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n. \tag{2}$$

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The concept of subordination plays a central role in the development of Geometric Function Theory (GFT). According to Miller and Mocanu (see [13]), given two analytic functions $g_1, g_2 \in \mathcal{H}(\Delta)$, g_1 is considered subordinate to g_2 (denoted $g_1 \prec g_2$) given the existence of Schwarz function, $\varpi \in \mathcal{H}(\Delta)$, for $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$ in which $g_1(\xi) = g_2(\varpi(\xi))$. Specifically, $g_1 \prec g_2$ implies that $g_1(0) = g_2(0)$ and $g_1(\Delta) \subset g_2(\Delta)$.

Ma and Minda [11] employed the idea of subordination to define subclasses of convex and starlike and functions. They initially presented the general expression for the class of starlike functions as follows:

$$\mathcal{S}^*(\partial) := \left\{ g \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec \partial(\xi) \right\},$$

and for convex functions as:

$$\mathcal{K}(\partial) := \left\{ g \in \mathcal{A} : 1 + \frac{\xi g''(\xi)}{fg(\xi)} \prec \partial(\xi) \right\}.$$

We consider a function ∂ normalized by $\partial(0) = 1$ and $\partial'(0) > 0$ whose real part remains positive in the unit disk. The functions map Δ onto regions that are starlike about point 1 and are symmetric along the real axis. The use of ∂ in both definitions allows a unified and generalized approach to the study of subclasses associated with starlikeness and convexity through subordination.

The study of analytic and univalent functions has also been extended through fuzzy set and neutrosophic concepts. Fuzzy subordination in GFT was introduced by Oros [16] and later developed into fuzzy differential subordinations [17], adapting Miller and Mocanu's framework. Subsequent works explored subordinations involving operators such as Wanas, Noor–Sălăgean, and Ruscheweyh (see Lupăș [10]). More recently, Oros [18] established univalence criteria, while Shah et al. [23] studied fuzzy α-convex functions. These developments illustrate how fuzzy and neutrosophic approaches enrich GFT. In particular, they demonstrate that uncertainty and indeterminacy can be systematically incorporated into classical subclasses of univalent functions, thereby offering a natural extension pathway for the present study.

Another classical direction of generalizations is the class of convex combinations of starlike and convex functions, introduced by Mocanu [14] and developed further by Miller et al. [12]. This class interpolates geometric behaviors through a real parameter $\varepsilon \in [0, 1]$, leading to the notion of ε-convex functions, which interpolate between starlike ($\varepsilon = 0$) and convex ($\varepsilon = 1$) cases. The general form of this class is defined as

$$\mathcal{M}_\varepsilon := \operatorname{Re}\left\{ (1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)} + \varepsilon\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \right\} > 0, \tag{3}$$

where $\varepsilon \in [0, 1]$. This inequality characterizes functions that exhibit behavior intermediate between starlikeness and convexity, depending on the value of ε.

Sizhuk and Chernikov [28] extended this framework by considering the ε-convex functions of order β, denoted $\mathcal{M}(\varepsilon, \beta)$, defined by the condition:

$$\operatorname{Re}\left\{ (1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)} + \varepsilon\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \right\} > \beta,$$

where $0 \leq \beta < 1$. These generalizations provide greater flexibility in controlling the geometric behavior of the mappings and have been widely investigated in the context of coefficient estimates, radius problems, and extremal function analysis.

In recent years, various subclasses have been investigated as specific cases of the class $S^*(\partial)$. Some authors replaced ∂ with functions such as Fibonacci numbers, Bell numbers and modified sigmoid function, along with shell-like curves and conic domains (see [8]). More broadly, Lupăș [10] introduced fuzzy set concepts into the geometric theory of analytic functions, providing a framework for addressing uncertainty in subclasses of univalent functions.

