



A Developed Non-Polynomial Spline Method for Solving Fuzzy Partial Differential Equations

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Abstract

This research introduces a novel approach to the non-polynomial spline dependent method for solving fuzzy partial differential equations. The tensor product of non-polynomial spline functions is derived in order to obtaining a solution to fuzzy partial differential equations, such as fuzzy hyperbolic and parabolic equations. The advantage of this method is that it simplifies the complex procedure that arises from the term of the typical product of a fuzzy number by fuzzy functions. Examples are presented to show that the outcomes of the research indicate that the technique is extremely useful to construct the solution to the desired fuzzy partial differential equations.

Keywords: Fuzzy partial differential equations; Cubic non-polynomial spline; Approximate solution

1. Introduction

Partial Differential Equations (PDEs) are fundamental to modeling and understanding complex real-world phenomena, from fluid dynamics and wave propagation to heat transfer and quantum mechanics, making them indispensable tools in various scientific and engineering applications [1]. We would severely limit our ability to comprehend and interact with the complex world around us without a detailed study of PDEs. Zadeh invented the concept of fuzzy sets in 1965 [2]. There have been instances where the behaviour of uncertainty in some system processes has been characterized as fuzzy rather than stochastic in nature. Recently, there has been a notable surge in the scholarly community's interest in investigating fuzzy applications [3]. Chang and Zadeh [4] proposed the idea of the fuzzy derivative. Then Dubois and Prade introduced the extension principle [5]. The development of differential and integral calculus for fuzzy-valued functions has been the subject of recent research, as evidenced by the literature [6, 7]. Khan and Aziz have proposed the cubic non-polynomial spline (CNPS) technique [8], and the authors of [9] to solve boundary value problems apply a mid-knot CNPS. In [10], Hasan converted the nonlinear initial value problem, known as the Lane-Emden equation, into a Volterra integral equation to solve it using NPS. Then, the method extended to tackle the 2-D linear fuzzy Fredholm integral equation [11]. Ibrahim and Hasan in [12] developed the non-polynomial spline (NPS) to get an approximate solution of the fuzzy nonlinear Volterra integral equation. This work focuses on developing an NPS-dependent method for solving FPDEs.

2. Preliminaries

We recall that $E^1 = \{\tilde{u}: R \rightarrow [0,1]\}$, \tilde{u} satisfies

- \tilde{u} is normal, i.e., $\exists x_0 \in R$ s.t. $u(x_0) = 1$
 - \tilde{u} is convex, i.e., $\forall r \in [0,1], x, y \in R$
- $$\tilde{u}(rx + (1-r)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}$$

Holds,

- \tilde{u} is an USC, i.e., $\forall x_0 \in R \quad \lim_{x \rightarrow x_0} \tilde{u}(x) \leq \tilde{u}(x_0)$
- $\text{supp}\tilde{u} = \{x \in R: \tilde{u}(x) > 0\}$ is the support of \tilde{u} , and $cl(\text{supp}\tilde{u})$ is compact

For $0 \leq \alpha \leq 1$, for $\tilde{u}, \tilde{v} \in E^1, k \in R$ defined

$$[\tilde{u} + \tilde{v}]_\alpha = \tilde{u}_\alpha + \tilde{v}_\alpha$$

$$[k\tilde{u}]_\alpha = k[\tilde{u}]_\alpha$$

Define $D: E^1 \times E^1 \rightarrow R^+ \cup \{0\}$ by the equation

$$D(\tilde{u}, \tilde{v}) = \sup d([\tilde{u}]_\alpha, [\tilde{v}]_\alpha)$$

where D is Hausdorff matrix

$$D([\tilde{u}]_\alpha, [\tilde{v}]_\alpha) = \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\bar{u}_\alpha - \bar{v}_\alpha|\}$$

Satisfies the properties, [13]

