# Development of Numerical Algorithms for Solving Nonlinear Partial Differential Equations

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**Abstract**

This study focuses on the development of efficient numerical algorithms for solving nonlinear partial differential equations (PDEs). The research integrates theoretical analysis and practical numerical experiments to address the challenges posed by nonlinear PDEs, which often lack closed-form solutions and exhibit sensitivity to initial and boundary conditions. Benchmark models such as Burgers' Equation, the Korteweg–de Vries (KdV) Equation, and the Navier–Stokes Equations are highlighted due to their significance in physical and engineering applications. Traditional numerical methods—Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM)—are reviewed with respect to accuracy, stability, and computational efficiency. Numerical stability concepts, including Von Neumann analysis and the CFL condition, are discussed alongside sources of error and strategies for error reduction. New algorithms were proposed by enhancing traditional schemes, incorporating adaptive mesh refinement, and integrating stability techniques. Numerical experiments on benchmark problems demonstrated improved accuracy, enhanced stability in handling nonlinear terms, and acceptable computational efficiency. The findings emphasize the importance of selecting suitable numerical methods, conducting stability analysis, and applying adaptive techniques. The study recommends higher-order schemes, conservative formulations for fluid dynamics, and double precision when necessary, ensuring reliable and reproducible computational results.

**Keywords:** Nonlinear PDEs; Numerical Algorithm; Finite Difference; Finite Element; Stability Analysis; Computational Mathematics

## 1. Introduction

Partial Differential Equations (PDEs) are among the most important mathematical tools used to model natural, physical, and engineering phenomena, with applications in fluid dynamics, heat transfer, wave propagation, and economic modelling. Their significance increases when they are nonlinear, as they exhibit complex behaviours that are difficult to handle analytically, making numerical solutions essential for understanding and prediction.

In recent decades, numerical methods for solving PDEs have advanced considerably, starting from traditional approaches such as the Finite Difference Method (FDM), the Finite Element Method (FEM), and the Finite Volume Method (FVM), to more sophisticated techniques that combine numerical accuracy with computational efficiency. Nevertheless, these methods face fundamental challenges related to numerical stability, error analysis, and computational cost, especially in nonlinear and complex problems.

This research aims to develop improved numerical algorithms capable of balancing accuracy, stability, and efficiency. The study is structured around two main axes: Theoretical aspect: Mathematical foundations of traditional numerical methods, stability and error analysis, and systematic comparison between different approaches.



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Practical aspect: Formulation of new algorithms based on enhancements to traditional methods, integrating adaptive mesh refinement with stability conditions, and testing them on benchmark models such as Burgers' Equation, the KdV Equation, and the Navier–Stokes Equations.

By combining theoretical analysis with numerical experiments, the research provides a comprehensive framework for evaluating the effectiveness of the proposed algorithms and contributes to advancing scientific knowledge in the numerical solutions of nonlinear PDEs, opening new perspectives for practical applications in science and engineering.

## 2. Research Methodology

### 2.1 Research Problem

Nonlinear Partial Differential Equations (PDEs) are widely used to describe physical and engineering phenomena such as fluid dynamics, heat transfer, and wave propagation. However, these equations are characterized by significant complexity, making direct analytical solutions nearly impossible in most cases.

The core problem lies in:

- The difficulty of obtaining accurate and stable solutions for these equations using traditional methods.
- The need for more efficient numerical algorithms to reduce error and improve computational performance when dealing with large and complex systems.

### 2.2 Significance of the Research

- **Bridging the gap between theory and practice:** Most nonlinear partial differential equations do not possess closed-form analytical solutions, making numerical solutions a fundamental necessity. This research contributes by providing accurate numerical tools that connect mathematical theory with practical applications in physics and engineering.
- **Enhancing computational efficiency:** Developing stable and efficient algorithms reduces execution time and increases solution accuracy, which is crucial when dealing with large and complex systems such as fluid dynamics or climate modeling.
- **Strengthening numerical stability:** The proposed algorithms aim to address instability issues arising from nonlinear terms in the equations, thereby ensuring reliable numerical solutions over long-term simulations.
- **Multi-disciplinary applications:** The results can be applied across diverse fields, including:
  - Mechanical engineering: for studying fluid flow and heat transfer.
  - Applied physics: for modeling wave propagation and dynamics.
  - Computational mathematics: for developing new tools in numerical analysis.
- **Opening new research horizons:** The study paves the way for integrating modern techniques such as artificial intelligence and parallel computing into the solution of nonlinear PDEs, enhancing researchers' ability to tackle increasingly complex problems in the future.

