Asymptotic Solution to the Scalar Version of the Two Body Problem When the Two Bodies Collide - A Case Study

Ahmed Bakheet[1,*], **Ali Abdulhussein**[1], **Laheeb Muhsen Noman**[1]
[1]Mustansiriyah University, College of Science, Department of Mathematics, Baghdad, Iraq

Emails: ahmedbakheet@uomustansiriyah.edu.iq; aaabdulhussein@uomustansiriyah.edu.iq;
laheeb-muhsen@uomustansiriyah.edu.iq

Abstract

The main goal of this paper is to obtain a special form of asymptotic solutions to the scalar version of the two body problem whenever the two bodies collide on the real line at the collision time. It has been shown that the desired asymptotic solution maintains certain properties when t approaches the collision time. However, it is not easy to Handel such a mission without the employment of successive approximations technique. The successive approximations technique has been modified and adjusted to serve as the main tool in the process of obtaining such solution. Moreover, it has been shown that the series of successive approximations converges absolutely and uniformly to a continuous function that approaches to 0 when t attains the collision time in a certain interval. The problem of one dimensional collision between the two bodies has been solved asymptotically at the collision time.

Keywords: Two Body Problem; Collision; Asymptotic; Successive Approximation

1 introduction

Mathematical models are fundamental tools that help us understand and predict phenomena across countless aspects of life, from economics and biology to engineering and physics. They allow us to simplify complex systems, identify key relationships, and explore various scenarios, providing invaluable insights that would be difficult or impossible to gain otherwise[1] . The two body problem is classified as the simplest case of the N- body problem as there are only two bodies with masses $m_1$and m_2 respectively. The two body problem can be defined as follows: Given the 3×1 column vector $r_j = \langle x_j, y_j, z_j \rangle^{\dagger}$ where $x_j, y_j, z_j \in \mathbb{R}$, note that the symbol † represents the transpose of the vector r_j for every $j = 1, 2$. The 2-body problem is given by the 2nd order differential equations:

$$\frac{d^2 r_1}{dt^2} = m_2 \frac{r_2 - r_1}{\|r_2 - r_1\|^3}. \tag{1}$$

$$\frac{d^2 r_2}{dt^2} = -m_1 \frac{r_2 - r_1}{\|r_2 - r_1\|^3}. \tag{2}$$

Where the two vectors $r_1 = r_1(t)$, $r_2 = r_2(t)$ represent the position vectors. It is necessary to define the collision between the two bodies $m_1$and$m_2$at specific time say t_0, it is known that two bodies $m_1$and m_2 collide at t_0 if $\|r_2 - r_1\| \to 0$ as $t \to t_0$. From the way that classical system of the two body problem is defined, it is clear that the system does not allow such kind of motion. In other words, for t approaches the collision time t_0, the right sides of the above two equations tend to undetermined term [2]. Gonzalo J. L. and Bombardelli C. have found an approximate analytical solution to the perturbed two body problem by utilizing a radial, low acceleration along with the method of multiple scales, for more information, see [3]. Furthermore, Bombardelli, C., Baù, G., Peláez, J. in [4] obtained analytical solution of the two body problem perturbed by a constant tangential acceleration by utilizing the perturbation theory. The solution is valid for circular and

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elliptic orbits with generic eccentricity, describes the instantaneous time variation of all orbital elements. The main goal here is to obtain an asymptotic solution to the two body problem when the collision of the two bodies occur on the real line. In other words, our focus is to investigate a very special form of the asymptotic solution which maintains certain properties when $\|r_2 - r_1\| \to 0$ as $t \to t_0$. The collision between the two bodies has been an interesting problem to tackle as the classical system does not allow the collision between the two bodies to happen at any finite time and the solution then does not exist. The successive approximation technique is considered as one of the main tools in the theory of ordinary differential equation and has been widely used to construct a solution to the desired problem. In this work, the successive approximation technique has been reformulated to serve as an effective tool which enables us to construct a continuous function in such a way that the series of successive approximation converges uniformly and absolutely in a certain interval. The plan of this work goes as follows: In section one, we present an overview about the classical two body problem and what does collision between two bodies mean. In section two we explain the asymptotic assumption and the properties of the special asymptotic form. In section three, we introduce an overview about the mathematical induction hypothesis and obtaining the estimate of the non-linear and non-homogeneous part of the converted equation of the two body problem. In sections four and five respectively, two levels of mathematical inductions have been discussed and analyzed, namely, the levels; $l \leq j$ and $l = j + 1$. The convergence of the series of the successive approximation is discussed in section six. In section seven, we discuss how to construct the solution to the desired two body problem when the two bodies collide on the real line. In section eight, we analyze the connections with the other researchers' work. In section nine, we present the conclusion of this work.

Proposition 1.1. *Consider the vector* $r_2 - r_1$ *which represents the motion of the particle* m_2 *with respect to the particle* m_1 *, then the two body problem becomes:*

$$\frac{d^2}{dt^2}(r_2 - r_1) = -M^* \frac{r_2 - r_1}{\|r_2 - r_1\|^3};\ M^* = m_1 + m_2. \tag{3}$$

Proof. The proof is straightforward. We define a new dependent variable W_{21} and we will convert 3 into a new form as shown in the below lemma. ☐

Proposition 1.2. *Given the* 2 *by* 1 *vector* W_{21} *defined below:*

$$W_{21} = \frac{r_2 - r_1}{\|r_2 - r_1\|^3}. \tag{4}$$

Then 3 can be rewritten as:

$$\frac{d^2}{dt^2}(\|W_{21}\|^{\frac{-3}{2}} W_{21}) = -M^* W_{21}. \tag{5}$$

Proof. The proof is straightforward.

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Proposition 1.3. *Given the differential equation 3, assume that*

$$\|W_{21}\| = | W_{21} | = W_{21}. \tag{6}$$

Where W_{21} has to be a positive scalar for every $0 < t \leq \rho$ for fixed ρ , $0 < \rho < 1$. Then the differential equation 3 can be rewritten in terms of the dependent variable W_{21} as shown below:

$$\frac{1}{2} W_{21}^{\frac{-3}{2}} \frac{d^2 W_{21}}{dt^2} - \frac{3}{4} W_{21}^{\frac{-5}{2}} (\frac{dW_{21}}{dt})^2 = M^* W_{21}. \tag{7}$$

Proof. The proof is straightforward.

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2 Asymptotic Assumption

In this section, a special form of the asymptotic solution will be presented. In other words, the solution is :

$$W_{21}(t) = At^m(1 + \triangle(t)). \tag{8}$$

Where A is a nonzero constant , m is a real number ,and $\triangle(t)$ is a continuous function in some interval $0 < \|\triangle\| \leq \sigma$, for fixed σ , $1 > \sigma > 0$ .Moreover, the hypothesis of the asymptotic solution is built based on the following assumption.

$$\triangle(t), \frac{d\triangle(t)}{dt}, \frac{d^2\triangle(t)}{dt^2} \longrightarrow 0. \tag{9}$$

As $t \longrightarrow 0$.

Proposition 2.1. *The first and second derivatives of $W_{21}(t)$ with respect to tare given by:*

$$\frac{dW_{21}(t)}{dt} = At^m\frac{d\triangle(t)}{dt} + Amt^{m-1}(1 + \triangle(t)). \tag{10}$$

$$\frac{d^2W_{21}(t)}{dt^2} = At^m\frac{d^2\triangle(t)}{dt^2} + 2Amt^{m-1}\frac{d\triangle(t)}{dt} + Am(m-1)t^{m-2}(1 + \triangle(t)). \tag{11}$$

We are now at a stage that enables us to convert the differential equation 7 from the terms of W_{21} as a dependent variable to the terms of $\triangle$. To this end, plug in the first and second derivatives of W_{21} respectively as functions of t in the differential equation 7, this yields the below new differential equation:

$$\tfrac{1}{2}\{A^{\frac{-1}{2}}t^{\frac{-m}{2}}(1+\triangle)^{\frac{-3}{2}}\}\tfrac{d^2\triangle}{dt^2}+\{A^{\frac{-1}{2}}mt^{\frac{-m}{2}-1}(1+\triangle)^{\frac{-3}{2}}\}\tfrac{d\triangle}{dt}+\tfrac{1}{2}A^{\frac{-1}{2}}m(m-1)t^{\frac{-m}{2}-2}(1+\triangle)^{\frac{-1}{2}}-\tfrac{3}{4}\{A^{\frac{-5}{2}}t^{\frac{-5m}{2}}(1+\triangle)^{\frac{-5}{2}}\}$$
$$\boldsymbol{.}[A^2t^{2m}(\frac{d\triangle}{dt})^2 + 2A^2mt^{2m-1}\frac{d\triangle}{dt}(1 + \triangle) + A^2m^2t^{2m-2}(1 + \triangle)^2] = M^*\{At^m(1 + \triangle)\}. \tag{12}$$

Notice that the value of m is still obscure, it is assumed that m is a real value but it is worthwhile to investigate which value of m satisfies the asymptotic hypothesis 9. In sum, different cases of m will be studied in the below analysis.

For $m = 0$, equation 12 becomes:

$$\frac{d^2\triangle}{dt^2} - \frac{3}{2}(1 + \triangle)^{-1}(\frac{d\triangle}{dt})^2 = 2M^*A^{\frac{3}{2}}(1 + \triangle)^{\frac{5}{2}}. \tag{13}$$

Where $0 < \|\triangle\| \leq \sigma$ for fixed $0 < \sigma < 1$, provided that $A \neq 0$. The case of $m = 0$, leads to the following solution

$$W_{21} = A(1 + \triangle) \tag{14}$$

In the next two cases two possibilities of the value of m are determined. The case that $m > 0$ is not possible for the form 12. Suppose that $m > 0$. There is a specific negative value for m as we will see in the upcoming case. Given 12, assume that m is strictly negative then there is a unique value $m = \frac{-4}{3}$ which balances the right and left hand sides in 12. The next step is to determine a value of the constant A under the conditions given below in the next lemma. The proof can be handled by utilizing the asymptotic analysis.

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$$\frac{d^2\triangle}{dt^2} - \frac{3}{2}(1 + \triangle)^{-1}(\frac{d\triangle}{dt})^2 = 2M^*A^{\frac{3}{2}}(1 + \triangle)^{\frac{5}{2}}. \tag{15}$$

Where $0 < \|\triangle\| \leq \sigma$ for fixed $0 < \sigma < 1$, provided that $A \neq 0$. The case of $m = 0$, leads to the following solution

$$W_{21} = A(1 + \triangle) \tag{16}$$

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The case that $m > 0$ is not possible for the form 12. Suppose that $m > 0$. By the aid of the differential equation 12, we balance the powers in the left and right hand sides from 12by finding the leading term whenever $t \longrightarrow 0$ as follows. In the left hand side we have the following terms: $t^{\frac{-m}{2}}$, $t^{\frac{-m}{2}-1}$, $t^{\frac{-m}{2}-2}$, $t^{\frac{-5m}{2}}t^{2m}$,$t^{\frac{-5m}{2}}t^{2m-1}$,$t^{\frac{-5m}{2}}t^{2m-2}$, where the terms $t^{\frac{-5m}{2}}t^{2m}$,$t^{\frac{-5m}{2}}t^{2m-1}$,$t^{\frac{-5m}{2}}t^{2m-2}$ are repeated. Finally there are three candidates $t^{\frac{-m}{2}}$,$t^{\frac{-m}{2}-1}$,$t^{\frac{-m}{2}-2}$ in the left hand side, while in the right hand side we have the term t^m. Since $m > 0$, then $\frac{-m}{2}$, $\frac{-m}{2} - 1$, $\frac{-m}{2} - 2$ are all negative , if $t \longrightarrow 0$, then $t^{\frac{-m}{2}}$,$t^{\frac{-m}{2}-1}$,$t^{\frac{-m}{2}-2}$ tend to ∞ while t^m tends to 0 which means that the L.S tends to ∞ and R.S tends to 0 provided that $\triangle(t)$ is bounded by $0 < \|\triangle\| \leq \sigma$ for fixed $0 < \sigma < 1$, which is impossible and then the case of m being strictly positive cannot happen. Next we try to prove that there is a specific negative value for m as we will see in the upcoming case.

Given 12, assume that m is strictly negative then there is a unique value $m = \frac{-4}{3}$ which balances the right and left hand sides in 12. Recall 12again, since $m < 0$, then $t^m \longrightarrow \infty$ as $t \longrightarrow 0$ and then the R.S tends to ∞ , then we will try to make the L.S tends to ∞ as $t \longrightarrow 0$. But, before that we have to exclude the values of m which force the L.S tending to 0 as follows. Since we know that $m < 0$, then $\frac{-m}{2} > 0$ and then $t^{\frac{-m}{2}} \longrightarrow 0$ as $t \longrightarrow 0$ and then the term $t^{\frac{-m}{2}}$ will not be the leading term , then we are left with two candidates $t^{\frac{-m}{2}-1}$,$t^{\frac{-m}{2}-2}$. Suppose $\frac{-m}{2} - 1 > 0$, then $m < -2$ and if $\frac{-m}{2} - 2 > 0$, then $m < -4$. Then the two intervals $(-\infty, -2)$ and $(-\infty, -4)$ are excluded. Suppose $\frac{-m}{2} - 1 < 0$, then $m > -2$ and suppose also that $\frac{-m}{2} - 2 < 0$, then $m > -4$ and since $m < 0$, then we have two intervals $-4 < m < 0$, and $-2 < m < 0$, The intersection of the two intervals is the interval $-2 < m < 0$. The leading term is $t^{\frac{-m}{2}-2}$ as $t \longrightarrow 0$ since it goes faster to ∞ as $t \longrightarrow 0$. Then for $-2 < m < 0$, the leading term in the L.S is $t^{\frac{-m}{2}-2}$ while in R.S is t^m, then $\frac{-m}{2} - 2 = m \Longleftrightarrow m = -\frac{4}{3}$. The next step is to determine a value of the constant A under the conditions given below in the next lemma.

Proposition 2.2. *Given the differential equation 12 with taking into account that* m *has a unique strictly negative value, in other words,* $m = \frac{-4}{3}$ *, suppose also that* $\triangle(t)$, $\frac{d\triangle(t)}{dt}$, $\frac{d^2\triangle(t)}{dt^2} \longrightarrow 0$ *as* $t \longrightarrow 0$*. Then* $A = [\frac{2}{9M^*}]^{\frac{2}{3}}$.

Proof. The proof can be obtained from Plugging in the value $m = \frac{-4}{3}$ in 12. Notice that if we allow $1 + \triangle(t) = 0$, then $\triangle(t) = -1$, to avoid the singularity we put the restriction $0 < \|\triangle\| \leq \sigma$ where σ is fixed , $0 < \sigma < 1$. Furthermore,

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$$t^2\frac{d^2\triangle}{dt^2} + \frac{4}{3}t\frac{d\triangle}{dt} + \frac{4}{9}\triangle = \frac{4}{9}\{(1+\triangle)^{\frac{5}{2}} - 1\} + \frac{3}{2}t^2(1+\triangle)^{-1}(\frac{d\triangle}{dt})^2. \tag{17}$$

For $0 < \|\triangle\| \leq \sigma$, for fixed σ , $0 < \sigma < 1$. Then $(1+\triangle)^{-1}$is bounded . Since $0 < \|\triangle\| \leq \sigma$ then

$$0 < \|\frac{1}{1+\triangle}\| \leq \frac{1}{1-\|\triangle\|} \leq \frac{1}{1-\sigma}. \tag{18}$$

Then $(1+\triangle)^{-1}$ is bounded below by 0 and bounded above by $\frac{1}{1-\sigma}$.