- (E^1, D) is complete metric space
- $D(\tilde{u}, \tilde{v}) = D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$
- $D(k\tilde{u}, k\tilde{v}) = |k|D(\tilde{u}, \tilde{v})$

Definition 1 [14]. Parametric form of fuzzy number u is a pair $(\underline{u}_\alpha, \bar{u}_\alpha)$ of functions

$\underline{u}_\alpha, \bar{u}_\alpha, 0 \leq \alpha \leq 1$, which satisfy:

- Both \underline{u}_α non-decreasing and \bar{u}_α non-increasing bounded left continuous functions in $(0,1]$, and right continuous at 0
- $\underline{u}_\alpha \leq \bar{u}_\alpha, 0 \leq \alpha \leq 1$

The α -level set of fuzzy numbers is a closed and bounded interval $[\underline{u}(x, \alpha), \bar{u}(x, \alpha)]$, where $\underline{u}(x, \alpha)$ denotes the left-hand endpoint of $[u(x)]_\alpha$ and $\bar{u}(x, \alpha)$ denotes the right-hand endpoint of $[u(x)]_\alpha$.

Definition 2 [15]. A fuzzy function f of 2-variables is assigned to each ordered pair (x, t) , in a set D , a unique fuzzy number $f(x, t)$. The set D is the domain of f , and its range is the set of values taken by f , i.e., $\{f(x, t): (x, t) \in D\}$. The fuzzy valued function $f: D \rightarrow E^1$ is expressed by parametric form $f(x, t, \alpha) = [\underline{f}(x, t, \alpha), \bar{f}(x, t, \alpha)]$, for all $(x, t) \in D$ and $\alpha \in [0,1]$.

Definition 3 [16]. The tensor product of matrices, if $S: R^M \rightarrow R^M$ and $T: R^N \rightarrow R^N$ are matrices, we define the linear mapping $S \otimes T: L_{M,N}(R) \rightarrow L_{M,N}(R)$

by linear extension of $(S \otimes T)(e_i \otimes e_j) = (Se_i) \otimes (Te_j)$.

The linear mapping $S \otimes T$ is called the tensor product of the matrices S and T , where e_i , and e_j basis of matrices S, T respectively.

Definition 4 [10]. Let the spline function $s(x)$ with parameter $k > 0$ has the $\text{span}\{1, x, \sin kx, \cos kx\}$ define on $\Omega: a = x_0 < x_1 < \dots < x_n = b$, with $x_j = a + jh, 0 \leq j \leq n$, where $h = (b - a)/n$.

restriction $s_j(x)$ on $[x_{j-1}, x_j], 1 \leq j \leq n - 1$ satisfies

$$s_j(x) = a_j \sin k(x - x_j) + b_j \cos k(x - x_j) + c_j (x - x_j) + d_j$$

where a_j, b_j, c_j , and d_j are constant and k , is frequency of the trigonometric functions which will be used to raise the accuracy of the method.

Now we develop CNPS $s(\cdot)$ by tensor product to get spline function of two dimension $s(\cdot, \cdot)$ to solve FPDE.

3. The method

The method of CNPS is used to solve 2-types of FPDE approximately

1) Fuzzy parabolic equation

$$(\partial_t - \beta^2 \partial_x \partial_x) \tilde{U}(x, t) = 0 \quad 0 < x < \zeta, t > 0$$

β constant, ∂_t, ∂_x partial derivative of \tilde{U} with respect to t and x respectively with conditions

$$\tilde{U}(0, t) = \tilde{k}_1, \tilde{U}(\zeta, t) = \tilde{k}_2, \quad t > 0$$

$$\tilde{U}(x, 0) = \tilde{f}(x), \quad 0 < x < \zeta$$

2) Fuzzy hyperbolic equation

$$(\partial_t \partial_t - \beta^2 \partial_x \partial_x) \tilde{U}(x, t) = 0, \quad 0 < x < \zeta, \quad t > 0$$

with conditions

$$\tilde{U}(0, t) = \tilde{U}(\zeta, t) = 0, \quad t > 0$$

$$\tilde{U}(x, 0) = \tilde{f}(x), \quad \frac{\partial \tilde{U}}{\partial t}(x, 0) = \tilde{g}(x), \quad 0 < x < \zeta$$

