



On Refined Neutrosophic Quotient Groups

A.A.A. Agboola¹

Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.

agboolaaaa@funaab.edu.ng

Abstract

This paper is devoted to the study of refined neutrosophic quotient groups. It is shown that the classical isomorphism theorems of groups do not hold for the refined neutrosophic groups. Moreover, the existence of classical morphisms between refined neutrosophic groups $G(I_1, I_2)$ and neutrosophic groups $G(I)$ is established.

Keywords: Neutrosophy, neutrosophic group, neutrosophic subgroup, refined neutrosophic group, refined neutrosophic subgroup, refined neutrosophic group homomorphism.

1 Introduction and Preliminaries

Agboola in [1] introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups in particular. Since the introduction of refined neutrosophic algebraic structures, many neutrosophic researchers have established and studied more refined neutrosophic algebraic structures. Adeleke et al. in [5] studied refined neutrosophic rings and refined neutrosophic subrings and presented their fundamental properties. Also in [6], Adeleke et al. studied refined neutrosophic ideals and refined neutrosophic homomorphisms and presented their basic properties. The present paper is devoted to the study of refined neutrosophic quotient groups. More properties of refined neutrosophic groups will be presented and it will be shown that the classical isomorphism theorems of groups do not hold in the refined neutrosophic groups. The existence of classical morphisms between refined neutrosophic groups $G(I_1, I_2)$ and neutrosophic groups $G(I)$ will be established. For more details about neutrosophy, refined neutrosophic logic, neutrosophic groups and refined neutrosophic groups, the readers should see [2,4,7,13].

Definition 1.1. ¹ Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where $+$ and \cdot are ordinary addition and multiplication respectively. I_1 and I_2 are the split components of the indeterminacy factor I that is $I = \alpha_1 I_1 + \alpha_2 I_2$ with $\alpha_i \in \mathbb{R}, \mathbb{C}, i = 1, 2$. Also, I_1 and I_2 are taken to have the properties $I_1^2 = I_1, I_2^2 = I_2$ and $I_1 I_2 = I_2 I_1 = I_1$ or I_2 . For the purposes of this paper, we will take $I_1 I_2 = I_2 I_1 = I_1$.

For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2), \quad (1)$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2). \quad (2)$$

Definition 1.2. ¹ Let $(G, *)$ be any group. The couple $(G(I_1, I_2), *)$ is called a refined neutrosophic group generated by G, I_1 and I_2 . $(G(I_1, I_2), *)$ is said to be commutative if for all $x, y \in G(I_1, I_2)$, we have $x * y = y * x$. Otherwise, we call $(G(I_1, I_2), *)$ a non-commutative refined neutrosophic group.

Example 1.3. ¹ Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +)$ is a commutative refined neutrosophic group of integers modulo 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +)$ is a finite commutative refined neutrosophic group of integers modulo n .

¹Correspondence:aaaola2003@yahoo.com

Theorem 1.4. ¹

- (1) Every refined neutrosophic group is a semigroup but not a group.
- (2) Every refined neutrosophic group contains a group.

Corollary 1.5. ¹ Every refined neutrosophic group $(G(I_1, I_2), +)$ is a group.

Definition 1.6. ¹ Let $(G(I_1, I_2), *)$ be a refined neutrosophic group and let $A(I_1, I_2)$ be a nonempty subset of $G(I_1, I_2)$. $A(I_1, I_2)$ is called a refined neutrosophic subgroup of $G(I_1, I_2)$ if $(A(I_1, I_2), *)$ is a refined neutrosophic group. It is essential that $A(I_1, I_2)$ contains a proper subset which is a group. Otherwise, $A(I_1, I_2)$ will be called a pseudo refined neutrosophic subgroup of $G(I_1, I_2)$.