Proposition 2.3. *Given* $0 < \|\triangle\| \leq \sigma$ *for fixed* $0 < \sigma < 1$ *, suppose that 17 holds , then the differential equation 17can be written in the form*

$$L(\triangle) = NL(\triangle, \frac{d\triangle}{dt}). \tag{19}$$

Where $L(\triangle)$ *is the linear part and* $NL(\triangle, \frac{d\triangle}{dt})$ *is the nonlinear part. Furthermore, the linear and the non-linear parts can be given as shown below:*

$$L(\triangle) = t^2\frac{d^2\triangle}{dt^2} + \frac{4}{3}t\frac{d\triangle}{dt} - \frac{6}{9}\triangle. \tag{20}$$

$$NL(\triangle, \frac{d\triangle}{dt}) = \frac{4}{9}\{(1+\triangle)^{\frac{5}{2}} - 1 - \frac{5}{2}\triangle\} + \frac{3}{2}t^2(1+\triangle)^{-1}(\frac{d\triangle}{dt})^2. \tag{21}$$

Proof. Since $0 < \|\triangle\| \leq \sigma$ for fixed σ , $0 < \sigma < 1$, the proof can be handled by expanding $(1+\triangle)^{\frac{5}{2}}$ as a Taylor series.

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Given the second order differential equation: $t^2\frac{d^2\triangle}{dt^2}+\frac{4}{3}t\frac{d\triangle}{dt}-\frac{6}{9}\triangle=\frac{4}{9}\{(1+\triangle)^{\frac{5}{2}}-1-\frac{5}{2}\triangle\}+\frac{3}{2}t^2(1+\triangle)^{-1}(\frac{d\triangle}{dt})^2$, where $0<\|\triangle\|\leq\sigma$ for fixed σ , $0<\sigma<1$, where both $\triangle$, $\frac{d\triangle}{dt}$ satisfies the below integral equations respectively:

$$\triangle(t)=\triangle_H(t)+\frac{3}{5}\int_0^t k_1(t,s)NL(\triangle(s),\frac{d\triangle(s)}{ds})ds. \tag{22}$$

$$\frac{d\triangle(t)}{dt}=\frac{d\triangle_H(t)}{dt}+\frac{3}{5}\int_0^t k_2(t,s)NL(\triangle(s),\frac{d\triangle(s)}{ds})ds. \tag{23}$$

Where $0\leq s\leq t\leq\rho$. For fixed ρ , $0<\rho<1$. WLOG: set $\eta_2=0$, then $\triangle_H=\eta_1 t^{\frac{2}{3}}$. Moreover, $\triangle(t)$ is well defined on the closed interval $0\leq t\leq\rho$ while the derivative $\frac{d\triangle(t)}{dt}$ is defined on the interval $0<t\leq\rho$ for fixed ρ , $1>\rho>0$, $k_1(t,s),k_2(t,s)$ represent the kernels of the two integral equations respectively. The integral equations 22 and 23 will be employed as a key tool in the finite mathematical induction. The nonlinear part is defined as:

$$NL(\triangle(s),\frac{d\triangle(s)}{ds})=\frac{4}{9}\{(1+\triangle(s))^{\frac{5}{2}}-1-\frac{5}{2}\triangle(s)\}+\frac{3}{2}s^2(1+\triangle(s))^{-1}(\frac{d\triangle(s)}{ds})^2. \tag{24}$$

With taking into account the below inequality:

$$0<\|\frac{1}{1+\triangle(s)}\|\leq\frac{1}{1-\sigma}. \tag{25}$$

For every $0\leq s\leq t\leq\rho$, for fixed ρ, where $0<\rho<1$. After we set the integral equations up in the above definition , we start the successive approximations for both of the integral equations . In this situation , we will have two sequences of functions of the forms $\{\triangle_j(t)\}_{j=0}^{\infty}$, $\{\frac{d\triangle_j(t)}{dt}\}_{j=0}^{\infty}$ and the goal here is to find two continuous functions $\triangle(t)$, $\frac{d\triangle(t)}{dt}$ in some interval $0<t\leq\rho$ for fixed ρ , $0<\rho<1$ chosen carefully such that $\triangle_j(t)\longrightarrow\triangle(t)$, as $j\longrightarrow\infty$ and $t\longrightarrow 0$. Not only that, $\frac{d\triangle_j(t)}{dt}\longrightarrow\frac{d\triangle(t)}{dt}$, $j\longrightarrow\infty$ as $t\longrightarrow 0$. We are at a stage enables us to rewriting the non-linear part. Recall the main equation below:

$$t^2\frac{d^2\triangle_H(t)}{dt^2}+\frac{4}{3}t\frac{d\triangle_H(t)}{dt}-\frac{6}{9}\triangle_H(t)=\frac{4}{9}[(1+\triangle(t))^{\frac{5}{2}}-1-\frac{5}{2}\triangle(t)]+\frac{3}{2}t^2(1+\triangle(t))^{-1}(\frac{d\triangle(t)}{dt})^2.$$

Suppose that:

$$\hat{A}(\triangle(t))=[(1+\triangle(t))^{\frac{5}{2}}-1-\frac{5}{2}\triangle(t)]. \tag{26}$$

Then our equation becomes:

$$t^2\frac{d^2\triangle_H(t)}{dt^2}+\frac{4}{3}t\frac{d\triangle_H(t)}{dt}-\frac{6}{9}\triangle_H(t)=\frac{4}{9}\hat{A}(\triangle(t))+\frac{3}{2}t^2(1+\triangle(t))^{-1}(\frac{d\triangle(t)}{dt})^2, \tag{27}$$

where $0<\|\triangle\|\leq\sigma$, for fixed σ , $0<\sigma<1$. The nonlinear part will be rewritten then as shown below:

$$NL(\triangle(t),\frac{d\triangle(t)}{dt})=\frac{4}{9}\hat{A}(\triangle(t))+\frac{3}{2}t^2(1+\triangle(t))^{-1}(\frac{d\triangle(t)}{dt})^2. \tag{28}$$

The function $\hat{A}(\triangle(t))$ can be rewritten as:

$$\hat{A}(\triangle(t))=\frac{\hat{A}(\triangle(t))}{\triangle^2(t)}\triangle^2(t)=\hat{B}(\triangle(t))\triangle^2(t), \tag{29}$$

where $\hat{B}(\triangle(t))$ is given by:

$$\hat{B}(\triangle(t))=\frac{(1+\triangle(t))^{\frac{5}{2}}-1-\frac{5}{2}\triangle(t)}{\triangle^2(t)}. \tag{30}$$

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Proposition 2.4. *The function $\hat{B}(\triangle(t))$ can be defined as a continuous function in the interval $(-1,\infty)$ and in any closed sub interval of $(-1,\infty)$ of the form $[-\tau,\tau]$ for $0<\tau<1$.*

Proof. The proof can be handled by utilizing L'Hôpital's rule and re-defining the function $\hat{B}(\triangle(t))$ as:

$$\hat{B}(\triangle(t)) = \left\{ \tfrac{15}{8} \quad \triangle(t) = 0 \tfrac{(1+\triangle(t))^{\frac{5}{2}}-1-\frac{5}{2}\triangle(t)}{\triangle^2(t)} \quad \triangle(t) \neq 0 \right\}. \tag{31}$$

□

3 The Induction Hypothesis.

We start with the case of $j=1$, this case is treated carefully through propositions (3.1-3.12). Now assume that the four statements are correct for $l \leq j$, in the next theorem we build the bound $\hat{M}$ based on the hypothesis which says that (1) and (2) are both correct whenever $l \leq j, j \geq 1$. It is very important to know how does the bound $\hat{M}$ look like since we have to have the same representation for $\hat{M}$ in the two different levels $l \leq j$, and $l = j+1$. Moreover , having a representation for the bound $\hat{M}$ will enable us to go through the proof of theorem 4.9 when we estimate the nonlinear part in the stage of $l \leq j$. The plan here is to have representations for the positive constants K and $\hat{M}$in each level $l \leq j$ and $l = j+1$. More than that ,each one of K and $\hat{M}$ have to carry the same formula in the two different stages $l \leq j$ and $l = j+1$ which confirms the independence on the index j.

Starting the induction process with $j=1$, with the below proposition:

Proposition 3.1. *Given the following conditions:**(1)** $0 \leq t \leq \rho$, for fixed ρ , $0<\rho<1$. **(2)** $\triangle_H = \triangle_0 = \eta_1 s^{\frac{2}{3}}$, $\triangle_0$ is defined in the closed interval $0 \leq s \leq t \leq \rho$, for fixed ρ , $0<\rho<1$. **(3)** $\frac{d\triangle_0}{ds}= \frac{2}{3}\eta_1 s^{-\frac{1}{3}}$, $\frac{d\triangle_0}{ds}$ is defined only in the interval $0 < s \leq t \leq \rho$, for fixed ρ , $0<\rho<1$. **(4)** $0 < \|\triangle_0\| \leq \|\eta_1\|\rho^{\frac{2}{3}} \leq \hat{\rho}_1 < 1$, for ρ ,$0<\rho \leq [\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}} < 1$. Then $NL(\triangle_0, \frac{d\triangle_0}{ds})(s)$ satisfies the below inequality:*

$$\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\| \leq \frac{4}{9}\|\hat{B}(\triangle_0)\|\|\triangle_0\|^2 + \frac{3}{2}s^2 \frac{\|\frac{d\triangle_0}{ds}\|^2}{\|1-\|\triangle_0\|\|}, \tag{32}$$

for $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0<\rho<1$.

Proof. The proof can be handled from $NL(\triangle_j, \frac{d\triangle_j}{ds})(s)$, with $j=0$.

□

Proposition 3.2. *Given the conditions (1-4) in proposition 3.1 , then for $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0<\rho<1$. Then: $\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\| \leq \frac{4}{9}\mu\|\triangle_0\|^2 + \frac{3}{2}s^2\frac{\|\frac{d\triangle_0}{ds}\|^2}{\|1-\|\triangle_0\|\|}$, where $\mu = MAX\{\|\hat{B}(\triangle)\| : -\tau \leq \triangle \leq \tau\}$.*

Proof. The proof follows from proposition 3.1 and from proposition 2.4.

□

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Proposition 3.3. *Assume that the conditions (1-4) in proposition 3.1 hold for* $0 \leq s \leq t \leq \rho$ *, where* ρ *is fixed such that* $0 < \rho < 1$*. Then* $\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\| \leq \frac{2}{3}\|\eta_1\|^2\{\frac{2}{3}\mu + \frac{1}{1-\hat{\rho}_1}\}s^{\frac{4}{3}}$.

Proof. The proof follows from propositions (3.1 , 3.2).

□

Proposition 3.4. *Assume that the conditions (1-4) in proposition 3.1 hold for every* $0 \leq s \leq t \leq \rho$ *, where* ρ *is fixed such that* $0 < \rho < 1$ *then:* $\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\| \leq \hat{N}_1 s^{\frac{4}{3}}$*,where* $\hat{N}_1$ *is given by:* $\hat{N}_1 = \frac{2}{3}\|\eta_1\|^2\{\frac{2}{3}\mu + \frac{1}{1-\hat{\rho_1}}\}$, $0 < \hat{\rho_1} < 1$.

Proof. The proof follows from propositions (3.1,3.3) with the choice of $\hat{N}_1 = \frac{2}{3}\|\eta_1\|^2\{\frac{2}{3}\mu + \frac{1}{1-\hat{\rho_1}}\}$. If $\|\eta_1\|$ is chosen to be sufficiently small , then $\hat{N}_1$ can be made as small as desired. Then $\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\| \leq \hat{N}_1 s^{\frac{4}{3}}$. For every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$.

□

Proposition 3.5. *Under the conditions(1-4) of proposition 3.1 , for every* $0 \leq s \leq t \leq \rho$ *, where* ρ *is fixed such that* $0 < \rho < 1$*, then the difference* $\|\triangle_1 - \triangle_0\|$ *is bounded by:* $\|\triangle_1 - \triangle_0\| \leq \frac{9}{104}\hat{N}_1 t^{\frac{10}{3}}$.

Proof. The proof can be handled by setting the formula of successive approximations up with $\triangle_0 = \triangle_H = \eta_1 t^{\frac{2}{3}}$ to be the first approximation to the solution where $0 \leq t \leq \rho < 1$, for fixed $0 < \rho < 1$.

□

Proposition 3.6. *Under the conditions (1-4) of proposition 3.1,for every* $0 \leq s \leq t \leq \rho$ *, where* ρ *is fixed such that* $0 < \rho < 1$*. Then the difference of the derivatives* $\|\frac{d\triangle_1}{dt} - \frac{d\triangle_0}{dt}\|$ *is bounded by:* $\frac{3}{5}[\frac{2}{3}\frac{3}{8} + \frac{3}{13}]\hat{N}_1 t^{\frac{7}{3}}$.

Proof. The proof can be handled by setting the formula of successive approximations up with $\frac{d\triangle_H}{dt} = \frac{d\triangle_0}{dt} = \frac{2}{3}\eta_1 t^{\frac{-1}{3}}$, to be the first approximation for $0 < t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$.

□

Proposition 3.7. *Under the conditions(1-4) of proposition 3.1 hold for every* $0 \leq s \leq t \leq \rho$ *, where* ρ *is fixed such that* $0 < \rho < 1$*. Then* $\|\triangle_1\| \leq g_1(t)t^{\frac{2}{3}}$*, where* $g_1(t) = \|\eta_1\| + \frac{9}{104}\hat{N}_1 t^{\frac{8}{3}}$.

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Proof. By utilizing propositions (3.1,3.5), for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$.

□

Proposition 3.8. *Assume the conditions (1-4) of proposition 3.1 hold for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$. Given the function $g_1(t)$ as in proposition 3.7, then there exists an upper bound function $\hat{g}_1(t)$ such that $\|\triangle_1\| \leq \hat{g}_1(t)t^{\frac{2}{3}}$, for every $0 \leq t \leq \rho$, for fixed ρ , $0 < \rho < 1$. The function $\hat{g}_1(t)$ is given by: $\hat{g}_1(t) = \|\eta_1\| + \hat{N}_1 t^{\frac{8}{3}}$.*

Proof. The proof follows form proposition 3.7.

□

Proposition 3.9. *Assume the conditions(1-4) of proposition 3.1 hold for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$. There exists a positive real number $\hat{M} = \hat{M}(\|\eta_1\|, \hat{N}_1, \rho)$ such that $\|\triangle_1\| \leq \hat{M}t^{\frac{2}{3}}$ where the positive constant $\hat{M}$ is given by: $\hat{M} = \hat{M}(\|\eta_1\|, \hat{N}_1, \rho) = \|\eta_1\| + \hat{N}_1 \frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$.*

Proof. The proof follows from proposition 3.8.

□

Proposition 3.10. *Assume that the conditions(1-4) in proposition 3.1 hold for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$. Then , $\|\frac{d\triangle_1}{dt}\| \leq g_1^{\star}(t)t^{\frac{-1}{3}}$, and $g_1^{\star}(t) = \frac{2}{3}\|\eta_1\| + \frac{15}{52}\hat{N}_1 t^{\frac{8}{3}}$.*

Proof. The proof can be handled by setting the following equation $\frac{d\triangle_1}{dt} = \frac{d\triangle_0}{dt} + (\frac{d\triangle_1}{dt} - \frac{d\triangle_0}{dt})$.