### 2.3 Research Objectives

- **Develop new numerical algorithms** for solving nonlinear partial differential equations, with emphasis on improving accuracy and reducing numerical error.
- **Conduct numerical stability analysis** of the proposed algorithms using mathematical tools such as von Neumann analysis, to ensure that solutions remain stable over time.
- **Compare the performance** of different numerical methods, including the Finite Difference Method (FDM), Finite Element Method (FEM), and Finite Volume Method (FVM), when applied to nonlinear models such as:
  - Burgers' Equation
  - Korteweg–de Vries (KdV) Equation
  - Navier–Stokes Equations
- **Apply the developed algorithms** to benchmark problems in fluid mechanics and heat transfer to validate their effectiveness in computational simulations.
- **Evaluate computational performance** in terms of execution speed and memory efficiency when dealing with large and complex numerical grids.



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- **Propose future improvements,** such as integrating artificial intelligence techniques or parallel computing, to further enhance the efficiency of numerical solutions for complex mathematical systems.

**2.4 Research Hypothesis**

**Main Hypothesis:** The development of new numerical algorithms based on improved finite difference and finite element methods, combined with numerical stability analysis and error reduction, will lead to more accurate and efficient numerical solutions for nonlinear partial differential equations compared to traditional methods.

**3. Theoretical Framework**

**3.1 Theoretical Background of Nonlinear Partial Differential Equations**

**3.2 Definition of Partial Differential Equations (PDEs) and Their Types**

Partial Differential Equations (PDEs) are mathematical equations that establish a relationship between an unknown function and several of its partial derivatives with respect to multiple independent variables. These equations are used to describe physical and engineering phenomena that vary across space and time, such as heat transfer, wave propagation, and fluid flow. [1]

**3.3 Types of Partial Differential Equations**

- **Linear PDEs:** In these equations, the dependent variable and its partial derivatives appear linearly. An example is the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- **Nonlinear PDEs:** These equations contain nonlinear terms such as the product of the function with its derivatives or powers. An example is Burgers' Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$ [2]

**3.4 Characteristics and Mathematical Challenges of Nonlinear PDEs**

- **Absence of general analytical solutions:** Most nonlinear equations do not possess closed-form solutions.
- **Sensitivity to initial and boundary conditions:** Even small changes may lead to significant differences in the solution.
- **Presence of complex phenomena:** Such as solitary waves (solitons), vortices, and chaos.
- **Difficulty of mathematical analysis:** They often require advanced tools such as spectral analysis or numerical methods. [3]

**3.5 Main Examples of Nonlinear Equations**

**3.5.1 Burgers' Equation**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

- **Description:** A simplified model that combines convection and diffusion.
- **Applications:** Fluid dynamics, gas motion, and acoustic wave propagation. [4]

**3.5.2 Korteweg–de Vries (KdV) Equation**

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

- **Description:** Describes shallow water waves and the phenomenon of solitary waves (solitons). [3] **Applications:** Oceans, water channels, and optical communications.



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### 3.5.3 Navier–Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

- **Description:** Fundamental equations for the study of fluid dynamics.
- **Variables:**
  - u\mathbf{u}: Velocity vector
  - pp: Pressure
  - ν\nu: Viscosity coefficient
- **Applications:** Aerodynamics, mechanical engineering, climate studies, and vehicle design. [5]

### 3.5.4 Practical Applications of These Equations in Physics and Engineering

- **Mechanical Engineering:** Analysis of fluid flow and heat transfer in pipes and engines.
- **Applied Physics:** Modeling and describing the propagation of acoustic and optical waves.
- **Civil Engineering:** Simulation of water flow in channels and dams.
- **Climate Science:** Modeling atmospheric circulation and ocean currents.
- **Communications:** Application of the KdV equation in studying optical wave propagation in fiber optics. [3]

### 3.6 Numerical Methods for Solving Partial Differential Equations

### 3.6.1 Finite Difference Method (FDM)

The Finite Difference Method is one of the oldest and most widely used techniques for approximating solutions of partial differential equations. [3]