Example 1.7. ¹

- (1) Let $G(I_1, I_2) = (\mathbb{Z}(I_1, I_2), +)$ and let $A(I_1, I_2) = (3\mathbb{Z}(I_1, I_2), +)$. Then $A(I_1, I_2)$ is a refined neutrosophic subgroup of $G(I_1, I_2)$.
- (2) Let $G(I_1, I_2) = (\mathbb{Z}_6(I_1, I_2), +)$ and let

$$A(I_1, I_2) = \{(0, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (0, 2I_1, 0), (0, 0, 2I_2), (0, 2I_1, 2I_2), (0, 3I_1, 0), (0, 0, 3I_2), (0, 3I_1, 3I_2), (0, 4I_1, 0), (0, 0, 4I_2), (0, 4I_1, 4I_2), (0, 5I_1, 0), (0, 0, 5I_2), (0, 5I_1, 5I_2)\}.$$

Then $A(I_1, I_2)$ is a pseudo refined neutrosophic subgroup of $G(I_1, I_2)$.

Definition 1.8. ¹ Let $A(I_1, I_2)$ and $B(I_1, I_2)$ be any two refined neutrosophic subgroups (pseudo refined neutrosophic subgroups) of a refined neutrosophic group $(G(I_1, I_2), +)$. We define the sum $A(I_1, I_2) + B(I_1, I_2)$ by the set

$$A(I_1, I_2) + B(I_1, I_2) = \{(a_1, a_2I_1, a_3I_2) + (b_1, b_2I_1, b_3I_2) : a_1, a_2, a_3 \in A(I_1, I_2), b_1, b_2, b_3 \in B(I_1, I_2)\} \quad (3)$$

which is a refined neutrosophic subgroup (pseudo refined neutrosophic subgroup) of $G(I_1, I_2)$.

Definition 1.9. ¹ Let $(G(I_1, I_2), *)$ and $(H(I_1, I_2), *')$ be two refined neutrosophic groups. The mapping $\phi : (G(I_1, I_2), *) \rightarrow (H(I_1, I_2), *')$ is called a neutrosophic homomorphism if the following conditions hold:

- (1) $\phi(x * y) = \phi(x) *' \phi(y)$.
- (2) $\phi(I_k) = I_k \quad \forall x, y \in G(I_1, I_2)$ and $k = 1, 2$.

The image of ϕ is defined by the set

$$Im\phi = \{y \in H(I_1, I_2) : y = \phi(x) \text{ for some } x \in G(I_1, I_2)\}. \quad (4)$$

If $G(I_1, I_2)$ and $H(I_1, I_2)$ are additive refined neutrosophic groups, then the kernel of the neutrosophic homomorphism $\phi : (G(I_1, I_2), +) \rightarrow (H(I_1, I_2), +)$ is defined by the set

$$Ker\phi = \{x \in G(I_1, I_2) : \phi(x) = (0, 0I_1, 0I_2)\}. \quad (5)$$

Epimorphism, monomorphism, isomorphism, endomorphism and automorphism of ϕ have the same definitions as those of the classical cases.

Theorem 1.10. ¹ Let $\phi : (G(I_1, I_2), *) \rightarrow (H(I_1, I_2), *')$ be a refined neutrosophic group homomorphism. Then $Im\phi$ is a neutrosophic subgroup of $H(I_1, I_2)$.

Theorem 1.11. ¹ Let $\phi : (G(I_1, I_2), +) \rightarrow (H(I_1, I_2), +)$ be a refined neutrosophic group homomorphism. Then $Ker\phi$ is a subgroup of G and not a neutrosophic subgroup of $G(I_1, I_2)$.

Example 1.12. ¹ Let $\phi : \mathbb{Z}_2(I_1, I_2) \times \mathbb{Z}_2(I_1, I_2) \rightarrow \mathbb{Z}_2(I_1, I_2)$ be a neutrosophic group homomorphism defined by $\phi(x, y) = x$ for all $x, y \in \mathbb{Z}_2(I_1, I_2)$. Then

$$Im\phi = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}.$$

$$Ker\phi = \{((0, 0, 0), (0, 0, 0)), ((0, 0, 0), (1, 0, 0)), ((0, 0, 0), (0, I_1, 0)), ((0, 0, 0), (0, I_1, I_2)), ((0, 0, 0), (0, 0, I_2)), ((0, 0, 0), (1, I_1, 0)), ((0, 0, 0), (1, 0, I_2)), ((0, 0, 0), (1, I_1, I_2))\}.$$

2 Main Results

Definition 2.1. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$ and let $x = (a, bI_1, cI_2) \in G(I_1, I_2)$.