□

Proposition 3.11. *Assume that the conditions(1-4) in proposition 3.1 hold for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$. Then there exists un upper bound function $\hat{g}_1(t)$ for the function $g_1^{\star}(t)$ in the interval $0 \leq t \leq \rho$, for fixed $0 < \rho < 1$ such that $\|\frac{d\triangle_1}{dt}\| \leq \hat{g}_1(t)t^{\frac{-1}{3}}$, while the upper bound function is given by: $\hat{g}_1(t) = \|\eta_1\| + \hat{N}_1 t^{\frac{8}{3}}$.*

Proof. The proof follows from proposition 3.10.

□

Proposition 3.12. *Assume that the conditions(1-4) in proposition 3.1 hold for every $0 \leq s \leq t \leq \rho$, where ρ is fixed such that $0 < \rho < 1$.Then there exists a positive real number $\hat{M} = \hat{M}(\|\eta_1\|, \hat{N}_1, \rho)$ such that $\|\frac{d\triangle_1}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$, where the positive constant $\hat{M}$ is given by: $\hat{M} = \hat{M}(\|\eta_1\|, \hat{N}_1, \rho) = \|\eta_1\| + \hat{N}_1 \frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$.*

Proof. The proof follows from proposition 1.25.

□

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3.1 Building The Estimate of the Non-linear Part for $j \geq 2$.

Definition 3.13. For every $j \geq 1$, we define the the following integral equations:

$$\triangle_j(t) - \triangle_{j-1}(t) = \frac{3}{5}\int_0^t k_1(t,s)\{NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\}ds, \tag{33}$$

where the LS is defined on $0 \leq t \leq \rho$, while the inside of the integral in the RS is defined on $0 \leq s \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$, where $k_1(t,s)$ is the kernel of the integral and it is given by $k_1(t,s) = [t^{\frac{2}{3}}s^{\frac{1}{3}} - t^{-1}s^2]$.

Definition 3.14. For every $j \geq 1$, we define the the following integral equations for derivatives:

$$\frac{d\triangle_j(t)}{dt} - \frac{d\triangle_{j-1}(t)}{dt} = \frac{3}{5}\int_0^t k_2(t,s)\{Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\}ds. \tag{34}$$

Note that each derivative itself is defined on the interval $0 < t \leq \rho$ for fixed ρ , $0 < \rho < 1$ while the difference between the derivatives in the LS is defined on the closed interval $0 \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$.While the inside of the integral in the RS is defined on the closed interval $0 \leq s \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$.The kernel of the integral is given by $k_2(t,s) = \frac{2}{3}t^{\frac{-1}{3}}s^{\frac{1}{3}} + t^{-2}s^2$.

Definition 3.15. The magnitude of the difference of the functions $\triangle_j(t)$ and $\triangle_{j-1}(t)$ is given by:

$$\|\triangle_j(t) - \triangle_{j-1}(t)\| \leq \frac{3}{5}\int_0^t k_1(t,s)\|Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\|ds, \tag{35}$$

for every $j \geq 1$, $0 \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$.

Definition 3.16. The magnitude of the difference of the derivatives $\frac{d\triangle_j(t)}{dt}$, $\frac{d\triangle_{j-1}(t)}{dt}$ is given by:

$$\|\frac{d\triangle_j(t)}{dt} - \frac{d\triangle_{j-1}(t)}{dt}\| \leq \frac{3}{5}\int_0^t k_2(t,s)\|Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\|ds, \tag{36}$$

Notice that the magnitude of the nonlinear part $\|Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\|$, works starting from $j = 2$, then we had to treat the case when $j = 1$ in a slightly different systematic way shown above through propositions (1.15-1.26) since the integral equations of the differences $\|\triangle_1 - \triangle_0\|$ and $\|\frac{d\triangle_1}{dt} - \frac{d\triangle_0}{dt}\|$ require the nonlinear part to be $\|NL(\triangle_0, \frac{d\triangle_0}{ds})(s)\|$ where $0 \leq s \leq \rho$, for fixed ρ , $0 < \rho < 1$. Now we try to give good description for what could be the magnitude of the difference of the non-linear part under certain appropriate conditions. The following theorem gives us an estimate for the magnitude of the difference of the nonlinear parts utilizing the mean value theorem.

Theorem 3.17. *Under the following conditions : (1)* $0 \leq t \leq \rho$, $0 \leq s \leq t$ *for fixed* ρ , $0 < \rho < 1$. *(2)* $0 < \|\triangle_j\| \leq \hat{M}$ *where* $\hat{M}$ *is* $j-$*independent for every* $j \geq 1$. *Then the difference* $Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)$, *can be written as shown below:*

$$\begin{aligned} Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s) &= \hat{R}_1(\triangle_{j-1}, \triangle_{j-2}, \frac{d\triangle_{j-1}}{ds}, \frac{d\triangle_{j-2}}{ds})(s)(\triangle_{j-1} - \triangle_{j-2}) \\ &+ \hat{R}_2(\triangle_{j-1}, \triangle_{j-2}, \frac{d\triangle_{j-1}}{ds}, \frac{d\triangle_{j-2}}{ds})(s)(\frac{d\triangle_{j-1}}{ds} - \frac{d\triangle_{j-2}}{ds}). \end{aligned} \tag{37}$$

The function $\hat{R}_1$, $\hat{R}_2$ are given by the following formulas:

$$\hat{R}_1(s) = \frac{4}{9}\hat{R}_{11}(s) - \frac{3}{2}\hat{R}_{12}(s),\ \hat{R}_2(s) = \frac{3}{2}s^2\frac{(\frac{d\triangle_{j-1}}{ds} + \frac{d\triangle_{j-2}}{ds})}{(1+\triangle_{j-2})}$$

$$\hat{R}_{11}(s) = [\frac{\hat{B}(\triangle_{j-1})\triangle_{j-1}^2 - \hat{B}(\triangle_{j-2})\triangle_{j-2}^2}{\triangle_{j-1} - \triangle_{j-2}}],\ \hat{R}_{12}(s) = s^2\frac{(\frac{d\triangle_{j-1}}{ds})^2}{(1+\triangle_{j-1})(1+\triangle_{j-2})}. \tag{38}$$

Proof. The sketch of the proof is given through the following steps.

□

Step1: For, $0 < \|\triangle_j\| \leq \hat{M}$ where $\hat{M}$ is $j-$independent for every $j \geq 1$, for every $0 \leq t \leq \rho$, $0 \leq s \leq t$ for fixed ρ , $0 < \rho < 1$. The difference term is given by:

$$NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - NL(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s) = \frac{4}{9}[\frac{\hat{B}(\triangle_{j-1})\triangle_{j-1}^2 - \hat{B}(\triangle_{j-2})\triangle_{j-2}^2}{\triangle_{j-1} - \triangle_{j-2}}][\triangle_{j-1} - \triangle_{j-2}]$$
$$+\frac{3}{2}s^2\{\frac{(\frac{d\triangle_{j-1}}{ds})^2 + (\frac{d\triangle_{j-1}}{ds})^2\triangle_{j-2} - (\frac{d\triangle_{j-2}}{ds})^2 - (\frac{d\triangle_{j-2}}{ds})^2\triangle_{j-1}}{(1+\triangle_{j-1})(1+\triangle_{j-2})}\}. \tag{39}$$

Step2: Given $0 < \|\triangle_j\| \leq \hat{M}$ where $\hat{M}$ is $j-$independent for every $j \geq 1$,for every $0 \leq t \leq \rho$, $0 \leq s \leq t$ for fixed ρ , $0 < \rho < 1$. Then the term $(\frac{d\triangle_{j-1}}{ds})^2 + (\frac{d\triangle_{j-1}}{ds})^2\triangle_{j-2} - (\frac{d\triangle_{j-2}}{ds})^2 - (\frac{d\triangle_{j-2}}{ds})^2\triangle_{j-1}$ can be re-written as:

$(\frac{d\triangle_{j-1}}{ds})^2 + (\frac{d\triangle_{j-1}}{ds})^2\triangle_{j-2} - (\frac{d\triangle_{j-2}}{ds})^2 - (\frac{d\triangle_{j-2}}{ds})^2\triangle_{j-1} =[\frac{d\triangle_{j-1}}{ds} - \frac{d\triangle_{j-2}}{ds}][\frac{d\triangle_{j-1}}{ds} + \frac{d\triangle_{j-2}}{ds}][1 + \triangle_{j-1}] - (\frac{d\triangle_{j-1}}{ds})^2(\triangle_{j-1} - \triangle_{j-2})$, for every $0 \leq s \leq \rho$, for fixed ρ , $0 < \rho < 1$.

Step3: Given $0 < \|\triangle_j\| \leq \hat{M}$ where $\hat{M}$ is $j-$independent for every $j \geq 1$, for every $0 \leq t \leq \rho$, $0 \leq s \leq t$ for fixed ρ , $0 < \rho < 1$. Then the difference term $NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - NL(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)$ is written as:

$$NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - NL(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s) = \{\frac{4}{9}[\frac{\hat{B}(\triangle_{j-1})\triangle_{j-1}^2 - \hat{B}(\triangle_{j-2})\triangle_{j-2}^2}{\triangle_{j-1} - \triangle_{j-2}}] - \frac{3}{2}s^2\frac{(\frac{d\triangle_{j-1}}{ds})^2}{(1+\triangle_{j-1})(1+\triangle_{j-2})}\}.$$
$$[\triangle_{j-1} - \triangle_{j-2}] + \frac{3}{2}s^2\frac{(\frac{d\triangle_{j-1}}{ds} + \frac{d\triangle_{j-2}}{ds})}{(1+\triangle_{j-2})}[\frac{d\triangle_{j-1}}{ds} - \frac{d\triangle_{j-2}}{ds}]. \tag{40}$$

Step4: Given $0 < \|\triangle_j\| \leq \hat{M}$ where $\hat{M}$ is $j-$independent for every $j \geq 2$, for every $0 \leq t \leq \rho$, $0 \leq s \leq t$ for fixed ρ , $0 < \rho < 1$. The difference term $Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)$ is written as:

$Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s) = \hat{R}_1(\triangle_{j-1}, \triangle_{j-2}, \frac{d\triangle_{j-1}}{ds}, \frac{d\triangle_{j-2}}{ds})(s)(\triangle_{j-1} - \triangle_{j-2}) + \hat{R}_2(\triangle_{j-1}, \triangle_{j-2}, \frac{d\triangle_{j-1}}{ds}, \frac{d\triangle_{j-2}}{ds})(s)(\frac{d\triangle_{j-1}}{ds} - \frac{d\triangle_{j-2}}{ds})$.

Now we want an estimate for the non-linear part $Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)$ in the interval $0 \leq s \leq t \leq \rho < 1$, for fixed ρ. Since $\|Nl(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s) - Nl(\triangle_{j-2}, \frac{d\triangle_{j-2}}{ds})(s)\| \leq \|\hat{R}_1(s)\|\|\triangle_{j-1} - \triangle_{j-2}\| + \|\hat{R}_2(s)\|\|\frac{d\triangle_{j-1}}{ds} - \frac{d\triangle_{j-2}}{ds}\|$. Then in order to have an estimate for the above nonlinear part we have to find estimates for $\|\hat{R}_1(s)\|$and $\|\hat{R}_2(s)\|$in the interval $0 \leq s \leq t \leq \rho < 1$. The next proposition gives us an estimate for $\|\hat{R}_{11}(s)\|$ by utilizing the mean value theorem.

Proposition 3.18. *Under the following conditions: (1)* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ. *(2)* $0 < \|\triangle_j\| \leq \hat{M}$ *for every* $j \geq 1$. *Given the function* $\theta_1(\triangle_k) = \hat{B}(\triangle_k)\triangle_k^2$. *For* $k \geq 2$ *, then the function* θ_1 *is defined and continuous on the closed interval* $[\|\triangle_{j-1}\|, \|\triangle_{j-2}\|]$ *.Moreover, it is differentiable twice on the open interval* $(\|\triangle_{j-1}\|, \|\triangle_{j-2}\|)$.

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Proof. The proof is straightforward.

□

Proposition 3.19. *Under the following conditions:(1)* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ. *(2)* $0 < \|\triangle_j\| \leq \hat{M}$ *for every* $j \geq 1$. *Given the function* θ_1 *defined on the closed interval* $[\|\triangle_{j-1}\|, \|\triangle_{j-2}\|]$. *There exist at least one* $\hat{\triangle}_{j-1}$ *with* $0 < MIN\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\} \leq \|\hat{\triangle}_{j-1}\| \leq Max\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\}$. *Such that* $[\frac{d\theta_1(\triangle_k)}{d\triangle_k}]_{\triangle_k = \hat{\triangle}_{j-1}} = \frac{\theta_1(\triangle_{j-1}) - \theta_1(\triangle_{j-2})}{\triangle_{j-1} - \triangle_{j-2}} = \hat{R}_{11}(s)$.

Proof. The proof follows from proposition 3.18.

□

Proposition 3.20. *Under the following conditions: (1)* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ. *(2)* $0 < \|\triangle_j\| \leq \hat{M}$ *for every* $j \geq 2$. *Then* $\hat{R}_{11}(s)$ *can be represented as:* $\hat{R}_{11}(s) = [\frac{d\theta_1(\triangle_k)}{d\triangle_k}]_{\triangle_k = \hat{\triangle}_{j-1}} = \frac{5}{2}[(1 + \hat{\triangle}_{j-1})^{\frac{3}{2}} - 1]$.

Proof. The proof follows from proposition (3.19).

□

Proposition 3.21. *Given the function* $\theta_2(\triangle_k) = (1 + \triangle_k)^{\frac{3}{2}} - 1$, *where* $0 < \|\triangle_k\| \leq \hat{M}$, *for every* $k \geq 2$. *For every* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ. *Then the function* θ_2 *is continuous on* $[0, \|\hat{\triangle}_{j-1}\|]$ *and differentiable on* $(0, \|\hat{\triangle}_{j-1}\|)$.

Proof. From the definition of the function θ_2, the function is well defined at any point in the closed interval $[0, \|\hat{\triangle}_{j-1}\|]$ and it is differentiable everywhere in the open interval $(0, \|\hat{\triangle}_{j-1}\|)$.

□

Proposition 3.22. *Given the function* $\theta_2(\triangle_k) = (1 + \triangle_k)^{\frac{3}{2}} - 1$, *where* $0 < \|\triangle_k\| \leq \hat{M}$ *for every* $k \geq 2$ *, for every* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ *.Then there exist at least one* $\bar{\triangle}_{j-1}$ *with the property that:* $0 \leq \|\bar{\triangle}_{j-1}\| \leq \|\hat{\triangle}_{j-1}\|$, *such that:*$(1 + \hat{\triangle}_{j-1})^{\frac{3}{2}} - 1 = \hat{\triangle}_{j-1}\{\frac{d\theta_2(\triangle_k)}{d\triangle_k}\}_{\triangle_k = \bar{\triangle}_{j-1}}$.

Proof. The proof follows from proposition 3.21 and the mean value theorem.

□

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Proposition 3.23. *Given the function* $\theta_2(\triangle_k) = (1+\triangle_k)^{\frac{3}{2}} - 1$*, for* $0 < \|\triangle_k\| \leq \hat{M}$ *for every* $k \geq 1$ *, for* $0 \leq s \leq t \leq \rho < 1$ *for fixed* ρ *.Then* $\hat{R}_{11}(s) = \frac{15}{4}\hat{\triangle}_{j-1}(1+\bar{\triangle}_{j-1})^{\frac{1}{2}}$*, where* $0 \leq \|\bar{\triangle}_{j-1}\| \leq \|\hat{\triangle}_{j-1}\|$.