Recently, attention has turned to subclasses of univalent functions associated with non-circular domains, particularly those related to epicycloids. An epicycloid is a curve on the plane formed by a point lying on the perimeter of a circle of radius b, rolling along the circumference of a stationary circle with radius a, without slipping [7]. In 2020, Gandhi [6], introduced the subclass of starlike functions defined in a domain with three cusps, as follows:

$$\mathcal{S}^*_{3,\mathrm{L}} := \left\{ g \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4 \right\}, \xi \in \Delta. \tag{4}$$

Later on, Gandhi et al. [7] introduced a generalized expression involving an $n-1$ cusps of epicycloid. This function satisfies all the criteria provided by Ma and Minda [11] regarding the properties of the unified class of starlike functions. Notably, for $n \geq 2$, Gandhi et al. [7] presented a subclass of starlike function, written in the following

$$\mathcal{S}^*_{n-1,\mathrm{L}} := \left\{ g \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec 1 + \frac{n}{n+1}\xi + \frac{1}{n+1}\xi^n \right\}, \xi \in \Delta. \tag{5}$$

Subsequently, Shi et al. [27] analyzed the subclass $\mathcal{S}^*_{n-1,\mathrm{L}}$ for $n = 5$, yielding the subclass $S^*_{4,\mathrm{L}}$ mapped onto a four-leaf domain, which can be represented as follows:

$$\mathcal{S}^*_{4,\mathrm{L}} := \left\{ g \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec 1 + \frac{5}{6}\xi + \frac{1}{6}\xi^5 \right\}, \xi \in \Delta. \tag{6}$$

Motivated by the convex-combination framework of Mocanu [14] and Miller et al. [12], as well as the more recent epicycloid-associated subclasses studied by Gandhi [6, 7] and Shi [26, 27], this paper proposes a new generelized subclass that integrates these approaches. Thus, we define the following class:

Definition 1.1. The function $f \in \mathcal{M}_{\varepsilon,4\mathrm{L}}$ if and only if

$$(1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)} + \varepsilon\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) \prec 1 + \frac{5}{6}\xi + \frac{1}{6}\xi^5, \quad \xi \in \Delta,$$

where $0 \leq \varepsilon \leq 1$.

From the concept of subordination, $f \in \mathcal{M}_{\varepsilon,4\mathrm{L}}$ holds if and only if

$$(1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)} + \varepsilon\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) = 1 + \frac{5}{6}\varpi(\xi) + \frac{1}{6}\left[\varpi(\xi)\right]^5 = p(\xi), \quad \varpi \in \mathcal{H}(\Delta). \tag{7}$$

If $\varepsilon = 0$, the class $\mathcal{M}_{\varepsilon,4\mathrm{L}}$ coincides with the starlike class $\mathcal{S}^*_{4,\mathrm{L}}$, while for $\varepsilon = 1$, it reduces to the convex class associated with the same four-leaf epicycloid domain.

In the context of univalent functions, a significant area of study involves the Hankel determinant. In [19] and [20], the concept of Hankel determinant of order q was originally formulated by Pommerenke, denoted by $H_q(n)$ for integers $n \geq 1$ and $q \geq 1$, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular, if $q = 2$ while $n = 1$, one obtains $H_2(1) = |a_3 - a_2^2|$. In the setting of univalent functions within in Δ, the inequality $|H_2(1)| = |a_3 - a_2^2| \leq 1$ holds, with sharpness attained by the Koebe function (see [3]). In a related direction, Fekete and Szegö [5] introduced a generalization by considering the functional $|a_3 - \mu a_2^2|$ where $\mu \in \mathbb{R}$, which is now commonly called the Fekete-Szegö functional. The problem under consideration involves finding the sharp maximal bound of the expression within the class $\mathcal{S}$. Next, for $q = n = 2$, the second Hankel determinant reduces to $|H_2(2)| = \left|a_2 a_4 - a_3^2\right|$. Shi et al. [26] obtained sharp estimates for the early coefficients and derived sharp bounds for the second and third Hankel determinants, together with a sharp Fekete–Szegö inequality for the subclass $\mathcal{S}^*_{3,L}$. Subsequently, Shi et al. [27] derived the Fekete–Szegö inequality and sharp bounds for Hankel determinants corresponding to the second and third orders in $\mathcal{S}^*_{n-1,L}$, and obtained exact results for the order three in $\mathcal{S}^*_{4,L}$. Additional studies on the Hankel determinant for various function classes can be found in [24], [29], [2], [25] and [30].