The parametric form of the fuzzy number \tilde{U} is denoted as

$$\tilde{U} = (\underline{U}, \overline{U})$$

$$\tilde{k}_1 = (\underline{k}_1, \overline{k}_1), \quad \tilde{k}_2 = (\underline{k}_2, \overline{k}_2)$$

$$\tilde{f}(x) = (\underline{f}(x), \overline{f}(x))$$

$$\tilde{g}(x) = (\underline{g}(x), \overline{g}(x))$$

The two-dimensional CNPS function is defined as follows

$$\tilde{S}_{m,n}(x, t) = \sum_{i=0}^m \sum_{j=0}^n c_{ij} (\tilde{S}_i(x) \otimes \tilde{S}_j(t)) \quad \forall (x, t) \in [a, b]^2$$

where \otimes denotes the Kronecker product, c_{ij} are constant coefficients to be determined $S_i(x)$ and $S_j(t)$ are CNPS described in definition 5.

The parametric form of $\tilde{S}_{m,n}$ denoted as $\tilde{S}_{m,n} = (\underline{S}_{m,n}, \overline{S}_{m,n})$,

$$\tilde{S}_i = (\underline{S}_i, \overline{S}_i) \text{ and } \tilde{S}_j = (\underline{S}_j, \overline{S}_j) \text{ with conditions}$$

$$\left. \begin{aligned} \sum_{i=1}^m \sum_{j=1}^n c_{ij} \tilde{S}_{m,n}(x, 0) &= \tilde{f}(x) \\ \sum_{i=1}^m \sum_{j=1}^n c_{ij} \tilde{S}_{m,n}(0, t) &= 0 \\ \text{and } \sum_{i=1}^m \sum_{j=1}^n c_{ij} \tilde{S}_{m,n}(\zeta, t) &= 0 \end{aligned} \right\} \quad (1)$$

For the parabolic equation with conditions (1), we have

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} \left(\frac{\partial}{\partial t} \tilde{S}_{m,n}(x, t) - \beta^2 \frac{\partial}{\partial x \partial x} \tilde{S}_{m,n}(x, t) \right) = 0 \quad (2)$$

For hyperbolic equation we

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} \left(\frac{\partial^2}{\partial t \partial t} \tilde{S}_{m,n}(x, t) - \beta^2 \frac{\partial}{\partial x \partial x} \tilde{S}_{m,n}(x, t) \right) = 0 \quad (3)$$

with conditions (1) and the derivative of the initial condition

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij} \frac{\partial}{\partial t} \tilde{S}_{m,n}(x, 0) = 0.$$

4. Illustrative Examples

For numerical results, we define the partition of the x-axis such that step size $h = 1/3$ and partition of the t-axis with step size $k = 0.01$ so we have the points

$$(0,0.01), (0,0.02), (0,0.03), \left(\frac{1}{3}, 0.01\right), \left(\frac{1}{3}, 0.02\right), \left(\frac{1}{3}, 0.03\right), \left(\frac{2}{3}, 0.01\right), \left(\frac{2}{3}, 0.02\right), \left(\frac{2}{3}, 0.03\right), (1,0.01), (1,0.02), (1,0.03).$$

Remark. We obtain different points for another partition, and the tensor product's dimension matrix also differs from our choice here.