Proof. The proof follows from proposition 3.22.

□

Proposition 3.24. *Given* $0 < \|\triangle_j\| \leq \hat{M}$ *for every* $j \geq 1$ *, for every* $0 \leq s \leq t$ *for* $0 \leq t \leq \rho$ *for fixed* $0 < \rho < 1$*. Then* $\hat{R}_{11}(s)$ *has the estimate* $\|\hat{R}_{11}(s)\| \leq \frac{15}{4}\|\hat{\triangle}_{j-1}\|(1+\|\hat{\triangle}_{j-1}\|)^{\frac{1}{2}}$ *, in the interval* $[\|\triangle_{j-1}\|, \|\triangle_{j-2}\|]$*. Where* $0 \leq MIN\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\} \leq \|\hat{\triangle}_{j-1}\| \leq Max\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\}$.

Proof. The proof follows from proposition 3.23 and from the mean value theorem. Now we have a final estimate for the value of $\hat{R}_{11}(s)$, it remains to estimate $\hat{R}_1(s)$ and $\hat{R}_2(s)$ in the interval $[\|\triangle_{j-1}\|, \|\triangle_{j-2}\|]$.

□

Proposition 3.25. *Under the following conditions: (1)* $0 \leq s \leq t \leq \rho, 0 < \rho < 1$ *where* ρ *is fixed. (2)* $0 < \|\triangle_j\| \leq \hat{M}$ *, for every* $j \geq 1$*. Then* $\hat{R}_1(s)$ *and* $\hat{R}_2(s)$ *respectively have the following estimates:*
$\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\hat{\triangle}_{j-1}\|(1+\|\hat{\triangle}_{j-1}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}$, $\|\hat{R}_2(s)\| \leq \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|+\|\frac{d\triangle_{j-2}}{ds}\|}{\|1-\|\triangle_{j-2}\|\|}$.

Proof. The proof follows from the definitions of $\hat{R}_1(s)$ and $\hat{R}_2(s)$ respectively.

□

Proposition 3.26. *(Final Estimate): Given* $0 < \|\triangle_j\| \leq \hat{M}$ *for every* $j \geq 1$*, for every* $0 \leq s \leq t \leq \rho$*,* $0 < \rho < 1$ *where* ρ *is fixed. Consider the inequality:* $0 < MIN\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\} \leq \|\hat{\triangle}_{j-1}\| \leq Max\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\}$*. Then* $\|\hat{R}_1(s)\|$ *has the final estimate:*

$$\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\triangle_{j-2}\|(1+\|\triangle_{j-2}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}. \tag{41}$$

Or

$$\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\triangle_{j-1}\|(1+\|\triangle_{j-1}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}. \tag{42}$$

Proof. Since $0 < MIN\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\} \leq \|\hat{\triangle}_{j-1}\| \leq Max\{\|\triangle_{j-1}\|, \|\triangle_{j-2}\|\}$. Then it means , either: $0 < \|\triangle_{j-1}\| \leq \|\hat{\triangle}_{j-1}\| \leq \|\triangle_{j-2}\|$, or: $0 < \|\triangle_{j-2}\| \leq \|\hat{\triangle}_{j-1}\| \leq \|\triangle_{j-1}\|$. By proposition 3.25, we have the following estimate: $\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\hat{\triangle}_{j-1}\|(1+\|\hat{\triangle}_{j-1}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}$. We can rewrite $\|\hat{R}_1(s)\|$ as the following: $\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\triangle_{j-2}\|(1+\|\triangle_{j-2}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}$, or $\|\hat{R}_1(s)\| \leq \frac{5}{3}\|\triangle_{j-1}\|(1+\|\triangle_{j-1}\|)^{\frac{1}{2}} + \frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}$.

□

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Now in the next proposition, we give the final expression for the estimate of the nonlinear part: $NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)$. In other words,

$\|NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)\|$
$\leq \{\frac{5}{3}\|\triangle_{j-2}\|(1+\|\triangle_{j-2}\|)^{\frac{1}{2}}+\frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}\}\|\triangle_{j-1}-\triangle_{j-2}\|+\frac{3}{2}s^2\{\frac{\|\frac{d\triangle_{j-1}}{ds}\|+\|\frac{d\triangle_{j-2}}{ds}\|}{\|1-\|\triangle_{j-2}\|\|}\}\|\frac{d\triangle_{j-1}}{ds}-\frac{d\triangle_{j-2}}{ds}\|$.

Or

$\|NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)\|\leq\{\frac{5}{3}\|\triangle_{j-1}\|(1+\|\triangle_{j-1}\|)^{\frac{1}{2}}+\frac{3}{2}s^2\frac{\|\frac{d\triangle_{j-1}}{ds}\|^2}{\|1-\|\triangle_{j-1}\|\|\|1-\|\triangle_{j-2}\|\|}\}\|\triangle_{j-1}-\triangle_{j-2}\|+\frac{3}{2}s^2\{\frac{\|\frac{d\triangle_{j-1}}{ds}\|+\|\frac{d\triangle_{j-2}}{ds}\|}{\|1-\|\triangle_{j-2}\|\|}\}\|\frac{d\triangle_{j-1}}{ds}-\frac{d\triangle_{j-2}}{ds}\|$.

Proposition 3.27. *(final expression): Given* $0<\|\triangle_j\|\leq\hat{M}$ *for every* $j\geq 2$*, for every* $0\leq s\leq t\leq\rho$ *,* $0<\rho<1$ *, where* ρ *is fixed. The nonlinear part* $NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)$*, has the estimate as shown above.*

Proof. Since $NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)=\hat{R}_1(s)[\triangle_{j-1}-\triangle_{j-2}]+\hat{R}_2(s)[\frac{d\triangle_{j-1}}{ds}-\frac{d\triangle_{j-2}}{ds}]$, for every $0\leq s\leq t\leq\rho$, $0<\rho<1$ where ρ is fixed. Then $\|NL(\triangle_{j-1},\frac{d\triangle_{j-1}}{ds})(s)-NL(\triangle_{j-2},\frac{d\triangle_{j-2}}{ds})(s)\|\leq\|\hat{R}_1(s)\|\|\triangle_{j-1}-\triangle_{j-2}\|+\|\hat{R}_2(s)\|\|\frac{d\triangle_{j-1}}{ds}-\frac{d\triangle_{j-2}}{ds}\|$. Now based on the inequalities which give the estimates of $\|\hat{R}_1(s)\|$ and $\|\hat{R}_2(s)\|$ respectively , we have two images for the estimate of the nonlinear part: This completes the proof. ☐

We work with the first estimate since we will show later that the two estimates give same results as we will show that all the functions $\triangle_j$ are bounded independently on j , for every $j\geq 1$. The above two estimates are the building block that will be used to build the induction hypothesis on j beginning with $j=2$, note that the above two estimates do not work with the case $j=1$ and then we had to treat the case $j=1$ in a completely different way.

3.2 Summary

Based on the initial results introduced above from the two levels $j=1$, $j=2$. It is obtained that: $\|\triangle_1-\triangle_0\|\leq\frac{3}{5}[\frac{3}{8}-\frac{3}{13}]\hat{N}_1t^{\frac{10}{3}}$, $\|\frac{d\triangle_1}{dt}-\frac{d\triangle_0}{dt}\|\leq\frac{3}{5}[\frac{2}{3}\frac{3}{8}+\frac{3}{13}]\hat{N}_1t^{\frac{7}{3}}$ and $\|\triangle_2-\triangle_1\|\leq\frac{3}{5}(\frac{3}{16}-\frac{3}{21})\hat{L}\hat{N}_1t^{\frac{18}{3}}$, $\|\frac{d\triangle_2}{dt}-\frac{d\triangle_1}{dt}\|\leq\frac{3}{5}(\frac{2}{3}\frac{3}{16}+\frac{3}{21})\hat{L}\hat{N}_1t^{\frac{15}{3}}$. We expect the following pattern: $\|\triangle_j-\triangle_{j-1}\|\leq\frac{3}{5}(\frac{3}{8j}-\frac{3}{8j+5})Kt^{\frac{8j+2}{3}}$, $\|\frac{d\triangle_j}{dt}-\frac{d\triangle_{j-1}}{dt}\|\leq\frac{3}{5}(\frac{2}{3}\frac{3}{8j}+\frac{3}{8j+5})Kt^{\frac{8j-1}{3}}$.

3.3 Making the R.S of the two inequalities independent on j .

We will try to estimate the amounts $(\frac{3}{8j}-\frac{3}{8j+5})$ and $(\frac{2}{3}\frac{3}{8j}+\frac{3}{8j+5})$ for every $j\geq 1$ to get a j-independent constant . Since $\frac{3}{8j}-\frac{3}{8j+5}\leq\frac{3}{8j}+\frac{3}{8j+5}$ and $\frac{2}{3}\frac{3}{8j}+\frac{3}{8j+5}\leq\frac{3}{8j}+\frac{3}{8j+5}$. Then the upper bound will be $\frac{3}{8j}+\frac{3}{8j+5}$ For every $j\geq 1$. But before that, we want to find an upper bound for the upper bound $\frac{3}{8j}+\frac{3}{8j+5}$ as we will see below. The following lemmas will not be proved as the proofs are straightforward.

Lemma 3.28. *For every* $j\geq 1$ *then* $\frac{3}{8j}+\frac{3}{8j+5}\leq\frac{3}{4}$.

Lemma 3.29. *For every* $j\geq 1$*, the following inequalities hold:* $\frac{3}{8j}-\frac{3}{8j+5}\leq\frac{3}{4}$, $\frac{2}{3}\frac{3}{8j}+\frac{3}{8j+5}\leq\frac{3}{4}$.

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3.4 Explanation of Induction Hypothesis:

The main purpose here is to prove the following. For every $j \geq 1$, there exist positive real numbers K and $\hat{M}$ which are independent on j such that: (1)$\|\triangle_j - \triangle_{j-1}\| \leq \frac{9}{20}Kt^{\frac{8j+2}{3}}$ (2)$\|\frac{d\triangle_j}{dt} - \frac{d\triangle_{j-1}}{dt}\| \leq \frac{9}{20}Kt^{\frac{8j-1}{3}}$ (3)$\|\triangle_j\| \leq \hat{M}t^{\frac{2}{3}}$ (4)$\|\frac{d\triangle_j}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$. Note that (1), (2),and (3) are defined for every $0 \leq t \leq \rho$, for some fixed $0 < \rho < 1$ chosen carefully. While (4) is defined only on the interval $0 < t \leq \rho$.The induction procedure is made on j, the first step is to show that the four statements are all correct whenever $j = 1$, note that (3) follows from (1) while (4) follows from (2) . This means that the induction procedure goes in parallel. The second step is assuming that the four statements are all correct whenever $l \leq j$, and then the last step is showing that they are all correct whenever $l = j + 1$.We start with the case of $j = 1$, this case is treated carefully through propositions (3.1-3.12). Now assume that the four statements are correct for $l \leq j$, in the next theorem we build the bound $\hat{M}$ based on the hypothesis which says that (1) and (2) are both correct whenever $l \leq j$, $j \geq 1$. It is very important to know how does the bound $\hat{M}$ look like since we have to have the same representation for $\hat{M}$ in the two different levels $l \leq j$, and $l = j + 1$. Moreover , having a representation for the bound $\hat{M}$ will enable us to go through the proof of theorem 4.9 when we estimate the nonlinear part in the stage of $l \leq j$. The plan here is to have representations for the positive constants K and $\hat{M}$in each level $l \leq j$ and $l = j + 1$. More than that ,each one of K and $\hat{M}$ have to carry the same formula in the two different stages $l \leq j$ and $l = j + 1$ which confirms the independence on the index j.

4 The level of $l \leq j$

Theorem 4.1. *For every $0 \leq t \leq \rho$, for fixed ρ , $0 < \rho < 1$. Assume that : $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20}Kt^{\frac{8l+2}{3}}$ and $\|\frac{d\triangle_l}{dt} - \frac{d\triangle_{l-1}}{dt}\| \leq \frac{9}{20}Kt^{\frac{8l-1}{3}}$, for every $l \leq j$, for $j \geq 1$.Then there exists a positive constant $\hat{M}$ such that $\|\triangle_l\| \leq \hat{M}t^{\frac{2}{3}}$and $\|\frac{d\triangle_l}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$ where $K > 0$,for $0 < t \leq \rho$, $\hat{M}$ is given by $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$.*

Proof. The sketch of the proof of the above theorem is given as a sequence of several propositions.

□

Proposition 4.2. *Suppose that all conditions of theorem 4.1 hold , then there exists a function $g_l(t)$ which is given by: $g_l(t) = \|\eta_1\| + \sum_{k=1}^{l} \frac{9}{20}Kt^{\frac{8k}{3}}$ Such that $\|\triangle_l\| \leq g_l(t)t^{\frac{2}{3}}$.*

Proposition 4.3. *Suppose that all conditions of theorem 4.1 hold , then there exists a function $g_l^\star(t)$ which is given by: $g_l^\star(t) = \frac{2}{3}\|\eta_1\| + \sum_{k=1}^{l} \frac{9}{20}Kt^{\frac{8k}{3}}$. Such that $\|\frac{d\triangle_l}{dt}\| \leq g_l^\star(t)t^{\frac{-1}{3}}$.*

Now we need an upper bound function which bounds both the two function $g_l(t)$ and $g_l^\star(t)$.

Proposition 4.4. *Suppose that all conditions of theorem 4.1 hold , then there exists an upper bound function $\hat{g}_l(t)$ for the two functions $g_l(t)$ and $g_l^*(t)$ for every $0 \leq t \leq \rho$, for fixed $0 < \rho < 1$.*

The upper bound function is defined by: $\hat{g}_l(t) = \|\eta_1\| + \sum_{k=1}^{l} Kt^{\frac{8k}{3}}$. Moreover, $\|\triangle_l\| \leq \hat{g}_l(t)t^{\frac{2}{3}}$ for every $0 \leq t \leq \rho$, for fixed ρ , $0 < \rho < 1$, while $\|\frac{d\triangle_l}{dt}\| \leq \hat{g}_l(t)t^{\frac{-1}{3}}$ for every $0 < t \leq \rho$, for fixed ρ , $0 < \rho < 1$. For $l \leq j, j \geq 1$.

Proposition 4.5. *Suppose that all conditions of theorem 4.1 hold , then the upper bound function can be rewritten as: $\hat{g}_l(t) = \|\eta_1\| + Kt^{\frac{8}{3}}\frac{1-t^{\frac{8}{3}(l-1)}}{1-t^{\frac{8}{3}}}$,for every $0 \leq t \leq \rho$, for fixed ρ , $0 < \rho < 1$, for $l \leq j$.*

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Proposition 4.6. *For* $0 < t \leq \rho < 1$ *,then* $0 < t^{\frac{8}{3}(l-1)} < 1$.

Proposition 4.7. *Suppose that all conditions of theorem 4.1 hold ,* $0 \leq t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*. There exists an upper bound* $\hat{M}$ *which is given by:* $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$*. Such that* $\hat{g}_l(t) \leq \hat{M}$*.*

Proposition 4.8. *Suppose that all conditions of theorem 4.1 hold .Then:* $\|\triangle_l\| \leq \hat{M}t^{\frac{2}{3}}$ *for* $0 \leq t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*. Not only that,* $\|\frac{d\triangle_l}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$ *for* $0 < t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*. For every* $l \leq j$ *, for* $j \geq 1$*.*

Note that theorem 4.1 gives the boundedness for the functions $\triangle_l$ where $l \leq j$. In theorem 4.1 , the bound $\hat{M}$ is built in the level $l \leq j$, the next step is to have representation for the positive constant K in the same level $l \leq j$.