The novelty of this work lies in the formulation of a subclass that not only extends coefficient problems and determinant inequalities from circular to epicycloidal domains but also unifies two fundamental cases, namely starlike and convex functions, under specific parameter choices. This study establishes bounds for the a_2, a_3, a_4, Fekete-Szegö inequalities, along with the second Hankel determinant corresponding to $\mathcal{M}_{\varepsilon,4\mathrm{L}}$.

2 Preliminary Results

The main results are supported by the lemmas presented in this section.

Lemma 2.1. ([4]) *Suppose $p \in P$ satisfies the representation in* (2) *with $\mu \in \mathbb{R}$, then*

$$|c_n - \mu c_k c_{n-k}| \leq 2 \max\{1, |2\mu - 1|\}, 1 \leq k \leq n-1. \tag{8}$$

Lemma 2.2. ([3]) *Every $p \in P$ given by* (2)*, the sharp estimate $|\mathrm{c}_n| \leq 2$ holds for every $n \geq 1$.*

Lemma 2.3. ([1,22]) *Suppose $p \in P$ as expressed in* (2)*. If $0 \leq \chi \leq 1$ and $\chi(2\chi - 1) \leq \delta \leq \chi$, then the following sharp inequality holds:*

$$\left|c_3 - 2\chi c_1 c_2 + \delta c_1^3\right| \leq 2. \tag{9}$$

Lemma 2.4. ([21], [9]) *For a function $p \in P$ expressed as in* (2)*, there exists x, δ satisfying $|x| \leq 1$, $|\delta| \leq 1$ with $c_1 \in [0, 2]$, such that*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right) x, \tag{10}$$

$$4c_3 = c_1^3 + 2c_1 x(4 - c_1^2) - x^2 c_1 (4 - c_1^2) + 2(1 - |x|^2)(4 - c_1^2)\delta. \tag{11}$$

3 Main Results

To commence, we derive the coefficient bounds corresponding to the class $\mathcal{M}_{\varepsilon,4\mathrm{L}}$.

Theorem 3.1. *Suppose $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$ expressed as in* (1)*. Then*

$$|a_2| \leq \frac{5}{6(1+\varepsilon)},$$

$$|a_3| \leq \frac{5}{12(1+2\varepsilon)},$$

$$|a_4| \leq \frac{5}{18(1+3\varepsilon)}.$$

Proof. Assumed that $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$. Then, using the concept of subordination, it follows that a Schwarz function $\varpi(z)$ exists, fulfilling the condition $\varpi(0) = 0$ and $\varpi(z) < 1$, leading to

$$(1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)} + \varepsilon\left(1 + \frac{\xi g''(\xi)}{g'(\xi)}\right) = 1 + \frac{5}{6}\varpi(\xi) + \frac{1}{6}\left[\varpi(\xi)\right]^5. \tag{12}$$

Let the function be defined as

$$p\left(\xi\right)=\frac{1+\varpi\left(\xi\right)}{1-\varpi\left(\xi\right)}=1+c_1\xi+c_2\xi^2+c_3\xi^3+\cdots. \tag{13}$$

As $p\in P$, we deduce that

$$\begin{aligned}\varpi(\xi)=\frac{p(\xi)-1}{p(\xi)+1}&=\frac{c_1\xi+c_2\xi^2+c_3\xi^3+c_4\xi^4+\cdots}{2+c_1\xi+c_2\xi^2+c_3\xi^3+c_4\xi^4+\cdots}\\&=\tfrac{1}{2}c_1\xi+\left(\tfrac{1}{2}c_2-\tfrac{1}{4}c_1^2\right)\xi^2+\left(\tfrac{1}{8}c_1^3-\tfrac{1}{2}c_1c_2+\tfrac{1}{2}c_3\right)\xi^3\\&\quad+\left(\tfrac{1}{2}c_4-\tfrac{1}{2}c_1c_3-\tfrac{1}{4}c_2^2-\tfrac{1}{16}c_1^4+\tfrac{3}{8}c_1^2c_2\right)\xi^4+\cdots\end{aligned} \tag{14}$$