(i) The set

$$xH(I_1, I_2) = \{xh : h \in H(I_1, I_2)\} \tag{6}$$

is called a refined left coset of $H(I_1, I_2)$ in $G(I_1, I_2)$.

(ii) The set

$$Hx(I_1, I_2) = \{hx : h \in H(I_1, I_2)\} \tag{7}$$

is called a refined right coset of $H(I_1, I_2)$ in $G(I_1, I_2)$.

If " \cdot " is commutative, there is no distinction between the refined left and refined right cosets of $H(I_1, I_2)$ in $G(I_1, I_2)$.

Definition 2.2. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), +)$ and let $x = (a, bI_1, cI_2) \in G(I_1, I_2)$.

(i) The set

$$x + H(I_1, I_2) = \{x + h : h \in H(I_1, I_2)\} \tag{8}$$

is called a refined left coset of $H(I_1, I_2)$ in $G(I_1, I_2)$.

(ii) The set

$$H(I_1, I_2) + x = \{h + x : h \in H(I_1, I_2)\} \tag{9}$$

is called a refined right coset of $H(I_1, I_2)$ in $G(I_1, I_2)$.

Definition 2.3. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$. $H(I_1, I_2)$ is said to be normal in $G(I_1, I_2)$ if for all $x = (a, bI_1, cI_2) \in G(I_1, I_2)$, we have

$$xH(I_1, I_2) = H(I_1, I_2)x. \tag{10}$$

Definition 2.4. Let $H(I_1, I_2)$ be a normal refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$. The quotient $G(I_1, I_2)/H(I_1, I_2)$ is defined by

$$G(I_1, I_2)/H(I_1, I_2) = \{xH(I_1, I_2) : x \in G(I_1, I_2)\}. \tag{11}$$

Definition 2.5. Let $H(I_1, I_2)$ be a normal refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), +)$. The quotient $G(I_1, I_2)/H(I_1, I_2)$ is defined by

$$G(I_1, I_2)/H(I_1, I_2) = \{x + H(I_1, I_2) : x \in G(I_1, I_2)\}. \tag{12}$$

Example 2.6. Let $G = \{e, a, b, c\}$ be the Klein-4 group and let $(G(I_1, I_2), \cdot)$ be a refined neutrosophic group given by

$$\begin{aligned} G(I_1, I_2) = & \{(e, 0I_1, 0I_2), (a, 0I_1, 0I_2), (b, 0I_1, 0I_2), (c, 0I_1, 0I_2), (e, eI_1, eI_2), \\ & (e, aI_1, aI_2), (e, bI_1, bI_2), (e, cI_1, cI_2), (a, eI_1, eI_2), (a, aI_1, aI_2), \\ & (a, bI_1, bI_2), (a, cI_1, cI_2), (b, eI_1, eI_2), (b, aI_1, aI_2), (b, bI_1, bI_2), \\ & (b, cI_1, cI_2), (c, eI_1, eI_2), (c, aI_1, aI_2), (c, bI_1, bI_2), \dots, (c, cI_1, cI_2)\} \end{aligned}$$

and let $H(I_1, I_2)$ be a subset of $G(I_1, I_2)$ given by

$$\begin{aligned} H(I_1, I_2) = & \{(e, 0I_1, 0I_2), (a, 0I_1, 0I_2), (e, eI_1, eI_2), (a, aI_1, aI_2), (e, aI_1, aI_2), \\ & (a, eI_1, eI_2), (e, aI_1, eI_2), (a, eI_1, aI_2)\}. \end{aligned}$$

The order of $G(I_1, I_2)$ is $4^3 = 64$ and it is clear that $H(I_1, I_2)$ is a refined neutrosophic subgroup of $G(I_1, I_2)$ of order 8. Since 8 is a divisor of 64, it follows that Lagrange's theorem holds in this case. It should be noted that Lagrange's theorem generally does not hold in refined neutrosophic groups. However, Lagrange's theorem holds in any refined neutrosophic group $G(I_1, I_2)$ whenever G is isomorphic to Klein-4 group.