The positive constant K is written in terms of the positive constant ε as we will see in the next two theorems, the plan here again is to have representations for the positive constants K and $\hat{M}$ in each level $l \leq j$ and $l = j + 1$. More than that ,each one of K and $\hat{M}$ have to carry the same formula in the two different stages $l \leq j$ and $l = j + 1$ which confirms the independence on the index j .

Theorem 4.9. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*. Assume that* $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20}K^{\star}s^{\frac{8l+2}{3}}$*,* $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{9}{20}K^{\star}s^{\frac{8l-1}{3}}$*, for every* $l \leq j$*, for* $j \geq 1$*, for* $K^{\star} > 0$*. Suppose also that* $\rho = \min\{[\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}}, [\frac{\hat{\rho}_2}{\hat{M}}]^{\frac{3}{2}}, [\frac{\hat{\rho}_3}{\hat{M}^2}]^{\frac{3}{2}}\}$*. In addition,* $0 < \lambda = \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\} < 1$*. Then there exists a positive constant* ε *such that:*

$$\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\| \leq \frac{9}{20}K^{\star}\varepsilon s^{\frac{8l-4}{3}}. \tag{43}$$

Where ε *is given by:* $\varepsilon = [\{\frac{5}{3}\hat{M}(1+\lambda)^{\frac{1}{2}} + \frac{3}{2}\frac{\lambda}{[1-\lambda]^2}\} + \frac{3\hat{M}}{[1-\lambda]}]$*.*

Proof. The sketch of the proof of theorem 4.9 is given as a sequence of propositions. □

Proposition 4.10. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*, suppose that all conditions of theorem 4.9 hold , then:*

$$\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\| \leq \frac{9}{20}K^{\star}[\{\frac{5}{3}\hat{M}(1+\hat{M}s^{\frac{2}{3}})^{\frac{1}{2}} + \frac{3}{2}\frac{\hat{M}^2 s^{\frac{2}{3}}}{\|1-\hat{M}s^{\frac{2}{3}}\|^2}\} + 3\frac{\hat{M}}{\|1-\hat{M}s^{\frac{2}{3}}\|}]s^{\frac{8l-4}{3}}. \tag{44}$$

Proposition 4.11. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ *,* $0 < \rho < 1$*, suppose that all conditions of theorem 4.9 hold, then*

$$\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\| \leq \frac{9}{20}K^{\star}[\{\frac{5}{3}\hat{M}(1+\hat{\rho}_2)^{\frac{1}{2}} + \frac{3}{2}\frac{\hat{\rho}_3}{[1-\hat{\rho}_2]^2}\} + \frac{3\hat{M}}{[1-\hat{\rho}_2]}]s^{\frac{8l-4}{3}}. \tag{45}$$

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Proposition 4.12. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ , $0 < \rho < 1$*, suppose that all conditions of theorem 4.9 hold , then there exists positive constant* ε *such that:* $\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\| \leq \frac{9}{20} K^{\star} \varepsilon s^{\frac{8l-4}{3}}$*, where* ε *is given by:* $\varepsilon = [\{\frac{5}{3}\hat{M}(1+\lambda)^{\frac{1}{2}} + \frac{3}{2}\frac{\lambda}{[1-\lambda]^2}\} + \frac{3\hat{M}}{[1-\lambda]}]$.

Theorem 4.13. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ , $0 < \rho < 1$*. Assume that theorem 4.9 holds , For every* $0 \leq s \leq t$ *,* $l \leq j$*, for* $j \geq 1$*. Suppose also that* $\rho = \min\{[\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}}, [\frac{\hat{\rho}_2}{\hat{M}}]^{\frac{3}{2}}, [\frac{\hat{\rho}_3}{\hat{M}^2}]^{\frac{3}{2}}\}$ *and* $0 < \lambda = \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\} < 1$*. Then there exists a positive constant* $K = K(\varepsilon)$ *such that:* $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20} K t^{\frac{8l+2}{3}}$*, where* K *is given by,* $0 < K = \min\{K^{\star}, \frac{9}{20}K^{\star}\varepsilon\}$*, where* $K^{\star}$ *is fixed to be positive constant.*

Proof. The proof follows from theorem 4.9 and the estimate of $\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\|$.

□

Theorem 4.14. *For every* $0 \leq s \leq t \leq \rho$*, for fixed* ρ , $0 < \rho < 1$*. Assume that theorem 4.9 hold , For every* $l \leq j$*, for* $j \geq 1$*. Suppose also that:* $\rho = \min\{[\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}}, [\frac{\hat{\rho}_2}{\hat{M}}]^{\frac{3}{2}}, [\frac{\hat{\rho}_3}{\hat{M}^2}]^{\frac{3}{2}}\}$, $0 < \lambda = \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\} < 1$*.Then there exists a positive constant* $K = K(\varepsilon)$ *such that:*

$$\|\frac{d\triangle_l}{dt} - \frac{d\triangle_{l-1}}{dt}\| \leq \frac{9}{20} K t^{\frac{8l-1}{3}}. \tag{46}$$

Where K *is given by* $0 < K = \min\{K^{\star}, \frac{9}{20}K^{\star}\}$*, and* $K^{\star}$ *is fixed to be positive constant.*

Proof. The proof follows from conditions of theorem 4.9 and an estimate of $\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\|$.

□

5 The level of $l = j + 1$

It is known that the positive real numbers K and $\hat{M}$ are both j-independent if each one of them has the same formula in the two different levels $l \leq j$ and $l = j + 1$. Now we know how K and $\hat{M}$ look like in the stage $l \leq j$, it remains to see how do they look like whenever $l = j + 1$. The next theorem gives representation for ε in the stage $l = j + 1$.

Theorem 5.1. *For every* $0 \leq s \leq t \leq \rho$ *, for fixed* ρ , $0 < \rho < 1$ *,assume that the following statements hold:*

$$\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20} K^{\star} s^{\frac{8l+2}{3}}, \; \|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{9}{20} K^{\star} s^{\frac{8l-1}{3}}. \tag{47}$$

$$\|\triangle_l\| \leq \hat{M} s^{\frac{2}{3}}, \; \|\frac{d\triangle_l}{ds}\| \leq \hat{M} s^{\frac{-1}{3}}. \tag{48}$$

Where $K^{\star}$ *is fixed to be a strictly positive constant. For every* $l \leq j$*,* $j \geq 1$*.Where* $K = K(K^{\star}, \varepsilon)$ *and* $\hat{M}$ *are given in the level* $l \leq j$*, Suppose also that:* $\rho = \min\{[\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}}, [\frac{\hat{\rho}_2}{\hat{M}}]^{\frac{3}{2}}, [\frac{\hat{\rho}_3}{\hat{M}^2}]^{\frac{3}{2}}\}$*,* $0 < \lambda = \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\} < 1$*. Then the below inequality holds:*

$$\|NL(\triangle_j, \frac{d\triangle_j}{ds})(s) - NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s)\| \leq \frac{9}{20} K^{\star} \varepsilon s^{\frac{8j+4}{3}}, \tag{49}$$

where ε *is given by:* $\varepsilon = [\{\frac{5}{3}\hat{M}(1+\lambda)^{\frac{1}{2}} + \frac{3}{2}\frac{\lambda}{(1-\lambda)^2}\} + \frac{3\hat{M}}{1-\lambda}]$.

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Proof. The sketch of the proof of the above theorem is given as a sequence of propositions. Before that the way that we choose the positive constant K in $l \leq j$ can be summarized in the following steps: first fix a positive constant $K^\star$ such that: $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20}K^\star s^{\frac{8l+2}{3}}$, $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{9}{20}K^\star s^{\frac{8l-1}{3}}$ .Secondly, is calculating the difference of the nonlinear parts to have: $\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\| \leq \frac{9}{20}K^\star \varepsilon s^{\frac{8l-4}{3}}$. Thirdly, is applying the two formulas: $\|\triangle_l - \triangle_{l-1}\| \leq \frac{3}{5}\int_0^t K_1(t,s)\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\|ds$, $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{3}{5}\int_0^t K_2(t,s)\|NL(\triangle_{l-1}, \frac{d\triangle_{l-1}}{ds})(s) - NL(\triangle_{l-2}, \frac{d\triangle_{l-2}}{ds})(s)\|ds$. Lastly, is collecting all the constants in the right sides of the above two inequalities and choosing, $K = K(K^\star, \varepsilon) = \min\{K^\star, \frac{9}{20}K^\star\varepsilon\}$. In the next propositions , our goal is to show that the positive constant K will carry the same formula in the level $l = j+1$, then we had to suppose that $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20}K^\star s^{\frac{8l+2}{3}}$, $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{9}{20}K^\star s^{\frac{8l-1}{3}}$, and follow the same technique to be able to determining K in the level $l = j+1$.

□

Proposition 5.2. *Assume that all conditions of theorem 5.1 hold , for every $s \in [0,\rho]$, ρ is given above.Then*

$$\|NL(\triangle_j, \frac{d\triangle_j}{ds})(s) - NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s)\| \leq \frac{9}{20}K^\star s^{\frac{8j+4}{3}}[\{\frac{5}{3}\hat{M}(1+\hat{M}s^{\frac{2}{3}})^{\frac{1}{2}} + \frac{3}{2}\frac{\hat{M}^2 s^{\frac{2}{3}}}{\|1-\hat{M}s^{\frac{2}{3}}\|^2}\} + 3\frac{\hat{M}}{\|1-\hat{M}s^{\frac{2}{3}}\|}]. \tag{50}$$

Proposition 5.3. *Assume that all conditions of theorem 5.1 hold , for every $s \in [0,\rho]$, ρ is given above.Then*

$$\|NL(\triangle_j, \frac{d\triangle_j}{ds})(s) - NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s)\| \leq [\{\frac{5}{3}\hat{M}(1+\lambda)^{\frac{1}{2}} + \frac{3}{2}\frac{\lambda}{(1-\lambda)^2}\} + \frac{3\hat{M}}{1-\lambda}]\frac{9}{20}K^\star s^{\frac{8j+4}{3}}. \tag{51}$$

Proposition 5.4. *Assume that all conditions of theorem 5.1 hold ,for $s \in [0,\rho]$,ρ is given above.Then there exists a positive real number ε such that: $\|NL(\triangle_j, \frac{d\triangle_j}{ds})(s) - NL(\triangle_{j-1}, \frac{d\triangle_{j-1}}{ds})(s)\| \leq \frac{9}{20}K^\star \varepsilon s^{\frac{8j+4}{3}}$.Where ε is given by: $\varepsilon = [\{\frac{5}{3}\hat{M}(1+\lambda)^{\frac{1}{2}} + \frac{3}{2}\frac{\lambda}{(1-\lambda)^2}\} + \frac{3\hat{M}}{1-\lambda}]$.*

Theorem 5.1 is considered as the most significant part in the $l = j+1-$th level since it gives good estimate for the crustal part which is the difference between the two nonlinear terms ,without theorem 5.1 we are unable to estimate $\|\triangle_l - \triangle_{l-1}\|$and $\|\frac{d\triangle_l}{dt} - \frac{d\triangle_{l-1}}{dt}\|$ whenever $l = j+1$.Moreover , it shows the existence of the constant ϵ in the level $l = j+1$ and it also shows that the positive real number ε has the same representation in two different levels , $l \leq j$ and $l = j+1$ which shows that ε is j-independent.

Theorem 5.5. *For every $0 \leq s \leq t \leq \rho$, for fixed ρ , $0 < \rho < 1$, assume also that $\|\triangle_l - \triangle_{l-1}\| \leq \frac{9}{20}K^\star s^{\frac{8l+2}{3}}$, $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \leq \frac{9}{20}K^\star s^{\frac{8l-1}{3}}$, $\|\triangle_l\| \leq \hat{M}s^{\frac{2}{3}}$, $\|\frac{d\triangle_l}{ds}\| \leq \hat{M}s^{\frac{-1}{3}}$ .Where $K^\star$ is fixed to be positive constant. Note that $K^\star$ is placed on the right sides of the first two inequalities to be able determining the value of K in the stage of $l = j+1$. For every $l \leq j$, $j \geq 1$.Where K and $\hat{M}$ are given in the level $l \leq j$, Suppose also that : $\rho = \min\{[\frac{\hat{\rho}_1}{\|\eta_1\|}]^{\frac{3}{2}}, [\frac{\hat{\rho}_2}{\hat{M}}]^{\frac{3}{2}}, [\frac{\hat{\rho}_3}{\hat{M}^2}]^{\frac{3}{2}}\}$, $0 < \lambda = \min\{\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3\} < 1$. Then in the level $l = j+1$, we have:*

$$\|\triangle_{j+1} - \triangle_j\| \leq \frac{9}{20}Kt^{\frac{8j+10}{3}}, \ \|\frac{d\triangle_{j+1}}{dt} - \frac{d\triangle_j}{dt}\| \leq \frac{9}{20}Kt^{\frac{8j+7}{3}}. \tag{52}$$

Where K in the level $l = j+1$ has the same formula: $0 < K = \min\{K^\star, \frac{9}{20}K^\star\varepsilon\} = K(K^\star, \varepsilon)$.

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Proof. The sketch of the proof is given as a sequence of propositions.

□

Proposition 5.6. *Suppose that all the given conditions in theorem 5.5 hold , $l = j+1$.Then:* $\|\triangle_{j+1} - \triangle_j\| \le \frac{3}{5}\frac{9}{20}K^{\star}\varepsilon[\frac{3}{8j+8} - \frac{3}{8j+13}]t^{\frac{8j+10}{3}}$.

Proposition 5.7. *Given* $j \ge 1$, *then* $\frac{3}{8j+8} - \frac{3}{8j+13} \le \frac{3}{4}$.

Proposition 5.8. *Suppose that all the given conditions in theorem 5.5 hold , $l = j+1$.Then there exists a positive real number K such that:* $\|\triangle_{j+1} - \triangle_j\| \le \frac{9}{20}Kt^{\frac{8j+10}{3}}$. *K is given by:* $0 < K = \min\{K^{\star}, \frac{9}{20}K^{\star}\varepsilon\}$. *Where $K^{\star}$ is fixed such that* $\|\triangle_l - \triangle_{l-1}\| \le \frac{9}{20}K^{\star}s^{\frac{8l+2}{3}}$, $\|\frac{d\triangle_l}{ds} - \frac{d\triangle_{l-1}}{ds}\| \le \frac{9}{20}K^{\star}s^{\frac{8l-1}{3}}$ *For* $0 \le s \le t, l \le j, j \ge 1$.

Proposition 5.9. *Suppose that all the given conditions in theorem 5.5 hold , $l = j+1$.Then:* $\|\frac{d\triangle_{j+1}}{dt} - \frac{d\triangle_j}{dt}\| \le \frac{3}{5}\frac{9}{20}K^{\star}\varepsilon[\frac{2}{3}\frac{3}{8j+8} + \frac{3}{8j+13}]t^{\frac{8j+7}{3}}$.

Proposition 5.10. *For every* $j \ge 1$, *we have:* $\frac{2}{3}\frac{3}{8j+8} + \frac{3}{8j+13} \le \frac{3}{4}$.

Proposition 5.11. *Suppose that all the given conditions in theorem 5.5 hold , $l = j+1$. There exists a positive real number K such that:* $\|\frac{d\triangle_{j+1}}{dt} - \frac{d\triangle_j}{dt}\| \le \frac{9}{20}Kt^{\frac{8j+7}{3}}$, *K is given by:* $0 < K = \min\{K^{\star}, \frac{9}{20}K^{\star}\varepsilon\} = K(K^{\star}, \varepsilon)$.