Now, using (1), we get

$$\begin{aligned}\frac{\xi g'(\xi)}{g(\xi)}&=1+a_2\xi+\left(2a_3-a_2^2\right)\xi^2+\left(a_2^3-3a_2a_3+3a_4\right)\xi^3\\&\quad+\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)\xi^4+\cdots\end{aligned} \tag{15}$$

and

$$\begin{aligned}1+\frac{\xi g''(\xi)}{g'(\xi)}&=1+2a_2\xi+(6a_3-4a_2^2)\xi^2+\left(12a_4-18a_2a_3+8a_3^2\right)\xi^3\\&\quad+\left(48a_2^2a_3-16a_2^4-32a_2a_4-18a_3^2+20a_5\right)\cdots.\end{aligned} \tag{16}$$

Next, the left-hand expression in equation (12) can be expressed as

$$\begin{aligned}&(1-\varepsilon)\frac{\xi g'(\xi)}{g(\xi)}+\varepsilon\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)\\&=1+\left(a_2+a_2\varepsilon\right)\xi+\left(2a_3-a_2^2+4a_3\varepsilon-3a_2^2\varepsilon\right)\xi^2\\&\quad+\left(a_2^3-3a_2a_3+3a_4+7a_3^2\varepsilon-15a_2a_3\varepsilon+9a_4\varepsilon\right)\xi^3+\cdots.\end{aligned} \tag{17}$$

By substituting the series expansion of (14) into (12), we obtain the right-hand side of (12) as follows

$$\begin{aligned}1+\frac{5}{6}\varpi(\xi)+\frac{1}{6}[\varpi(\xi)]^5&=1+\frac{5}{12}c_1\xi+\left(\frac{5}{12}c_2-\frac{5}{24}c_1^2\right)\xi^2\\&\quad+\left(\frac{5}{12}c_3-\frac{5}{12}c_1c_2+\frac{5}{48}c_1^3\right)\xi^3+\cdots.\end{aligned} \tag{18}$$

From the comparison of (17) and (18), we derive

$$a_2=\frac{5}{12\left(1+\varepsilon\right)}c_1, \tag{19}$$

$$a_3=\frac{5}{288}\left[\frac{12}{(1+2\varepsilon)}c_2-\left(\frac{6(1+\varepsilon)^2-5\left(1+3\varepsilon\right)}{\left(1+\varepsilon\right)^2\left(1+2\varepsilon\right)}\right)c_1^2\right], \tag{20}$$

and

$$\begin{aligned}a_4=\frac{5}{10368(1+3\varepsilon)}&\left[\left(72-\frac{50(1+7\varepsilon)}{(1+\varepsilon)^2}-\frac{30(1+\varepsilon)^2(3+15\varepsilon)-25(1+3\varepsilon)(3+15\varepsilon)}{(1+\varepsilon)^3(1+2\varepsilon)}\right)c_1^3\right.\\&\left.-\left(288-\frac{360(1+\varepsilon)^2(3+15\varepsilon)-300(1+3\varepsilon)(3+15\varepsilon)}{(1+\varepsilon)(1+2\varepsilon)\left(6(1+\varepsilon)^2-5(1+3\varepsilon)\right)}\right)c_1c_2+288c_3\right].\end{aligned} \tag{21}$$

By applying **Lemma 2.2**, we deduce that

$$|a_2| \leq \frac{5}{6\left(1+\varepsilon\right)}. \tag{22}$$

From (20), we rearrange and get

$$|a_3| = \frac{5}{24\left(1+2\varepsilon\right)}\left|c_2 - \left(\frac{6(1+\varepsilon)^2 - 5\left(1+3\varepsilon\right)}{12(1+\varepsilon)^2}\right)c_1^2\right|.$$