Example 4.1. Consider the fuzzy parabolic problem

$$\frac{\partial \tilde{u}}{\partial t}(x, t) - \frac{\partial^2 \tilde{u}}{\partial x^2}(x, t) = 0, \quad 0 < x < \zeta, \quad t > 0 \quad (4)$$

with conditions

$$\tilde{u}(0, t) = \tilde{u}(\zeta, t) = 0, \quad t > 0 \quad (5)$$

$$\tilde{u}(x, 0) = \tilde{k} \cos(\pi x - \frac{\pi}{2}) \quad (6)$$

$$\text{where } \tilde{k}(\alpha) = [\underline{k}(\alpha), \overline{k}(\alpha)] = [\alpha - 1, 1 - \alpha]$$

The exact solutions are

$$\underline{u}(x, t; \alpha) = \underline{k}(\alpha) e^{-\pi^2 t} \cos(\pi x - \pi/2)$$

$$\overline{u}(x, t; \alpha) = \overline{k}(\alpha) e^{-\pi^2 t} \cos(\pi x - \pi/2)$$

Substitute the points $(0,0), (\frac{1}{3}, 0), (\frac{2}{3}, 0), (1,0)$ in initial condition (6) and the points $(0,0.01), (0,0.02), (0,0.03)$ and $(1,0.01), (1,0.02), (1,0.03)$ in boundary condition (5).

Substitute the points $(\frac{1}{3}, 0.01), (\frac{1}{3}, 0.02), (\frac{1}{3}, 0.03), (\frac{2}{3}, 0.01), (\frac{2}{3}, 0.02), (\frac{2}{3}, 0.03)$ in equation (4). By tensor product illustrated in section 3, we have in matrix form $AC = b$

where C vector of dim. 16×1 to be determined, the matrix A of dim. 16×16 conformed by tensor product the points of initial condition (6), boundary condition (5) and equation (4).

For fuzzy parameter $\alpha = 0, 0.1, \dots, 1$. The results of Example 4.1 are shown in Table 1 :

Table 1: Results of Example 4.1

$\underline{u}(x, \alpha)$	$\underline{s}(x, \alpha)$	Error	$\bar{u}(x, \alpha)$	$\bar{s}(x, \alpha)$	Error
-0.711	-0.721	-0.01	0.711	0.721	0.01
-0.64	-0.649	-9.096e-3	0.64	0.649	9.096e-3
-0.569	-0.577	-8.085e-3	0.569	0.577	8.085e-3
-0.498	-0.505	-7.075e-3	0.498	0.505	7.075e-3
-0.427	-0.433	-6.064e-3	0.427	0.433	6.064e-3
-0.355	-0.361	-5.053e-3	0.355	0.361	5.053e-3
-0.284	-0.288	-4.043e-3	0.284	0.288	4.043e-3
-0.213	-0.216	-3.032e-3	0.213	0.216	3.032e-3
-0.142	-0.144	-2.021e-3	0.142	0.144	2.021e-3
-0.071	-0.072	-1.011e-3	0.071	0.072	1.011e-3
0	0	0	0	0	0

Example 4.2. Consider the fuzzy hyperbolic problem

$$\frac{\partial^2 \bar{u}}{\partial t^2}(x, t) - 4 \frac{\partial^2 \bar{u}}{\partial x^2}(x, t) = 0, \quad 0 < x < \zeta, t > 0 \quad (7)$$

with conditions

$$\bar{u}(0, t) = \bar{u}(\zeta, t) = 0, \quad t > 0 \quad (8)$$

$$\bar{u}(x, 0) = \bar{k} \sin(\pi x), \quad 0 < x < \zeta \quad (9)$$

where $\bar{k}(\alpha) = [\underline{k}(\alpha), \bar{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$

The exact solutions are

$$\underline{u}(x, t; \alpha) = \underline{k}(\alpha) \sin(\pi x) \cos(2\pi t)$$

$$\bar{u}(x, t; \alpha) = \bar{k}(\alpha) \sin(\pi x) \cos(2\pi t)$$

As in example 4.1 substitutes the partition points in initial condition (9), boundary condition (8) and equation (7).