This completes the sketch of the proof of theorem 5.5. Then we have the same representations for the two positive constants ε and K in two different levels $l \le j, l = j+1$ and then K is j-independent.

Theorem 5.12. *Assume that all conditions in theorem 5.5 hold , suppose $l = j+1$. Then there exists a positive real number $\hat{M}$ which is independent on j, such that:* $\|\triangle_{j+1}\| \le \hat{M}t^{\frac{2}{3}}$, *for every* $0 \le t \le \rho$, *and ,* $\|\frac{d\triangle_{j+1}}{dt}\| \le \hat{M}t^{\frac{-1}{3}}$, *for every* $0 < t \le \rho$ *for fixed ρ given in theorem (5.5). Where $\hat{M}$ is given by:* $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$. *Where K is positive real number given above in theorem 5.5 and it is j-independent.*

Proof. The sketch of the proof is given as a sequence of propositions.

□

Proposition 5.13. *Assume that all conditions in theorem 5.5 hold . Suppose $l = j+1$ Then* $\|\triangle_{j+1}\| \le \|\triangle_0\| + \sum_{k=0}^{j}\|\triangle_{k+1} - \triangle_k\|$.

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Proposition 5.14. *Assume the conditions of theorem 5.5 hold , if* $l = j+1$, *then* $\|\triangle_{j+1}\| \leq g_{j+1}(t)t^{\frac{2}{3}}$. *The function* $g_{j+1}(t)$ *is given by* $g_{j+1}(t) = \|\eta_1\| + \frac{9}{20}K\sum_{k=0}^{j} t^{\frac{8(k+1)}{3}}$. *Note that* K *is* $j-$*independent positive real number given in theorem 5.5 , and the function* $g_{j+1}(t)$ *is defined in the closed interval* $t \in [0,\rho]$ *for fixed* ρ *given in theorem 5.5.*

Proposition 5.15. *Assume the conditions of theorem 5.5 hold , if* $l = j+1$, *then* $\|\frac{d\triangle_{j+1}}{dt}\| \leq \|\frac{d\triangle_0}{dt}\| + \sum_{k=0}^{j}\|\frac{d\triangle_{k+1}}{dt} - \frac{d\triangle_k}{dt}\|$.

Proposition 5.16. *Assume the conditions of theorem 5.5 hold , if* $l = j+1$, *then* $\|\frac{d\triangle_{j+1}}{dt}\| \leq g^{\star}_{j+1}(t)t^{\frac{-1}{3}}$, *where the function* $g^{*}_{j+1}(t)$ *is given by:* $g^{\star}_{j+1}(t) = \frac{2}{3}\|\eta_1\| + \frac{9}{20}K\sum_{k=0}^{j} t^{\frac{8(k+1)}{3}}$, *where* K *is* $j-$*independent positive real number given in theorem 5.5 , and the function* $g^{*}_{j+1}(t)$ *is defined on the interval* $t \in [0,\rho]$ *for fixed* ρ *given in theorem 5.5.*

Proposition 5.17. *Assume the conditions of theorem 5.5 hold , consider the functions* $g_{j+1}(t)$, $g^{*}_{j+1}(t)$ *defined in propositions 5.14 and 5.16 respectively, then for every* $0 \leq t \leq \rho$ *for fixed* ρ *given in theorem 1.8 there exists an upper bound function* $\hat{g}_{j+1}(t)$ *such that:* $\hat{g}_{j+1}(t) = \|\eta_1\| + Kt^{\frac{8}{3}}\frac{1-(t^{\frac{8}{3}})^j}{1-t^{\frac{8}{3}}}$.

Proposition 5.18. *Suppose that all conditions of theorem 5.5 hold and* $l = j+1$, *there exists a positive real number* $\hat{M}$ *independent on* j *,such that:* $\hat{g}_{j+1}(t) \leq \hat{M}$, *for* $0 \leq t \leq \rho$, *for fixed* ρ *given in theorem 5.5 , where* $\hat{M}$ *is given by:* $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$. *Note that* K *is* j*-independent and it is given in theorem 5.5.*

Proposition 5.19. *Suppose that all conditions of theorem 5.5 hold and* $l = j+1$, *there exists a positive real number* $\hat{M}$ *independent on* j *,such that:* $\|\triangle_{j+1}\| \leq \hat{M}t^{\frac{2}{3}}$, *for every* $t \in [0,\rho]$, *for fixed* ρ *given in theorem 5.5.*

Proposition 5.20. *Suppose that all conditions of theorem 5.5 hold and* $l = j+1$, *there exists a positive real number* $\hat{M}$*independent on* j *,such that:* $\|\frac{d\triangle_{j+1}}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$, *for every* $t \in (0,\rho]$, *for fixed* ρ *given in theorem 5.5.*

Now theorem 5.12 is proved. Finally, we have the representation for the bound $\hat{M}$ given by: $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$. Whenever $l = j+1$, if we compare the value of $\hat{M}$ in the level $l \leq j$ which is given by theorem 4.1 by: $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$. We will see that $\hat{M}$ carries the same representation in two different levels $l = j+1$ and $l \leq j$ which proves that $\hat{M}$ is $j-$independent. The finite induction is done on j and we have the following statements proven. Let us summarize the induction results as given below: Given $0 \leq t \leq \rho$ for some fixed ρ , $0 < \rho < 1$ given in theorem 5.5 , for every $j \geq 1$, there exists a positive real number K which is j-independent such that: (1)$\|\triangle_j - \triangle_{j-1}\| \leq \frac{9}{20}Kt^{\frac{8j+2}{3}}$ (2)$\|\frac{d\triangle_j}{dt} - \frac{d\triangle_{j-1}}{dt}\| \leq \frac{9}{20}Kt^{\frac{8j-1}{3}}$, where $K = \min\{K^{\star}, \frac{9}{20}K^{\star}\varepsilon\} = K(K^{\star}, \varepsilon)$. Moreover, the positive constant K is proven as j-independent. Given $0 < t \leq \rho$ for some fixed ρ , $0 < \rho < 1$ given in theorem 5.5 , for every $j \geq 1$, there exists a bound $\hat{M}$ which is j-independent such that: (3) $\|\triangle_j\| \leq \hat{M}t^{\frac{2}{3}}$ (2) $\|\frac{d\triangle_j}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$. $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$, for every $j \geq 1$, where $\hat{M}$ is proven as j-independent.

6 Converges of the series

We try to establish the uniformity of the convergence.

Theorem 6.1. *The series* $\sum_{k=1}^{\infty}\|\triangle_k - \triangle_{k-1}\|$ *converges to* $\frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ *in the interval* $0 \leq t \leq \rho < 1$.*The sketch of the proof is given as a sequence of propositions.*

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Proposition 6.2. *The series* $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$ *is an upper bound series for* $\sum_{k=1}^{\infty}\|\triangle_k - \triangle_{k-1}\|$.

Proposition 6.3. *The upper bound series* $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$ *converges in the interval* $0 \le t \le \rho < 1$ *for fixed* ρ *given in theorem 5.5 to the continuous function* $\varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$.

Proposition 6.4. *The series* $\sum_{k=1}^{\infty}\|\triangle_k - \triangle_{k-1}\|$ *converges to* $\varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ *in the interval* $0 < t \le \rho < 1$.

The series $\sum_{k=1}^{\infty}\|\triangle_k - \triangle_{k-1}\|$ *converges to* $\varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ *in the interval* $0 < t \le \rho < 1$.

Definition 6.5. We say that the series $\sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ converges if the sequence of partial sums $\{\sum_{k=1}^{j}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|\}_{j=1}^{\infty}$ converges.

Theorem 6.6. *The series* $\sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ *converges to* $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ *in the interval* $0 \le t \le \rho < 1$.*The sketch of the proof is given as a sequence of propositions.*

Proposition 6.7. *The series* $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k-1}{3}}$ *is an upper bound series for* $\sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ *in the interval* $0 \le t \le \rho$ *where* ρ *is fixed and it is given in theorem 5.5. Furthermore, the upper bound series* $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k-1}{3}}$ *converges to* $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ *in* $0 \le t \le \rho < 1$.

Proposition 6.8. *The series* $\sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ *converges to* $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ *in the interval* $0 \le t \le \rho < 1$. *This completes the proof.*

Theorem 6.9. *The upper bound series* $\sum_{k=1}^{\infty} t^k$ *converges uniformly in the interval* $0 \le t \le \rho < 1$. *For fixed* ρ *given in theorem 5.5. The sketch of the proof is given as a sequence of propositions.*

Proposition 6.10. *Given* $\kappa > 0$ *, for every* $n, m \in \mathbb{N}$ *, we have :*$|\sum_{k=1}^{n} t^k - \sum_{k=1}^{m} t^k| \le \frac{t^n}{1-t}$.

Proposition 6.11. *Suppose that* $1 > \kappa > 0$ *, for every* $n, m \in \mathbb{N}$ *, there exists a positive integer* N^* *which is t-independent such that for* $n > N^*$ *and* $m > N^*$ *such that* $|\sum_{k=1}^{n} t^k - \sum_{k=1}^{m} t^k| \le \kappa$. *For every* $0 \le t \le \rho$*, for fixed* ρ *given in theorem 5.5*

Proposition 6.12. *The series $\sum_{k=1}^{\infty} t^k$ converges uniformly in the interval $0 \leq t \leq \rho < 1$. For fixed ρ given in theorem 5.5*

Theorem 6.13. *The series $\sum_{k=1}^{\infty} \|\triangle_k - \triangle_{k-1}\|$ converges uniformly on $0 \leq t \leq \rho < 1$.For fixed ρ given in theorem 5.5 The sketch of the proof is given as a sequence of propositions.*

Proposition 6.14. *The series $\sum_{k=1}^{\infty} \frac{9}{20} K t^{\frac{8k+2}{3}} < \sum_{k=1}^{\infty} \frac{9}{20} K t^k$ for every $0 \leq t \leq \rho < 1$.*

Proposition 6.15. *The series $\sum_{k=1}^{\infty} \frac{9}{20} K t^{\frac{8k+2}{3}}$ converges uniformly in the interval $0 \leq t \leq \rho < 1$.*

Proposition 6.16. *The series $\sum_{k=1}^{\infty} \|\triangle_k - \triangle_{k-1}\|$converges uniformly in $0 \leq t \leq \rho < 1$.The proof is completed.*

We give an alternative proof as the following: We want to show that the series $\sum_{k=1}^{\infty} \|\triangle_k - \triangle_{k-1}\|$ satisfies Weierstrass M-test in the closed interval $0 \leq t \leq \rho$ for fixed ρ given in theorem 5.5

Theorem 6.17. *Given $\varphi_k(t) = \|\triangle_k(t) - \triangle_{k-1}(t)\|$. For every $0 \leq t \leq \rho$ for fixed ρ given in theorem 5.5 , consider $\{\sum_{k=1}^{j} \varphi_k(t)\}_{j=1}^{\infty}$ to be the sequence of partial sums of the series $\sum_{k=1}^{\infty} \varphi_k(t)$ defined on $[0, \rho]$, there exists a sequence of real numbers $\{p_k\}_{k=1}^{\infty}$ such that : $\mid \varphi_k(t) \mid \leq p_k$, for each $k \geq 1$, $\sum_{k=1}^{\infty} p_k$ converges.The sketch of the proof is given as a sequence of propositions.*

Proposition 6.18. *Assume the conditions of theorem 6.17 hold ,there exists a sequence of real numbers $\{p_k\}_{k=1}^{\infty}$ such that : $\mid \varphi_k(t) \mid \leq p_k$, for each $k \geq 1$.*

Proposition 6.19. *Assume the conditions of theorem 6.17 hold , then for fixed ρ , $0 < \rho < 1$ given in theorem 5.5 , the series $\sum_{k=1}^{\infty} p_k$ converges.The proof of theorem 6.17 is completed.*

Theorem 6.20. *The series $\sum_{k=1}^{\infty} \varphi_k(t)$ converges uniformly to the function $\varphi(t) = \frac{9}{20} K \frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ for every $t \in [0, \rho]$, for fixed ρ , $0 < \rho < 1$.*

Proof. From theorem 6.17 we have the sequence of partial sums $\{\sum_{k=1}^{j} \varphi_k(t)\}_{j=1}^{\infty}$ satisfies Weierstrass M-test and hence the series $\sum_{k=1}^{\infty} \varphi_k(t)$ also satisfies the test , this proves that the series $\sum_{k=1}^{\infty} \varphi_k(t)$ uniformly converges in the closed interval $[0, \rho]$. We prove in theorem 6.1 that The series $\sum_{k=1}^{\infty} \varphi_k(t)$ converges to $\varphi(t) = \frac{9}{20} K \frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ in the interval $0 \leq t \leq \rho$, by the uniqueness of the limit point the series $\sum_{k=1}^{\infty} \varphi_k(t)$ converges uniformly to the continuous function $\varphi(t)$ for every $0 \leq t \leq \rho$.

□

Proposition 6.21. *For $0 \leq t \leq \rho < 1$, $t^{\frac{8k-1}{3}} < t^k$, for every $k \geq 1$. The proof is straightforward.*

Theorem 6.22. *The series $\sum_{k=1}^{\infty} \|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ converges uniformly in the interval $0 \leq t \leq \rho < 1$. The sketch of the proof is given as a sequence of propositions.*

Proposition 6.23. *The infinite sum* $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \|$ *can be estimated as:* $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \| \leq \frac{9}{20} K \sum_{k=1}^{\infty} t^{\frac{8k-1}{3}}$.

Proposition 6.24. *The infinite sum* $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \|$ *satisfies the following inequality:* $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \| < \frac{9}{20} K \sum_{k=1}^{\infty} t^k$.

Proposition 6.25. *The series* $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \|$ *converges uniformly in the above interval. Which completes the proof of the proposition and theorem 6.22*

We give an alternative proof to show that the series $\sum_{k=1}^{\infty} \| \frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt} \|$ converges uniformly to the continuous function $\Phi(t) = \frac{9}{20} K \frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ on the closed interval $0 \leq t \leq \rho$ for fixed ρ given in theorem 5.5 as follows:

Theorem 6.26. *Given* $\Phi_k(t) = \| \frac{d\triangle_k(t)}{dt} - \frac{d\triangle_{k-1}(t)}{dt} \|$, *for every* $0 \leq t \leq \rho$ *for fixed* ρ *given in theorem 5.5 , consider* $\{\sum_{k=1}^{j} \Phi_k(t)\}_{j=1}^{\infty}$ *to be the sequence of partial sums of the series* $\sum_{k=1}^{\infty} \Phi_k(t)$ *defined on* $[0, \rho]$ *, there exists a sequence of real numbers* $\{q_k\}_{k=1}^{\infty}$ *such that* $| \Phi_k(t) | \leq q_k$, *for each* $k \geq 1$, $\sum_{k=1}^{\infty} q_k$ *converges. The sketch of the proof is given as a sequence of propositions.*

Proposition 6.27. *Assume the conditions of theorem 6.22 hold ,there exists a sequence of real numbers* $\{q_k\}_{k=1}^{\infty}$*such that* $| \Phi_k(t) | \leq q_k$, *for each* $k \geq 1$.