Utilizing **Lemma 2.1** with $\mu = \frac{6(1+\varepsilon)^2-5(1+3\varepsilon)}{12(1+\varepsilon)^2}$, we obtain

$$|a_3| \leq \frac{5}{12\left(1+2\varepsilon\right)}. \tag{23}$$

To estimate the bound of $|a_4|$, we consider the expression

$$\begin{aligned}|a_4| \leq \frac{5}{36(1+3\varepsilon)}\Bigg|c_3 &- \left(1 - \frac{30(1+\varepsilon)^2(3+15\varepsilon) - 25(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)(1+2\varepsilon)\left(6(1+\varepsilon)^2 - 5(1+3\varepsilon)\right)}\right)c_1c_2 \\ &+ \left(\frac{1}{4} - \frac{25(1+7\varepsilon)}{144(1+\varepsilon)^2} - \frac{30(1+\varepsilon)^2(3+15\varepsilon) - 25(1+3\varepsilon)(3+15\varepsilon)}{288(1+\varepsilon)^3(1+2\varepsilon)}\right)c_1^3\Bigg|.\end{aligned}$$

By selecting the parameters

$$\chi = \frac{1}{2}\left(1 - \frac{30(1+\varepsilon)^2\left(3+15\varepsilon\right) - 25\left(1+3\varepsilon\right)\left(3+15\varepsilon\right)}{24\left(1+\varepsilon\right)\left(1+2\varepsilon\right)\left(6(1+\varepsilon)^2 - 5\left(1+3\varepsilon\right)\right)}\right)$$

and

$$\delta = \frac{1}{4} - \frac{25\left(1+7\varepsilon\right)}{144(1+\varepsilon)^2} - \frac{30(1+\varepsilon)^2\left(3+15\varepsilon\right) - 25\left(1+3\varepsilon\right)\left(3+15\varepsilon\right)}{288(1+\varepsilon)^3\left(1+2\varepsilon\right)},$$

one checks that for $0 \leq \varepsilon \leq 1$, the inequalities $0 \leq \chi \leq 1$, $\chi - \delta > 0$ and $\chi\left(2\chi - 1\right) < 0 \leq \delta \leq \chi$ hold. Consequently, the requirements of **Lemma 2.3** are satisfied, which leads to the bound

$$|a_4| \leq \frac{5}{18\left(1+3\varepsilon\right)}. \tag{24}$$

Proof of **Theorem 3.1** completes. □

For $\varepsilon = 0$, the following corollary holds:

Corollary 3.2. *Suppose $g \in \mathcal{S}^*_{4,\mathrm{L}}$. Then,*

$$|a_2| \leq \frac{5}{6}, \tag{25}$$

$$|a_3| \leq \frac{5}{12}, \tag{26}$$

and

$$|a_4| \leq \frac{5}{18}. \tag{27}$$

The inequality of (25), (26) *and* (27) *from* ***Corollary 3.2*** *coincides with the sharp bounds obtained by Gandhi et al. [7] for the class of $\mathcal{S}^*_{4,\mathrm{L}}$.*

Next, we prove for the Fekete–Szegö inequality within the class $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$.

Theorem 3.3. *If* $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$ *be given by the form* (1)*. Then for any* $\lambda \in \mathbb{R}$,

$$\left|a_3 - \lambda a_2^2\right| \le \frac{5}{12\,(1+2\varepsilon)} \max\left\{1, \frac{5\,(2\lambda-1) + 5\varepsilon\,(4\lambda-3)}{6(1+\varepsilon)^2}\right\}.$$