Example 2.7. Let $G(I_1, I_2)$ and $H(I_1, I_2)$ be as defined in **Example 2.6**. Simple computations show that $xH(I_1, I_2)$ the set of all left cosets of $H(I_1, I_2)$ in $G(I_1, I_2)$ for all $x \in G(I_1, I_2)$ is a partition of $G(I_1, I_2)$.

Generally, the left(right) cosets of a refined neutrosophic subgroup in a refined neutrosophic group does not partition the refined neutrosophic group.

Theorem 2.8. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$ and let $G(I_1, I_2)/H(I_1, I_2)$ be a set defined by

$$G(I_1, I_2)/H(I_1, I_2) = \{xH(I_1, I_2) : x \in G(I_1, I_2)\}. \tag{13}$$

Let \odot be a binary operation defined on $G(I_1, I_2)/H(I_1, I_2)$ by

$$\begin{aligned} xH(I_1, I_2) \odot yH(I_1, I_2) &= xyH(I_1, I_2) \quad \forall xH(I_1, I_2), \\ & \quad yH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2). \end{aligned} \tag{14}$$

Then

- (i) $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is a refined neutrosophic monoid.
- (ii) $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is not a refined neutrosophic group but contains a proper subset which is a group.

Proof. (i) It is clear that \odot is well defined. Let $xH(I_1, I_2), yH(I_1, I_2), zH(I_1, I_2)$ be arbitrary elements in $G(I_1, I_2)/H(I_1, I_2)$ with $x, y, z \in G(I_1, I_2)$. Now,

$$\begin{aligned} xH(I_1, I_2) \odot (yH(I_1, I_2) \odot zH(I_1, I_2)) &= xH(I_1, I_2) \odot (yzH(I_1, I_2)) \\ &= x(yz)H(I_1, I_2) \\ &= (xy)zH(I_1, I_2) \\ &= (xH(I_1, I_2) \odot yH(I_1, I_2)) \odot zH(I_1, I_2). \end{aligned}$$

For all $xH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$, there exists $eH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ with $e \in G(I_1, I_2)$ such that

$$\begin{aligned} xH(I_1, I_2) \odot eH(I_1, I_2) &= eH(I_1, I_2) \odot xH(I_1, I_2) \\ &= exH(I_1, I_2) \\ &= xH(I_1, I_2) \end{aligned}$$

$\therefore (G(I_1, I_2)/H(I_1, I_2), \odot)$ is a refined neutrosophic monoid.

(ii) For all $xH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ there does not exist any $yH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ such that

$$xH(I_1, I_2) \odot yH(I_1, I_2) = yH(I_1, I_2) \odot xH(I_1, I_2) = eH(I_1, I_2)$$

this shows that $x^{-1}H(I_1, I_2)$ does not exist and therefore, $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is not a refined neutrosophic group. However, it contains a proper subset which is a group. □

Example 2.9. Let $G(I_1, I_2)$ and $H(I_1, I_2)$ be as defined in **Example 2.6**. It can be shown that $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is not a refined neutrosophic group but a refined neutrosophic monoid.

Theorem 2.10. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$. Then $(G(I_1, I_2)/H(I_1, I_2), \cdot)$ is a commutative refined neutrosophic monoid if and only if G is an abelian group.