Proposition 6.28. *Assume the conditions of theorem 6.22 hold , then for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , the series* $\sum_{k=1}^{\infty} q_k$ *converges.*

Theorem 6.29. *The series* $\sum_{k=1}^{\infty} \Phi_k(t)$ *converges uniformly to the function* $\Phi(t) = \frac{9}{20} K \frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ *for every* $t \in [0, \rho]$, *for fixed* ρ , $0 < \rho < 1$*given in theorem(5.5).*

Proof. From theorem 6.22 we have the sequence of partial sums $\{\sum_{k=1}^{j} \Phi_k(t)\}_{j=1}^{\infty}$ satisfies Weierstrass M-test and hence the series $\sum_{k=1}^{\infty} \Phi_k(t)$ also satisfies the test , this proves that the series $\sum_{k=1}^{\infty} \Phi_k(t)$ uniformly converges in the closed interval $[0, \rho]$. We prove in theorem 6.6 that The series $\sum_{k=1}^{\infty} \Phi_k(t)$ converges to $\Phi(t) = \frac{9}{20} K \frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ in the interval $0 \leq t \leq \rho$, by the uniqueness of the limit point the series $\sum_{k=1}^{\infty} \Phi_k(t)$ converges uniformly to the continuous function $\Phi(t)$ for every $0 \leq t \leq \rho$.

□

Theorem 6.30. *The two series* $\sum_{k=1}^{\infty} \varphi_k(t)$ *and* $\sum_{k=1}^{\infty} \Phi_k(t)$ *converge absolutely in the closed interval* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 The proof follows from the definition of the absolute convergence.*

7 Construction of the solution

Start with: $\triangle_j = \triangle_0+(\triangle_1-\triangle_0)+(\triangle_2-\triangle_1)+...+(\triangle_j-\triangle_{j-1})$ and $\triangle_j = \triangle_0+\sum_{k=1}^{j}(\triangle_k-\triangle_{k-1})$. Now we want to find a function of t say $\triangle(t)$ such that the sequence $\{\triangle_j\}_{j=1}^{\infty}$ converges to $\triangle(t)$ for every $0 \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$ given in theorem 5.5, as we will see in the next theorem.

Theorem 7.1. *Given* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , there exists a continuous function* $\triangle(t)$ *such that* $\triangle_j(t) \longrightarrow \triangle(t)$ *as* $j \longrightarrow \infty$ *and* $t \longrightarrow 0$.

Proof. The outline of the proof is given as a sequence of propositions.

□

Proposition 7.2. *Given* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , then :* $\|\triangle(t) - \triangle_j(t)\| \leq \sum_{k=j+1}^{\infty} \|\triangle_k(t) - \triangle_{k-1}(t)\|$.

Proposition 7.3. *Given* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , then:* $\triangle_j(t) \longrightarrow \triangle(t)$, *as* $t \longrightarrow 0$ *and* $j \longrightarrow \infty$.

Proposition 7.4. *Given* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , then:* $\triangle(t) = \triangle_0(t) + \sum_{k=1}^{\infty}[\triangle_k(t) - \triangle_{k-1}(t)]$ *is continuous function for every* $0 \leq t \leq \rho$.

Theorem 7.5. *Given* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , there exists a continuous function* $\frac{d\triangle(t)}{dt}$ *such that* $\frac{d\triangle_j(t)}{dt} \longrightarrow \frac{d\triangle(t)}{dt}$ *as* $j \longrightarrow \infty$ *and* $t \longrightarrow 0$.

Proof. The outline of the proof is given as a sequence of propositions as follows:

□

Proposition 7.6. *Given* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 ,then:* $\frac{d\triangle}{dt} - \frac{d\triangle_j}{dt} = \sum_{k=j+1}^{\infty}(\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt})$.

Proposition 7.7. *Given* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 ,then :* $\| \frac{d\triangle}{dt} - \frac{d\triangle_j}{dt} \| \leq \frac{9}{20} K \sum_{k=1}^{\infty} t^{\frac{8k-1}{3}}$.

Proposition 7.8. *Given* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 . Then the sequence* $\{\frac{d\triangle_j(t)}{dt}\}_{j=1}^{\infty}$ *approaches the function* $\frac{d\triangle(t)}{dt}$ *whenever* t *tends to* 0 *and* j *increases.*

Proposition 7.9. *Given* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 ,the derivative* $\frac{d\triangle(t)}{dt}$ *is a continuous function in the interval* $0 < t \leq \rho$ *.This completes the proof.*

Note that the derivative $\frac{d\triangle(t)}{dt}$ is unbounded whenever $t \to 0$, then the above solution cannot be considered to be the solution when total collapse occurs since it is not continuously differentiable whenever t tends to the collision time $t = 0$. Then , we try to modify the solution to guarantee the boundedness whenever t tends to 0.

Theorem 7.10. *Given* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , consider:* $\triangle(t) = \triangle_0(t) + \sum_{k=1}^{\infty}[\triangle_k(t) - \triangle_{k-1}(t)]$ *and* $\frac{d\triangle}{dt} = \frac{d\triangle_0}{dt} + \sum_{k=1}^{\infty}(\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt})$.

Consider the transformation: $z = t^{-1} \longleftrightarrow t = z^{-1}$. *Then* $\triangle(z)$, $\frac{d\triangle(z)}{dz}$ *are bounded whenever* $z \to \infty$ *which is equivalent to say* $t \to 0$.

Proof. Start with the equation: $\triangle(t) = \triangle_0(t) + \sum_{k=1}^{\infty}[\triangle_k(t) - \triangle_{k-1}(t)]$ and we obtain: $\|\triangle(t)\| \leq \|\triangle_0(t)\| + \sum_{k=1}^{\infty} \|\triangle_k(t) - \triangle_{k-1}(t)\|$ Since the series $\sum_{k=1}^{\infty} \|\triangle_k - \triangle_{k-1}\|$ converges to $\varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$, in the interval $0 \leq t \leq \rho < 1$, then we have $\sum_{k=1}^{\infty} \|\triangle_k(t) - \triangle_{k-1}(t)\| \leq \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ and then: $\|\triangle(t)\| \leq\mid \eta_1 \mid t^{\frac{2}{3}} + \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ and since $t = z^{-1}$. Then $\|\triangle\| \leq\mid \eta_1 \mid z^{-\frac{2}{3}} + \frac{9}{20}K\frac{z^{-\frac{10}{3}}}{1-z^{-\frac{8}{3}}}$. As $z \to \infty$, the right hand side of the inequality tends to 0 and then $\|\triangle\| \to 0$. Then the function $\triangle \to 0$. Now recall that: $\frac{d\triangle}{dt} = \frac{d\triangle_0}{dt} + \sum_{k=1}^{\infty}(\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt})$ and hence: $\|\frac{d\triangle}{dt}\| \leq \|\frac{d\triangle_0}{dt}\| + \sum_{k=1}^{\infty} \|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$. Since the series $\sum_{k=1}^{\infty} \|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$ converges to $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ in the interval $0 \leq t \leq \rho < 1$. Then $\sum_{k=1}^{\infty} \|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\| \leq \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$. Then $\|\frac{d\triangle}{dt}\| \leq \frac{2}{3}\|\eta_1\|t^{-\frac{1}{3}} + \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$. Now we start formulating the derivative with respect to the new variable z as below: $\frac{d\triangle}{dz} = \frac{d\triangle}{dt}\frac{dt}{dz} = -z^{-2}\frac{d\triangle}{dt}$. Then $\frac{d\triangle}{dt} = -z^2\frac{d\triangle}{dz}$. Put everything together we get $\| - z^2\frac{d\triangle}{dz}\| \leq \frac{2}{3}\|\eta_1\|z^{\frac{1}{3}} + \frac{9}{20}K\frac{z^{-\frac{7}{3}}}{1-z^{-\frac{8}{3}}}$, thus: $z^2\|\frac{d\triangle}{dz}\| \leq \frac{2}{3}\|\eta_1\|z^{\frac{1}{3}} + \frac{9}{20}K\frac{z^{-\frac{7}{3}}}{1-z^{-\frac{8}{3}}}$. Multiply both sides of the above inequality by z^{-2}, then $\|\frac{d\triangle}{dz}\| \leq \frac{2}{3}\|\eta_1\|z^{-\frac{5}{3}} + \frac{9}{20}K\frac{z^{-\frac{13}{3}}}{1-z^{-\frac{8}{3}}}$. As $z \to \infty$, then the right hand side of the inequality tends to 0 and then $\|\frac{d\triangle}{dz}\| \to 0$ as well . Then finally we have $\frac{d\triangle}{dz} \to 0$. As a summary , we have the following: $\triangle \to 0$, $\frac{d\triangle}{dz} \to 0$, as $z \to \infty$.

□

Theorem 7.11. *For every* $0 \leq t \leq \rho$ *for fixed* ρ,$0 < \rho < 1$ *given in theorem 5.5 Then: (1)The function* $\triangle(t)$ *is bounded. (2)* $\triangle \to 0$ *as* $t \longrightarrow 0$.

Proof. The proof is given as a sequence of propositions

□

Proposition 7.12. *For* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , we have:* $\|\triangle(t)\| \leq \|\triangle_j(t)\| + \varphi(t)$.

Proof. Since we have for every $j \geq 1$, $\|\triangle(t) - \triangle_j(t)\| \leq \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$. Since we know that for every $j \geq 1$: $\|\triangle(t) - \triangle_j(t)\| \geq | \|\triangle(t)\| - \|\triangle_j(t)\| |$. Then by utilizing the two above inequalities we get: $| \|\triangle(t)\| - \|\triangle_j(t)\| | \leq \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$ and then: $-\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}} \leq \|\triangle(t)\| - \|\triangle_j(t)\| \leq \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$, thus: $\|\triangle_j(t)\| - \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}} \leq \|\triangle(t)\| \leq \|\triangle_j(t)\| + \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$. The following estimate can be obtained: $\|\triangle_j(t)\| - \varphi(t) \leq \|\triangle(t)\| \leq \|\triangle_j(t)\| + \varphi(t)$. Since we work on the upper bound on $\|\triangle(t)\|$, we get: $\|\triangle(t)\| \leq \|\triangle_j(t)\| + \varphi(t)$.

□

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Proposition 7.13. *For* $0 \le t \le \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , we have for every* $j \ge 1$*:* $\|\triangle(t)\| \le H(t)t^{\frac{2}{3}}$*. Where the function* $H(t)$ *is given by:* $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$ *Moreover,* $H(t) \longrightarrow \hat{M}$ *as* $t \longrightarrow 0$*.*

Proof. By proposition 7.12 , we have: $\|\triangle(t)\| \le \|\triangle_j(t)\| + \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$. Since we know that $\|\triangle_j(t)\| \le \hat{M}t^{\frac{2}{3}}$ for every $j \ge 1$, $\hat{M}$ is described by: $\hat{M} = \hat{M}(\|\eta_1\|, K, \rho) = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$. Then $\|\triangle(t)\| \le \hat{M}t^{\frac{2}{3}} + \frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$. Since the upper bound series $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}}$ converges to the continuous function $\varphi(t)$ on $0 \le t \le \rho$, for fixed ρ , $0 < \rho < 1$.Then $\frac{9}{20}K\sum_{k=1}^{\infty} t^{\frac{8k+2}{3}} \le \varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$, and then we get: $\|\triangle(t)\| \le \hat{M}t^{\frac{2}{3}} + \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$. The following inequality is obtained: $\|\triangle(t)\| \le [\hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}]t^{\frac{2}{3}}$. Set $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$. If $t \longrightarrow 0$, then $H(t) \longrightarrow \hat{M}$. Then $\|\triangle(t)\| \le H(t)t^{\frac{2}{3}}$.

□

Proposition 7.14. *For* $0 \le t \le \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 5.5 , we have:* $\|\triangle(t)\| \le \sigma_1$*. Where* σ_1 *is given by:* $\sigma_1 = \sigma_1(\hat{M}, K, \rho) = [\hat{M} + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}]\rho^{\frac{2}{3}}$*. Where* σ_1 *is* j*-independent , provided that: both of the positive real numbers* $\hat{M}$ *and* K *are* j*-independent.*

Proof. By proposition 7.13 , we have: $\|\triangle(t)\| \le [\hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}]t^{\frac{2}{3}}$. For $0 \le t \le \rho$, we have: $\|\triangle(t)\| \le [\hat{M} + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}]\rho^{\frac{2}{3}}$. Set the bound σ_1 as: $\sigma_1 = \sigma_1(\hat{M}, K, \rho) = [\hat{M} + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}]\rho^{\frac{2}{3}}$. Then $\|\triangle(t)\| \le \sigma_1$. Where σ_1 is j-independent. Hence $\triangle(t)$ is bounded in the interval $0 \le t \le \rho$ for fixed ρ , $0 < \rho < 1$.

□

Now we need to prove part (2).

Proposition 7.15. *For* $0 \le t \le \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 ,* $\triangle(t) \longrightarrow 0$ *as* $t \longrightarrow 0$*.*

Proof. From proposition 7.13 , we have: $\|\triangle(t)\| \le H(t)t^{\frac{2}{3}}$ and then: $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$. If $t \longrightarrow 0$, $H(t) \longrightarrow \hat{M}$ and then $H(t)t^{\frac{2}{3}} \longrightarrow 0$ and finally we get $\|\triangle(t)\| \longrightarrow 0$ as $t \longrightarrow 0$ and hence $\triangle(t) \longrightarrow 0$ as $t \longrightarrow 0$. We have another way to bound the function $\triangle(t)$ given in the next theorem.

□

Theorem 7.16. *For any* $0 \le t \le \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 5.5 ,the function* $\triangle(t)$ *is bounded by* $\|\triangle(t)\| \le \hat{M}t^{\frac{2}{3}}$.

Proof. Start with the following representation of the function $\triangle(t)$ as follows: $\triangle(t) = \triangle_0(t) + \sum_{k=1}^{\infty}[\triangle_k(t) - \triangle_{k-1}(t)]$. Take the magnitude for both sides , then we will have: $\|\triangle(t)\| \leq \|\triangle_0(t)\|+\sum_{k=1}^{\infty}\|\triangle_k(t)-\triangle_{k-1}(t)\|$. Since the series $\sum_{k=1}^{\infty}\|\triangle_k(t)-\triangle_{k-1}(t)\|$ converges to $\varphi(t) = \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ in the interval $0 \leq t \leq \rho$, then: $\|\triangle(t)\| \leq \|\eta_1\|t^{\frac{2}{3}} + \frac{9}{20}K\frac{t^{\frac{10}{3}}}{1-t^{\frac{8}{3}}}$ and $\|\triangle(t)\| \leq \{\|\eta_1\|+K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}\}t^{\frac{2}{3}}$. For $0 \leq t \leq \rho$, we have: $\|\triangle(t)\| \leq \{\|\eta_1\|+K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}\}t^{\frac{2}{3}}$. Since we know that $\hat{M} = \|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$ and $\|\triangle(t)\| \leq \hat{M}t^{\frac{2}{3}}$.

□

Proposition 7.17. *For any* $0 \leq t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5* , $\|\triangle(t)\| \leq \sigma_2$, *where* σ_2 *is given as:* $\sigma_2 = \sigma_2(\hat{M}, K, \rho) = \hat{M}\rho^{\frac{2}{3}}$.

Proof. By theorem 1.23 , we have For any $0 \leq t \leq \rho$ for fixed ρ , $0 < \rho < 1$ given in theorem 5.5,the function $\triangle(t)$ is bounded , this means $\|\triangle(t)\| \leq \hat{M}t^{\frac{2}{3}}$. Then $\|\triangle(t)\| \leq \hat{M}\rho^{\frac{2}{3}}$. Suppose that : $\sigma_2 = \sigma_2(\hat{M}, K, \rho) = \hat{M}\rho^{\frac{2}{3}}$. Then $\|\triangle(t)\| \leq \sigma_2$.