Proof. Using (19) and (20) yields

$$\begin{aligned} a_3 - \lambda a_2^2 &= \frac{5}{288}\left[\frac{12}{1+2\varepsilon}c_2 - \left(\frac{6(1+\varepsilon)^2 - 5(1+3\varepsilon)}{(1+\varepsilon)^2(1+2\varepsilon)}\right)c_1^2\right] - \frac{25\lambda}{144(1+\varepsilon)^2}c_1^2 \\ &= \frac{5}{24(1+2\varepsilon)}\left[c_2 - \left(\frac{6(1+\varepsilon)^2 - 5(1+3\varepsilon)}{12(1+\varepsilon)^2}\right)c_1^2\right] - \frac{25\lambda}{144(1+\varepsilon)^2}c_1^2. \end{aligned}$$

After simplifying, we get

$$\left|a_3 - \lambda a_2^2\right| \le \frac{5}{24(1+2\varepsilon)}\left|c_2 - \left(\frac{6(1+\varepsilon)^2 - 5(1+3\varepsilon) + 10\lambda(1+2\varepsilon)}{12(1+\varepsilon)^2}\right)c_1^2\right|.$$

Invoking **Lemma 2.1** with $\mu = \frac{6(1+\varepsilon)^2 - 5(1+3\varepsilon) + 10\lambda(1+2\varepsilon)}{12(1+\varepsilon)^2}$, we obtain

$$\left|a_3 - \lambda a_2^2\right| \le 2\left|\frac{5}{24(1+2\varepsilon)}\right| \max\left\{1, \left|\ \frac{12\lambda(1+2\varepsilon) + 6(1+\varepsilon)^2}{6(1+\varepsilon)^2} - \frac{5(1+3\varepsilon)}{6(1+\varepsilon)^2} - 1\ \right|\right\}.$$

By simplifying, we get

$$\left|a_3 - \lambda a_2^2\right| \le \frac{5}{12(1+2\varepsilon)} \max\left\{1, \frac{5(2\lambda-1) + 5\varepsilon(4\lambda-3)}{6(1+\varepsilon)^2}\right\}. \tag{28}$$

Proof of **Theorem 3.3** completes. ☐

Setting $\lambda = 1$ and $\varepsilon = 0$, the corollary below follows.

Corollary 3.4. *Let* $g \in \mathcal{S}^*_{4,\mathrm{L}}$*. Then,*

$$\left|a_3 - \lambda a_2^2\right| \le \frac{5}{12}.$$

The inequality presented in ***Corollary 3.4*** *aligns with the sharp inequality obtained by Shi et al. [27] for the class* $\mathcal{S}^*_{4,\mathrm{L}}$.

Next, we prove for the second Hankel determinant within the class $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$.

Theorem 3.5. *Suppose* $g \in \mathcal{M}_{\varepsilon,4\mathrm{L}}(\varepsilon)$ *has the form outlined in* (1)*, then*

$$\left|a_2 a_4 - a_3^2\right| \le \frac{25}{144(1+2\varepsilon)^2}.$$

Proof. Using (19)–(21) yields

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{248832(1+3\varepsilon)} \Bigg[& \frac{14400}{1+\varepsilon} c_1c_3 - \frac{10800(1+3\varepsilon)}{(1+2\varepsilon)^2} c_2^2 \\ & + \Bigg(\frac{3600}{1+\varepsilon} - \frac{18000(1+\varepsilon)^2(3+15\varepsilon) - 15000(1+3\varepsilon)(3+15\varepsilon)}{12(1+\varepsilon)^4(1+2\varepsilon)} \\ & \quad - \frac{2500(1+7\varepsilon)}{(1+\varepsilon)^3} - 75(1+3\varepsilon) \left(\frac{6(1+\varepsilon)^2(1+3\varepsilon) - 5(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)} \right)^2 \Bigg) c_1^4 \\ & - \Bigg(\frac{14400}{1+\varepsilon} - \frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{12(1+\varepsilon)^2(1+2\varepsilon)\left(6(1+\varepsilon)^2 - 5(1+3\varepsilon)\right)} \\ & \quad - \frac{10800(1+\varepsilon)^2(1+3\varepsilon) - 9000(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) c_1^2c_2 \Bigg]. \end{aligned}$$