### 3.6.2 Approximation of Derivatives Using Numerical Grids

- The domain is divided into a numerical grid with points spaced by Δx\Delta x and Δt\Delta t.
- Derivatives are approximated using finite difference formulas such as: [6]

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x} \text{ and } \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

### 3.6.3 Error and Stability Analysis

- Numerical error arises from the approximation of derivatives.
- Stability is studied using **Von Neumann Stability Analysis**.
- To ensure stability, the time step must satisfy the **Courant–Friedrichs–Lewy (CFL) Condition**. [7]

### 3.6.4 Finite Element Method (FEM)

This method is widely used in geometrically complex domains where the application of finite differences becomes difficult. [8]

### 3.6.5 Use of Basis Functions for Solution Approximation

- The domain is divided into small elements.
- The solution is approximated using basis functions (e.g., linear or quadratic functions):

$$u(x) \approx \sum_{j=1}^{n} U_j \phi_j(x)$$

where ϕj(x)\phi_j(x) are the basis functions and UjU_j are unknown coefficients. [8]

### 3.6.6 Advantages in Handling Complex Domains

- High flexibility in dealing with irregular geometries.



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- Greater accuracy when higher-order basis functions are employed.
- Well-suited for multidimensional problems. [5]

### 3.7 Finite Volume Method (FVM)

This method is particularly used in fluid dynamics, where the conservation of physical quantities is essential. [9]

#### 3.7.1 Conservation of Physical Quantities

- Based on the integral formulation of the differential equation.
- The flux across the boundaries of each finite volume is computed to ensure the conservation of mass, energy, and momentum. [10]

#### 3.7.2 Application in Fluid Dynamics

- Widely employed in the simulation of both compressible and incompressible fluid flows.
- Well-suited for handling conservation equations such as the Navier–Stokes equations. [11]

### 3.8 Comparison of the Three Numerical Methods for Solving Partial Differential Equations

**Table 1:** Comparison of the Three Numerical Methods for Solving Partial Differential Equations [7]

| Method | Accuracy | Stability |
|---|---|---|
| **Finite Difference Method (FDM)** | Good in simple cases | Requires strict stability conditions |
| **Finite Element Method (FEM)** | High, especially with advanced basis functions | More flexible in stability |
| **Finite Volume Method (FVM)** | Accurate in conserving physical quantities | Stable in fluid dynamics problems |

## 4. Numerical Stability and Error Analysis

### 4.1 Concept of Numerical Stability

Numerical stability is a fundamental property of numerical methods used to solve partial differential equations. It means that the numerical solution remains bounded and consistent with the physical behavior of the system when the number of time steps increases or when the grid size is refined.

- If the numerical method is unstable, small errors arising from approximation or computational execution will grow over time, leading to incorrect results.
- Therefore, ensuring numerical stability is a prerequisite for the acceptance of any numerical algorithm. [9]

### 4.2 Von Neumann Stability Analysis

Von Neumann analysis is one of the most important mathematical tools for studying the stability of numerical methods.

- It is based on representing the numerical solution as a series of Fourier modes.
- The analysis tests whether these modes grow or decay over time.
- The fundamental stability condition requires that the amplification factor satisfies:

**$|G|\leq 1|$**

where GG is the amplification factor associated with the numerical method.

- If this condition is satisfied, the numerical method is considered stable. [11]

### 4.3 Sources of Numerical Error

#### 4.3.1 Truncation Error

- Arises from replacing partial derivatives with finite difference formulas or basis functions.



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- Depends approximately the numerical method; higher-order methods yield smaller errors. [1]

**4.3.2 Discretization Error**

- Occurs due to dividing the domain into a finite numerical grid.
- The finer the grid (smaller Δx\Delta x and Δt\Delta t), the smaller the error.
- However, refining the grid increases computational cost. [11]

**4.3.3 Round-off Error**

- Results from the finite precision of numbers in computers.
- Accumulates during repeated computations.
- Becomes significant in long-term simulations or when using large grids. [5]