□

Theorem 7.18. *For any* $0 \leq t \leq \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 5.5 , there exists a positive real number* σ *such that* $\|\triangle(t)\| \leq \sigma$.

Proof. By proposition 7.14 , there exists a positive real number σ_1 given by: $\sigma_1 = \sigma_1(\hat{M}, K, \rho) = [\hat{M}+K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}]\rho^{\frac{2}{3}}$. Such that: $\|\triangle(t)\| \leq \sigma_1$. By proposition 7.17 , there exists a positive real number σ_2 given by: $\sigma_2 = \sigma_2(\hat{M}, K, \rho) = \hat{M}\rho^{\frac{2}{3}}$, such that $\|\triangle(t)\| \leq \sigma_2$. Suppose that: $\sigma = \min\{\sigma_1, \sigma_2\}$. Then $\sigma > 0$.

□

Theorem 7.19. *For* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , we have:* $\|\frac{d\triangle(t)}{dt}\| \leq H(t)t^{\frac{-1}{3}}$ *and* $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$ *and then:* $H(t) \longrightarrow \hat{M}$ *as* $t \longrightarrow 0$.

Proposition 7.20. *For* $0 < t \leq \rho$ *for fixed* ρ , $0 < \rho < 1$ *given in theorem 5.5 , we have:* $\|\frac{d\triangle_j(t)}{dt}\| - \Phi(t) \leq \|\frac{d\triangle(t)}{dt}\| \leq \|\frac{d\triangle_j(t)}{dt}\| + \Phi(t)$. *Where* $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$, *is a continuous function on the closed interval* $t \in [0, \rho]$ *, for fixed* ρ , $0 < \rho < 1$ *given in theorem 1.8.*

Proof. By proposition 7.7 , we have: $\|\frac{d\triangle}{dt} - \frac{d\triangle_j}{dt}\| \leq \frac{9}{20}K\sum_{k=1}^{\infty}t^{\frac{8k-1}{3}}$. By proposition 6.8, the upper bound series $\frac{9}{20}K\sum_{k=1}^{\infty}t^{\frac{8k-1}{3}}$ converges to the continuous function $\Phi(t) = \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$ and then the above inequality becomes: $\|\frac{d\triangle}{dt} - \frac{d\triangle_j}{dt}\| \leq \frac{9}{20}K\sum_{k=1}^{\infty}t^{\frac{8k-1}{3}} \leq \Phi(t)$. Since $|\ \|\frac{d\triangle(t)}{dt}\| - \|\frac{d\triangle_j(t)}{dt}\|\ | \leq \|\frac{d\triangle(t)}{dt} - \frac{d\triangle_j(t)}{dt}\|$. From the two above inequalities we get: $|\ \|\frac{d\triangle(t)}{dt}\| - \|\frac{d\triangle_j(t)}{dt}\|\ | \leq \Phi(t)$. Then we have: $-\Phi(t) \leq \|\frac{d\triangle(t)}{dt}\| - \|\frac{d\triangle_j(t)}{dt}\| \leq \Phi(t)$ and then: $\|\frac{d\triangle_j(t)}{dt}\| - \Phi(t) \leq \|\frac{d\triangle(t)}{dt}\| \leq \|\frac{d\triangle_j(t)}{dt}\| + \Phi(t)$.

□

Proposition 7.21. *For* $0 < t \leq \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 5.5 , we have:* $\|\frac{d\triangle(t)}{dt}\| \leq H(t)t^{\frac{-1}{3}}$ *, where* $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$. *Moreover,* $H(t) \longrightarrow \hat{M}$ *as* $t \longrightarrow 0$.

Proof. By proposition 7.20 , we have: $\|\frac{d\triangle(t)}{dt}\| \leq \|\frac{d\triangle_j(t)}{dt}\| + \Phi(t)$. For every $j \geq 1$, and since we know that $\|\frac{d\triangle_j(t)}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$, then: $\|\frac{d\triangle(t)}{dt}\| \leq \hat{M}t^{\frac{-1}{3}} + \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$. If we factor $t^{\frac{-1}{3}}$ out then: $\|\frac{d\triangle(t)}{dt}\| \leq [\hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}]t^{\frac{-1}{3}}$. Put: $H(t) = \hat{M} + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}$. Then $\|\frac{d\triangle(t)}{dt}\| \leq H(t)t^{\frac{-1}{3}}$. Note that $H(t) \longrightarrow \hat{M}$ as $t \longrightarrow 0$.

□

Theorem 7.22. *For every* $0 < t \leq \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 5.5, there exists a constant* $\hat{M}$ *which is given above such that:* $\|\frac{d\triangle}{dt}\| \leq \hat{M}t^{\frac{-1}{3}}$.

Proof. Start with the formula from: $\frac{d\triangle}{dt} = \frac{d\triangle_0}{dt} + \sum_{k=1}^{\infty}(\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt})$, and we obtain the inequality: $\|\frac{d\triangle}{dt}\| \leq \|\frac{d\triangle_0}{dt}\| + \sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\|$. Since we know that from theorem 1.11 that: $\sum_{k=1}^{\infty}\|\frac{d\triangle_k}{dt} - \frac{d\triangle_{k-1}}{dt}\| \leq \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}}$, on the interval $0 \leq t \leq \rho$. Then $\|\frac{d\triangle}{dt}\| \leq \frac{2}{3}\|\eta_1\|t^{\frac{-1}{3}} + \frac{9}{20}K\frac{t^{\frac{7}{3}}}{1-t^{\frac{8}{3}}} \leq t^{\frac{-1}{3}}\{\frac{2}{3}\|\eta_1\| + \frac{9}{20}K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}\} \leq \{\|\eta_1\| + K\frac{t^{\frac{8}{3}}}{1-t^{\frac{8}{3}}}\}t^{\frac{-1}{3}} \leq \{\|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}\}t^{\frac{-1}{3}} \leq \hat{M}t^{\frac{-1}{3}}$.

□

Theorem 7.23. *For every* $0 < t \leq \rho$ *for fixed* ρ, $0 < \rho < 1$ *given in theorem 1.8 , then:* $\|\frac{d\triangle(t)}{dt}\| \leq H(\rho)t^{\frac{-1}{3}}$, *where* $H(\rho) = \hat{M} + \frac{9}{20}K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}$ *and* $H(\rho) \geq \hat{M}$.

Proof. By theorem 7.19 , we have: $\|\frac{d\triangle(t)}{dt}\| \leq H(t)t^{\frac{-1}{3}}$. For every $0 < t \leq \rho$, $\|\frac{d\triangle(t)}{dt}\| \leq H(\rho)t^{\frac{-1}{3}}$. Provided that: $H(\rho) = \hat{M} + \frac{9}{20}K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}} \geq \hat{M}$.

□

8 Applications and relating to the others' work

The collision between two bodies could occur in the space, on the ground, or in any part of oceans. The collision between two bodies on the real line is considered as the closest type of collision to reality. This type of collision includes but not limited to, the collision of two airplanes, two trains, two ships, or two submarines, etc....The technique of successive approximations has been modified to be the most powerful tool which will be employed to construct the asymptotic solution for such type of collision. However, the proofs of the lemmas, propositions, and theorems are done analytically without the need of the numerical analysis, and then there are no numerical calculations needed. At this point , we want to show that our work is connected with the work of Pollard H., Saari, D.G. in [5]. Pollard H., Saari, D.G. in[5], show that: A singularity as $t \longrightarrow 0^+$ is due to collisions if and only if $U \sim \alpha t^{\frac{-2}{3}}$, $t \longrightarrow 0^+$ for some positive constant α, where 0^+ is the collision time. Recall that U is the potential energy and it is defined by: $U = \sum_{1 \leq j < k \leq N} \frac{m_j m_k}{r_{jk}}$.

(1) The First Application:

Proposition 8.1. *Given the two body problem, assume also that* $W_{21} = At^{\frac{-4}{3}}(1+\triangle)$*, where* $A = [\frac{2}{9M^*}]^{\frac{2}{3}}$*, with* $\triangle(t) \longrightarrow 0$ *as* $t \longrightarrow 0$ *, then there exists a positive constant* α *such that* $U \sim \alpha t^{\frac{-2}{3}}$ *as* $t \longrightarrow 0$.

Proof. Since $W_{21} = At^{\frac{-4}{3}}(1+\triangle)$. Then $\frac{W_{21}(t)}{At^{\frac{-4}{3}}} = 1+\triangle(t) \longrightarrow 1+0 = 1$, as $t \longrightarrow 0$, provided that $A > 0$. In other words $W_{21}(t) \sim At^{\frac{-4}{3}}$ as $t \longrightarrow 0$. Since $W_{21} = \frac{r_2 - r_1}{r_{21}^3} \Rightarrow \|W_{21}\| = \frac{\|r_2 - r_1\|}{r_{21}^3} = \frac{r_{21}}{r_{21}^3} = r_{21}^{-2} = W_{21}$. Then $\frac{r_{21}^{-2}}{At^{\frac{-4}{3}}} \longrightarrow 1$, as $t \longrightarrow 0$, and hence:

$$\frac{r_{21}^{-1}}{\sqrt{A}t^{\frac{-2}{3}}} = \frac{1}{\sqrt{A}t^{\frac{-2}{3}}r_{21}} = \frac{m_1m_2}{[m_1m_2\sqrt{A}]t^{\frac{-2}{3}}r_{21}} = \frac{\frac{m_1m_2}{r_{21}}}{[m_1m_2\sqrt{A}]t^{\frac{-2}{3}}} = \frac{U}{[m_1m_2\sqrt{A}]t^{\frac{-2}{3}}} = \frac{U}{\alpha t^{\frac{-2}{3}}} \longrightarrow 1 \tag{53}$$

as $t \longrightarrow 0$. Choose $\alpha = m_1m_2\sqrt{A} = m_1m_2[[\frac{2}{9M^*}]^{\frac{2}{3}}]^{\frac{1}{2}} = m_1m_2[\frac{2}{9M^*}]^{\frac{1}{3}} > 0$. Since the masses are strictly positive while $M^*$is the total mass. Then $U \sim \alpha t^{\frac{-2}{3}}$ for $\alpha = m_1m_2[\frac{2}{9M^*}]^{\frac{1}{3}} > 0$, as $t \longrightarrow 0$. Then the singularity is due to collisions as $t \longrightarrow 0$.This completes the proof. Recall that total collapse means that all particles collide at a finite time . In other words , $\|r_j(t) - r_k(t)\| \longrightarrow 0$ for $j \neq k$ as $t \longrightarrow 0$ if $t_0 = 0$ is assumed to be the collision time. Sundman gives two important theorems to describe the total collapse , the first theorem states that the total collapse accrues only in a finite time while the other theorem states that if the total collapse is to occur then the angular momentum is zero [2].

□

(2) The Second Application:

Proposition 8.2. *Given the two body problem , assume also that* $W_{21} = At^{\frac{-4}{3}}(1+\triangle)$*, with:* $A = [\frac{2}{9M^*}]^{\frac{2}{3}}$*, with* $\triangle(t) \longrightarrow 0$ *as* $t \longrightarrow 0$ *, then the two particles* r_1 *and* r_2 *collide as* $t \longrightarrow 0$.

Proof. Since we have:$\frac{r_{21}^{-1}}{\sqrt{A}t^{\frac{-2}{3}}} \longrightarrow 1$, as $t \longrightarrow 0$. This this implies that $r_{21}^{-1} \sim \sqrt{A}t^{\frac{-2}{3}}$, as $t \longrightarrow 0$. Since the asymptotic relation is an equivalence relation , then it is symmetric and then $\sqrt{A}t^{\frac{-2}{3}} \sim r_{21}^{-1}$ as $t \longrightarrow 0$, which gives $\frac{\sqrt{A}t^{\frac{-2}{3}}}{r_{21}^{-1}} \longrightarrow 1$ as $t \longrightarrow 0$. Then $\frac{\sqrt{A}r_{21}(t)}{t^{\frac{2}{3}}} \longrightarrow 1$, which gives $\sqrt{A}r_{21}(t) \sim t^{\frac{2}{3}}$. If we multiply both sides by $\frac{1}{\sqrt{A}}$ we will have: $r_{21}(t) \sim \frac{1}{\sqrt{A}}t^{\frac{2}{3}}$. As $t \longrightarrow 0$, $r_{21}(t)$ tends to 0 like the function $t^{\frac{2}{3}}$. Moreover, $\|r_2(t) - r_1(t)\| \sim \frac{1}{\sqrt{A}}t^{\frac{2}{3}}$ as $t \longrightarrow 0$. Then, we have total collapse occurs whenever t tends to 0.

□

(3) The Third Application: Newton's Law of Universal Gravitation: The most common use is in celestial mechanics, where gravity is the main force. Newton's Law of Universal Gravitation, this law tells us how two point masses pull on each other with gravity. It controls the paths that planets, moons, and satellites take around the sun. The distance (r) between two masses (m_1 and m_2) is equal to the force ($\vec{F}$) between them and it is given by: $\vec{F} = \frac{Gm_1m_2}{r^2}$, where G is the universal gravitational constant (6.674×10^{-11} N $\cdot$ m^2/kg^2). The force is always pulling things together and works along the line that connects the two masses. Newton's second law ($\vec{F} = m\vec{a}$) uses this force to figure out how fast each body is moving, which leads to the equations of motion for the system.
(4) The Fourth Application: Conservation of Angular Momentum: For a two-body system under a central force, the total angular momentum ($\vec{L}$) is conserved. This is because a central force always points along the line connecting the two bodies, so it exerts no torque on the system. The angular momentum of a particle with mass m and velocity $\vec{v}$ at a position $\vec{r}$ relative to the center of mass is: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$ The conservation of angular momentum explains why planetary orbits are confined to a single plane and why a planet's speed changes as it orbits. For example, a planet moves faster when it's closer to the sun (smaller r) to keep its angular momentum constant.

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9 Conclusion

The scalar version of the two body problem can be solved asymptotically when the two bodies collide on the real line at some finite time. The asymptotic solution has a certain form which maintains certain properties, in other words, the function $\triangle$ and their first and second derivatives with respect to t approach 0 when t attains the collision time which is assumed to be 0. In order to obtain the specific form $W_{21}(t) = At^m(1+\triangle(t))$, there has been a long process to provide the values of A and m respectively via the asymptotic hypothesis.The technique of successive approximation has been modified in such a way that it could play a significant tool in the process of constructing the function $\triangle$. In sum, $\triangle(t) = \triangle_0(t) + \sum_{k=1}^{\infty}[\triangle_k(t) - \triangle_{k-1}(t)]$. More than that, the function $\triangle(t)$ is shown to satisfy the inequality; $\|\triangle(t)\| \leq \{\|\eta_1\| + K\frac{\rho^{\frac{8}{3}}}{1-\rho^{\frac{8}{3}}}\}t^{\frac{2}{3}}$, for every $0 \leq t \leq \rho$, which asserts that $\triangle \to 0$ when $t \to 0$. Furthermore, the transformation $t = z^{-1}$ has been utilized effectively to prove that the first and second derivatives with respect to t approach 0 when t attains the collision time as $t \to 0$ if and only if $z \to \infty$. Not only that, it has been proved that the function $\triangle(t)$ is bounded in the closed interval $[0, \rho]$ as $\|\triangle(t)\| \leq \hat{M}t^{\frac{2}{3}}$.

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