By **Lemma 2.4**, c_2 and c_3 may be represented using c_1 and by **Lemma 2.2**, assuming that $c_1 = c \in [0,2]$. Applying modulus operation together with the triangle inequality, it follows that

$$\begin{aligned} \left| a_2a_4 - a_3^2 \right| \le \frac{1}{248832(1+3\varepsilon)} \Bigg[& \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} |x|^2 (4-c^2)^2 \\ & + \frac{7200}{1+\varepsilon} (1-|x|^2)(4-c^2)c\delta \\ & + \frac{3600}{1+\varepsilon} |x|^2 c^2 (4-c^2) \\ & + \Bigg(\frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) |x|(4-c^2)c^2 \\ & + \Bigg(75(1+3\varepsilon) \left(\frac{6(1+\varepsilon)^2(1+3\varepsilon) - 5(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)} \right)^2 \\ & + \frac{2500(1+7\varepsilon)}{(1+\varepsilon)^3} + \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} \\ & + \frac{18000(1+\varepsilon)^2(3+15\varepsilon) - 15000(1+3\varepsilon)(3+15\varepsilon)}{12(1+\varepsilon)^4(1+2\varepsilon)} \\ & + \frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{10800(1+\varepsilon)^2(1+3\varepsilon) - 9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) c^4 \Bigg]. \end{aligned}$$

Let $|x| = y$ with y ≤ 1 and by taking $|\delta| \leq 1$, we derive $G\left(c, y\right)$ as

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leq \frac{1}{248832(1+3\varepsilon)} \Bigg[& \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} y^2(4-c^2)^2 \\ & + \frac{7200}{1+\varepsilon}(1-y^2)(4-c^2)c \\ & + \frac{3600}{1+\varepsilon} y^2c^2(4-c^2) \\ & + \Bigg(\frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) y(4-c^2)c^2 \\ & + \Bigg(75(1+3\varepsilon) \left(\frac{6(1+\varepsilon)^2(1+3\varepsilon) - 5(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)} \right)^2 \\ & + \frac{2500(1+7\varepsilon)}{(1+\varepsilon)^3} + \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} \\ & + \frac{18000(1+\varepsilon)^2(3+15\varepsilon) - 15000(1+3\varepsilon)(3+15\varepsilon)}{12(1+\varepsilon)^4(1+2\varepsilon)} \\ & + \frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{10800(1+\varepsilon)^2(1+3\varepsilon) - 9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) c^4 \Bigg]. \end{aligned} \tag{29}$$

For every fixed $c \in [0, 2]$ and $y \in [0, 1]$, from (29) confirms that $G'(c, y) > 0$. Consequently, $G(c, y)$ increases with y which implies $G\left(c, y\right) \leq G\left(c, 1\right) = G\left(c\right)$. Thus, we have

$$\begin{aligned} G(c) = \frac{1}{248832(1+3\varepsilon)} \Bigg[& \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} (4-c^2)^2 \\ & + \frac{3600}{1+\varepsilon} c^2(4-c^2) \\ & + \Bigg(\frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) (4-c^2)c^2 \\ & + \Bigg(75(1+3\varepsilon) \left(\frac{6(1+\varepsilon)^2(1+3\varepsilon) - 5(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)} \right)^2 \\ & + \frac{2500(1+7\varepsilon)}{(1+\varepsilon)^3} + \frac{2700(1+3\varepsilon)}{(1+2\varepsilon)^2} \\ & + \frac{18000(1+\varepsilon)^2(3+15\varepsilon) - 15000(1+3\varepsilon)(3+15\varepsilon)}{12(1+\varepsilon)^4(1+2\varepsilon)} \\ & + \frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2 - 5(1+3\varepsilon))} \\ & + \frac{10800(1+\varepsilon)^2(1+3\varepsilon) - 9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg) c^4 \Bigg]. \end{aligned} \tag{30}$$