Proof. That $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is a refined neutrosophic monoid follows from **Theorem 2.8**. Suppose that G is an abelian group. Then $((G(I_1, I_2), \cdot))$ is a commutative refined neutrosophic monoid. Suppose that $xH(I_1, I_2), yH(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ are arbitrary with $x, y \in G(I_1, I_2)$. Then

$$\begin{aligned} xH(I_1, I_2).yH(I_1, I_2) &= x.yH(I_1, I_2) \\ &= y.xH(I_1, I_2) \\ &= yH(I_1, I_2).xH(I_1, I_2). \end{aligned}$$

Conversely, suppose that $(G(I_1, I_2)/H(I_1, I_2), \odot)$ is a commutative refined neutrosophic monoid. Then $G(I_1, I_2)$ is a commutative refined neutrosophic group and consequently, G is an abelian group. □

Theorem 2.11. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), +)$ and let $G(I_1, I_2)/H(I_1, I_2)$ be a set defined by

$$G(I_1, I_2)/H(I_1, I_2) = \{x + H(I_1, I_2) : x \in G(I_1, I_2)\}. \tag{15}$$

Let \oplus be a binary operation defined on $G(I_1, I_2)/H(I_1, I_2)$ by

$$(x + H(I_1, I_2)) \oplus (y + H(I_1, I_2)) = (x + y) + H(I_1, I_2) \quad \forall x + H(I_1, I_2), y + H(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2). \tag{16}$$

Then $(G(I_1, I_2)/H(I_1, I_2), \oplus)$ is a refined neutrosophic group.

Proof. It is clear that \oplus is well defined. Let $x + H(I_1, I_2), y + H(I_1, I_2), z + H(I_1, I_2)$ be arbitrary elements in $G(I_1, I_2)/H(I_1, I_2)$ with $x, y, z \in G(I_1, I_2)$. Now,

$$\begin{aligned} (x + H(I_1, I_2)) \oplus ((y + H(I_1, I_2)) \oplus (z + H(I_1, I_2))) &= (x + H(I_1, I_2)) \oplus ((y + z) + H(I_1, I_2)) \\ &= (x + (y + z)) + H(I_1, I_2) \\ &= ((x + y) + z) + H(I_1, I_2) \\ &= ((x + H(I_1, I_2)) \oplus (y + H(I_1, I_2))) \\ &\quad \oplus (z + H(I_1, I_2)). \end{aligned}$$

Next, for all $x + H(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$, there exists $(0, 0I_1, 0I_2) + H(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ such that

$$\begin{aligned} x + H(I_1, I_2) \oplus (0, 0I_1, 0I_2) + H(I_1, I_2) &= (0, 0I_1, 0I_2) + H(I_1, I_2) \oplus x + H(I_1, I_2) \\ &= (x + (0, 0I_1, 0I_2)) + H(I_1, I_2) \\ &= x + H(I_1, I_2). \end{aligned}$$

$\therefore (0, 0I_1, 0I_2) + H(I_1, I_2) = H(I_1, I_2)$ is the additive identity element.

Lastly, for all $x + H(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$, there exists $(-x) + H(I_1, I_2) \in G(I_1, I_2)/H(I_1, I_2)$ with $-x \in G(I_1, I_2)$ such that

$$\begin{aligned} (x + H(I_1, I_2)) \oplus ((-x) + H(I_1, I_2)) &= ((-x) + H(I_1, I_2)) \oplus (x + H(I_1, I_2)) \\ &= (x + (-x)) + H(I_1, I_2) \\ &= H(I_1, I_2). \end{aligned}$$

$\therefore (-x) + H(I_1, I_2)$ is the additive inverse of $x + H(I_1, I_2)$.