For fuzzy parameter = 0, 0.1, ..., 1. The outcomes of Example 4.2 are shown Table 2:

Table 2: Results of Example 4.2

$\underline{u}(x, \alpha)$	$\underline{s}(x, \alpha)$	Error	$\bar{u}(x, \alpha)$	$\bar{s}(x, \alpha)$	Error
0.644	0.644	-1.474e-4	1.074	1.074	-2.457e-4
0.666	0.666	-1.523e-4	1.053	1.052	-2.408e-4
0.687	0.687	-1.572e-4	1.031	1.031	-2.358e-4
0.709	0.709	-1.621e-4	1.01	1.009	-2.309e-4
0.73	0.73	-1.671e-4	0.988	0.988	-2.26e-4
0.752	0.752	-1.72e-4	0.967	0.966	-2.211e-4
0.773	0.773	-1.769e-4	0.945	0.945	-2.162e-4
0.795	0.795	-1.818e-4	0.924	0.923	-2.113e-4
0.816	0.816	-1.867e-4	0.902	0.902	-2.064e-4
0.838	0.838	-1.916e-4	0.881	0.88	-2.014e-4
0.859	0.859	-1.965e-4	0.859	0.859	-1.965e-4

Figures 1 and 2 illustrate the accurate approximate upper and lower solutions of Example 4.1 and Example 4.2, respectively, one can see from the figures that the results of the method and the exact nearly overlap with each other with small errors.

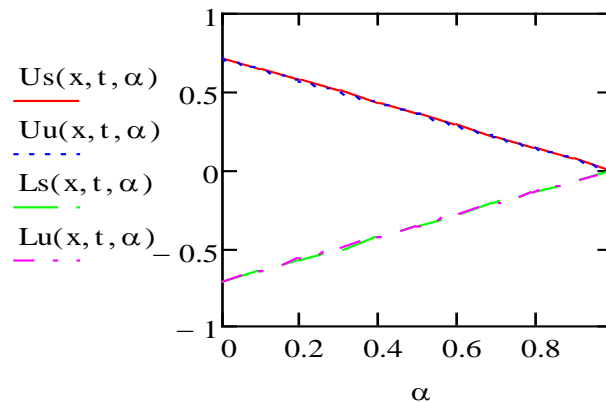


Figure 1. Results of Example 4.1

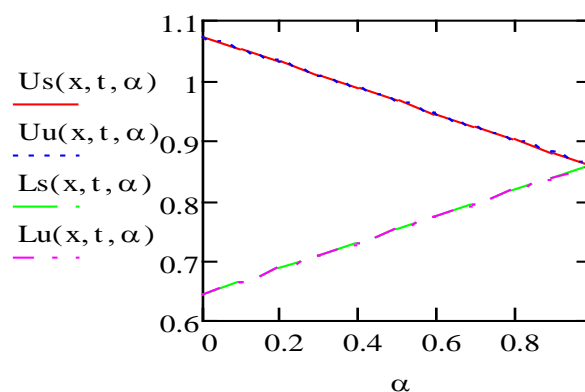


Figure 2. Results of Example 4.2

5. Conclusion

We present CNPS to approximate solutions of parabolic and hyperbolic fuzzy partial differential equations. The results in tables and figures demonstrate the method's efficiency, particularly regarding trigonometric property that allows the spline method to have an infinite derivative. In Figure 1 and Figure 2, we plot the graphs of Example 4.1 and Example 4.2 of the approximate and exact solutions for fuzzy parameter α , $\alpha \in [0,1]$, where $Us(x, t; \alpha)$ and $Ls(x, t; \alpha)$ upper and lower approximate solutions. $Uu(x, t; \alpha)$ and $Lu(x, t; \alpha)$ upper and lower exact solutions. A key insight of this work is the recognition that fuzzy PDEs can be conceptualized as a special case of neutrosophic PDEs where the indeterminacy and falsity components are constrained. This understanding not only validates the proposed method but also lays a robust foundation for future work, allowing for the generalization of this method to solve a broader range of neutrosophic PDEs, thereby addressing more complex systems with inherent uncertainty.

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