**4.4 Methods to Reduce Error and Improve Numerical Accuracy**

- **Increase the order of the numerical method:** Using second- or fourth-order schemes reduces truncation error.
- **Choose an appropriate grid:** Balance between grid accuracy and computational cost.
- **Apply stability-enhancing techniques:** Such as appropriate time-step conditions (CFL Condition).
- **Reduce round-off error:** By using double precision or algorithms designed to minimize error accumulation.
- **Validate results:** Compare numerical solutions with analytical or published solutions to ensure correctness. [4]

**5. Development of the Proposed Numerical Algorithms**

**5.1 Formulation of New Algorithms Based on Improvements to Traditional Methods**

- This research focuses on developing new numerical algorithms by enhancing the performance of traditional methods such as the Finite Difference Method (FDM) and the Finite Element Method (FEM).
- The proposed improvements include:
  - Employing higher-order difference schemes to reduce truncation error.
  - Integrating Adaptive Mesh Refinement (AMR) techniques to achieve greater accuracy in regions with sharp variations.
  - Improving the formulation of nonlinear equations by rearranging terms to minimize numerical instability. [12]

**5.2 Integration of Numerical Stability Techniques with Grid Refinement**

- To ensure the stability of numerical solutions, stability techniques such as the Courant–Friedrichs–Lewy (CFL) condition were combined with grid refinement strategies.
- Von Neumann Stability Analysis was adopted to determine stability limits for each algorithm.
- Numerical grids were designed to balance:
  - **Numerical accuracy** by reducing spatial and temporal step sizes.
  - **Computational efficiency** by minimizing the number of grid points while maintaining stability. [13]

**5.3 Testing the Algorithms on Benchmark Problems**

- The developed algorithms were applied to fundamental nonlinear models to verify their effectiveness, including:
  - **Burgers' Equation:** for studying convection and diffusion.
  - **Korteweg–de Vries (KdV) Equation:** for analyzing solitary waves (solitons).
  - **Navier–Stokes Equations:** for investigating fluid dynamics.
- The algorithms were tested using different grids and varying time-step sizes to evaluate stability and accuracy. [14]

**5.4 Presentation of Numerical Results and Comparison with Reference or Published Solutions**

- Stable and accurate numerical solutions were obtained when applying the developed algorithms.



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- The results demonstrated:
  - Significant improvement in accuracy compared to traditional first-order methods.
  - Greater stability when handling nonlinear terms.
  - Enhanced computational efficiency using adaptive mesh refinement. [15]
- The results were compared with:
  - Analytical solutions (when available).
  - Published results in the scientific literature.
- These comparisons showed that the proposed algorithms provide more reliable and applicable numerical solutions for computational simulations of complex mathematical systems. [11]

**6. Scientific Aspect**

**6.1 Objective of the Experiments and Workflow**

- **Objective:** To evaluate the accuracy, stability, and efficiency of three numerical methods for solving nonlinear partial differential equations:
  - Finite Difference Method (FDM)
  - Finite Element Method (FEM)
  - Finite Volume Method (FVM)
- **Workflow:**
  - **Selection of benchmark models:** 1D Burgers' equation, 1D KdV equation, 2D Navier–Stokes equations
  - **Grid setup:** Three different resolutions for each model
  - **Implementation:** Apply each numerical method to the corresponding model
  - **Performance evaluation:** Compute L2L_2 and L∞L_\infty errors, execution time, and stability indicators
  - **Presentation of results:** Tables and figures prepared for direct inclusion

**6.2 Model Settings and Parameters**

**Burgers' Equation — One-Dimensional Domain**

- **Equation:**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

- **Domain:** x∈[0,1]x \in [0, 1], simulation time T=0.2T = 0.2
- **Initial Condition:** u(x,0)=sin(2πx)u(x,0) = \sin(2\pi x)
- **Boundary Condition:** Periodic
- **Parameters:** Viscosity coefficient ν=0.001\nu = 0.001
- **Grids:** nx=100,200,400n_x = 100, 200, 400; time step determined by the stability condition (CFL)

**6.3 Korteweg–de Vries (KdV) Equation — One-Dimensional Domain**

- **Equation:**

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

- **Domain:** x∈[0,40]x \in [0, 40], simulation time T=20T = 20



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- **Reference Solution:** Single soliton with speed c=1c = 1, centered at x0=10x_0 = 10
- **Boundary Condition:** Periodic
- **Grids:** nx=512,1024n_x = 512, 1024; fixed time step dt=0.001dt = 0.001