Hence, $(G(I_1, I_2)/H(I_1, I_2), \oplus)$ is a refined neutrosophic group. □

Example 2.12. Let $G(I_1, I_2) = (\mathbb{Z}(I_1, I_2), +)$ be a refined neutrosophic group of integers and let $H(I_1, I_2) = 2\mathbb{Z}(I_1, I_2)$ be its refined neutrosophic subgroup. By definition,

$$\begin{aligned} G(I_1, I_2) &= \{(a, bI_1, cI_2) : a, b, c \in \mathbb{Z}\} \\ H(I_1, I_2) &= \{(x, yI_1, zI_2) : x, y, z \in 2\mathbb{Z}\} \end{aligned}$$

Simple computations show that

$$\begin{aligned} G(I_1, I_2)/H(I_1, I_2) &= \{(0, 0I_1, 0I_2) + H(I_1, I_2), (1, 0I_1, 0I_2) + H(I_1, I_2), \\ &\quad (0, 1I_1, 0I_2) + H(I_1, I_2), (0, 0I_1, 1I_2) + H(I_1, I_2), (1, 1I_1, 0I_2) + H(I_1, I_2), \\ &\quad (1, 0I_1, 1I_2) + H(I_1, I_2), (0, 1I_1, 1I_2) + H(I_1, I_2), (1, 1I_1, 1I_2) + H(I_1, I_2)\}. \end{aligned}$$

It can easily be shown that $(G(I_1, I_2)/H(I_1, I_2), +)$ is a refined neutrosophic group.

Theorem 2.13. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), \cdot)$. Then the mapping $\phi : G(I_1, I_2) \rightarrow G(I_1, I_2)/H(I_1, I_2)$ defined by

$$\phi(x) = xH(I_1, I_2) \quad \forall x \in G(I_1, I_2)$$

is not a neutrosophic homomorphism.

Proof. Let $x, y \in G(I_1, I_2)$. Then

$$\begin{aligned}\phi(xy) &= xyH(I_1, I_2) \\ &= xH(I_1, I_2)yH(I_1, I_2) \\ &= \phi(x)\phi(y).\end{aligned}$$

But then,

$$\begin{aligned}\phi(I_i) &= I_iH(I_1, I_2), i = 1, 2 \\ &\neq I_i.\end{aligned}$$

These show that ϕ is not a neutrosophic homomorphism. \square

Remark 2.14. Since ϕ in **Theorem 2.13** is not a neutrosophic homomorphism, it follows that its kernel, $Ker\phi$ does not exist and the quotient $G(I_1, I_2)/Ker\phi$ cannot be found. Hence,

$$G(I_1, I_2)/Ker\phi \not\cong \phi(G(I_1, I_2)).$$

This shows that 1st isomorphism theorem for classical groups does not hold in the refined neutrosophic groups. 2nd and 3rd isomorphism theorems for classical groups equally do not hold in the refined neutrosophic groups.

Theorem 2.15. Let $H(I_1, I_2)$ be a refined neutrosophic subgroup of a refined neutrosophic group $(G(I_1, I_2), +)$. Then the mapping $\phi : G(I_1, I_2) \rightarrow G(I_1, I_2)/H(I_1, I_2)$ defined by

$$\phi(x) = x + H(I_1, I_2) \quad \forall x \in G(I_1, I_2)$$

is not a neutrosophic homomorphism.

Theorem 2.16. Let $(G(I_1, I_2), +)$ be a refined neutrosophic group and let $(G(I), +)$ be a neutrosophic group where $I = xI_1 + yI_2$ with $x, y \in \mathbb{R}, \mathbb{C}$. Let $\psi : G(I_1, I_2) \rightarrow G(I)$ be a mapping defined by

$$\psi((a, xI_1, yI_2)) = (a, (x + y)I) \quad \forall (a, xI_1, yI_2) \in (G(I_1, I_2)) \quad \text{with } a, x, y \in G.$$

Then ψ is a group homomorphism.