Then, by differentiating the function (30), we get

$$\begin{aligned} G'(c) = \frac{1}{248832(1+3\varepsilon)} \Bigg\{ & \frac{10800(1+3\varepsilon)}{(1+2\varepsilon)^2}(c^3-4c) \\ & + \frac{3600}{1+\varepsilon}(8c-4c^3) \\ & + \Bigg[\frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{24(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2-5(1+3\varepsilon))} \\ & \quad + \frac{9000(1+3\varepsilon)^2}{2(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg] (8c-4c^3) \\ & + \Bigg[300(1+3\varepsilon)\left(\frac{6(1+\varepsilon)^2(1+3\varepsilon) - 5(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)}\right)^2 \\ & + \frac{10000(1+7\varepsilon)}{(1+\varepsilon)^3} + \frac{10800(1+3\varepsilon)}{(1+2\varepsilon)^2} \\ & + \frac{18000(1+\varepsilon)^2(3+15\varepsilon) - 15000(1+3\varepsilon)(3+15\varepsilon)}{3(1+\varepsilon)^4(1+2\varepsilon)} \\ & + \frac{216000(1+\varepsilon)^2(3+15\varepsilon) - 180000(1+3\varepsilon)(3+15\varepsilon)}{6(1+\varepsilon)^2(1+2\varepsilon)(6(1+\varepsilon)^2-5(1+3\varepsilon))} \\ & + \frac{21600(1+\varepsilon)^2(1+3\varepsilon) - 18000(1+3\varepsilon)^2}{(1+\varepsilon)^2(1+2\varepsilon)^2} \Bigg] c^3 \Bigg\}. \end{aligned}$$

By solving the equation $G'(c) = 0$ and considering the assumption $c \in [0, 2]$, it follows that the maximal value of $G(c)$ is reached when $c = 0$. Consequently, based on equation (29), $G(c, y)$ reaches its greatest value when $c = 0$ and $y = 1$, yields the corresponding upper bound

$$\left|a_2a_4 - a_3^2\right| \leq \frac{25}{144(1+2\varepsilon)^2}. \tag{31}$$

Proof of **Theorem 3.5** completes. □

By setting $\varepsilon = 0$, the corollary below holds.

Corollary 3.6. *Let $g \in \mathcal{S}^*_{4,\mathrm{L}}$. Then,*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{25}{144}. \tag{32}$$

The inequality presented in (32) *of* ***Corollary 3.6*** *aligns with the sharp bound established by Shi et al. [27] for the class of $\mathcal{S}^*_{4,\mathrm{L}}$.*

Conclusion

This paper defined a new subclass, denoted as $\mathcal{M}_{\varepsilon,4\mathrm{L}}$, defined through a convex combination of two classical families: the starlike type and the convex type, associated with the epicycloid domain having four cusps. For this class, general coefficient bounds were established, and sharp initial coefficients, the Fekete–Szegö inequality, and the Hankel determinant of order two were obtained as corollaries for specific parameter choices. In particular, by assigning suitable parameter values, the coefficient bounds reduce to the sharp results previously obtained by Gandhi et al. [7], as shown in Corollary 3.2. Similarly, Corollaries 3.4 and 3.6 illustrate that the outcomes of the Fekete–Szegö functional together with the second order of Hankel determinant coincide with the sharp estimates established by Shi et al. [27]. These corollaries highlight the unifying role of the present class, directly connecting it to several well-studied subclasses of analytic functions. Overall, the unified framework presented here captures the transitional geometric behavior of the convex-combination class and deepens the understanding of its analytic properties. It also provides a basis for further studies on related functionals, including inverse coefficient estimates, higher-order Hankel and Toeplitz determinants, and mappings associated with more general epicycloidal or other non-circular domains. In addition, future research may consider neutrosophic extensions of the present class, where uncertainty and indeterminacy can be incorporated into the parameter choices and domain conditions. Such an approach could lead to novel applications of univalent function theory within decision-making frameworks, optimization problems, and systems influenced by incomplete or indeterminate information.

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