Proof. Obviously, ψ well defined. Suppose that $(a, xI_1, yI_2), (b, uI_1, vI_2) \in (G(I_1, I_2))$. Then

$$\begin{aligned}\psi((a, xI_1, yI_2) + (b, uI_1, vI_2)) &= \psi((a + b, (x + u)I_1, (y + v)I_2)) \\ &= (a + b, (x + u + y + v)I) \\ &= (a, (x + u)I) + (b, (y + v)I) \\ &= \psi((a, xI_1, yI_2)) + \psi((b, uI_1, vI_2)).\end{aligned}$$

This shows that ψ is a group homomorphism. \square

Remark 2.17. From **Theorem 2.11** it follows that

$$\begin{aligned}Ker\psi &= \{(a, xI_1, yI_2) : \psi((a, xI_1, yI_2)) = (0, 0I)\} \\ &= \{(a, xI_1, yI_2) : (a, (x + y)I) = (0, 0I)\} \\ &= \{(0, xI_1, -xI_2) : x \in G\}\end{aligned}$$

which is a subgroup of $G(I_1, I_2)$. The mapping $\phi : Ker\psi \rightarrow G$ defined by $\phi((0, xI_1, -xI_2)) = x$ is a group isomorphism that is $Ker\phi \cong G$.

3 Conclusion

In this paper, we have studied refined neutrosophic quotient groups. We have shown that the classical isomorphism theorems of groups do not hold in the refined neutrosophic groups. Also, we have established existence of classical morphisms between refined neutrosophic groups $G(I_1, I_2)$ and neutrosophic groups $G(I)$.

4 Appreciation

The author appreciates all anonymous reviewers for their useful comments and suggestions.

References

- [1] A.A.A. Agboola, On Refined Neutrosophic Algebraic Structures, Neutrosophic Sets and Systems, vol.10, pp. 99-101, 2015.
- [2] A.A.A. Agboola, A.D. Akinola and O.Y. Oyebola, Neutrosophic Rings I, Int. J. of Math. Comb., vol. 4, pp. 1-14, 2011.
- [3] A.A.A. Agboola, E.O. Adeleke and S.A. Akinleye, Neutrosophic Rings II, Int. J. of Math. Comb., vol. 2, pp. 1-8, 2012.
- [4] A.A.A. Agboola, Akwu A.O. and Y.T. Oyebo, Neutrosophic Groups and Neutrosopic Subgroups, Int. J. of Math. Comb., vol. 3, pp. 1-9, 2012.
- [5] E.O. Adeleke, A.A.A. Agboola and F. Smarandache, Refined Neutrosophic Rings I, International Journal of Neutrosophic Science (IJNS), vol. 2(2), pp. 77-81, 2020. (DOI:10.5281/zenodo.3728222)
- [6] E.O. Adeleke, A.A.A. Agboola and F. Smarandache, Refined Neutrosophic Rings II, International Journal of Neutrosophic Science (IJNS), vol. 2(2), pp. 89-94, 2020. (DOI:10.5281/zenodo.3728235)
- [7] T. Bera and N.K. Mahapatra, On neutrosophic normal soft groups, Int. J. Appl. Comput. Math, Vol. 2(4), 2016. (DOI:10.1007/s40819-016-0284-2).
- [8] T. Bera and N.K. Mahapatra, Introduction to neutrosophic soft groups, Neutrosophic Sets and Systems, Vol. 13, pp. 118-127, 2016. (DOI: 10.5281/zenodo.570845)
- [9] T. Bera and N.K. Mahapatra, Study of the group theory in neutrosophic soft sense, Asia Mathematica, Vol. 3(2), pp. 1-18, 2019.
- [10] F. Smarandache(2003), A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (3rd edition), American Research Press, Rehoboth, <http://fs.gallup.unm.edu/eBook-Neutrosophic4.pdf>.
- [11] F. Smarandache, n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Progress in Physics, USA, vol. 4, pp. 143-146, 2013.
- [12] F. Smarandache, (T,I,F)- Neutrosophic Structures, Neutrosophic Sets and Systems, vol. 8, pp. 3-10, 2015.
- [13] W.B. Vasantha Kandasamy and F. Smarandache, Neutrosophic Rings, Hexis, Phoenix, Arizona, 2006 <http://fs.gallup.unm.edu/NeutrosophicRings.pdf>