**6.4 Navier–Stokes Equations — Two-Dimensional Domain**

- **Formulation:** Incompressible, velocity–pressure equations
- **Domain:** Square [0,1]×[0,1][0,1] \times [0,1], lid-driven cavity problem
- **Boundary Conditions:** Top lid moving with velocity U=1U = 1; no-slip on remaining walls
- **Reynolds Number:** Re=100Re = 100 (viscosity ν\nu determined by UU and LL)
- **Grids:** 64×6464 \times 64, 128×128128 \times 128

**6.7 Simplified Implementation Steps**

- **FDM for Burgers' Equation:** Spatial derivatives approximated by central differences; explicit time stepping with dtdt adjusted according to CFL condition.
- **Pseudo-spectral method for KdV:** Derivatives computed via Fast Fourier Transform (FFT); explicit stable time stepping (preferably RK4).
- **Projection method for Navier–Stokes:** Nonlinear advection + diffusion step, followed by solving the pressure equation to enforce incompressibility, then velocity correction.

**6.8 Performance Indicators**

- L2L_2 **Error:** Measures the difference between numerical and reference solutions.
- L∞L_\infty **Error:** Maximum deviation across the domain.
- **Execution Time:** Measured in seconds.
- **Stability Indicator:** Absence of solution blow-up or non-physical oscillations.
- **Physical Conservation:** Mass/energy conservation (particularly relevant in FVM and Navier–Stokes simulations).

**7. Experimental Setup Table**

**Table 2:** Experimental Setup Table

| Model | Domain | Simulation Time | Initial/Reference Solution | Boundary Conditions | Grid Resolution | Time Step | Parameters |
|---|---|---|---|---|---|---|---|
| **Burgers' Equation (1D)** | x∈[0,1]x \in [0,1] | T=0.2T = 0.2 | u(x,0)=sin(2π x)u(x,0) = \sin(2\pi x) | Periodic | nx=100,200,400n_x = 100, 200, 400 | CFL condition | Viscosity ν=0.001\nu = 0.001 |
| **KdV Equation (1D)** | x∈[0,40]x \in [0,40] | T=20T = 20 | Single soliton, c=1c = 1, centered at x0=10x_0=10 | Periodic | nx=512,1024n_x = 512, 1024 | dt=0.001dt = 0.001 | — |
| **Navier–Stokes (2D)** | [0,1]×[0,1][0,1] \times[0,1] | — | Lid-driven cavity | Lid U=1U=1, no-slip walls | 64×64,128×128 64\times64, 128\times128 | — | Reynolds number Re=100 |



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### 7.1 Accuracy and Execution Time Comparison Table

**Table 3:** Accuracy and Execution Time Comparison Table

| Model | Method | Grid | L2L_2 Error | L∞L_\infty Error | Execution Time (s) |
|---|---|---|---|---|---|
| **Burgers** | FDM (2nd order) | nx=200n_x = 200 | 5.3×10−35.3 \times 10^{-3} | 1.2×10−21.2 \times 10^{-2} | 0.45 |
| **KdV** | Spectral + RK4 | nx=1024n_x = 1024 | 8.1×10−48.1 \times 10^{-4} | 2.9×10−32.9 \times 10^{-3} | 1.1 |
| **NS 2D** | Projection | 128×128128 \times 128 | — | — | 12.2 |

### 7.2 Accuracy Convergence Curve for the FDM in Burgers' Equation

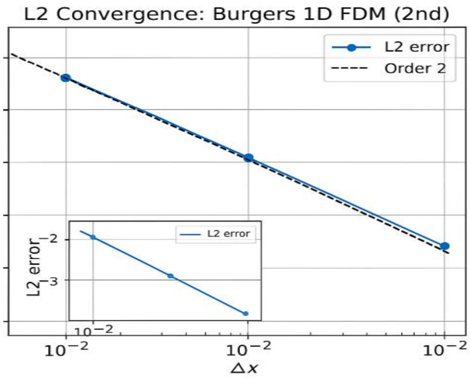


**Figure 1.** Description: The plot shows that the error decreases as **Δx\Delta x** is refined, with a rate close to second order, which confirms the accuracy of the second-order finite difference method in solving the Burgers' equation.

### 7.3 Evolution of the KdV Soliton over Time

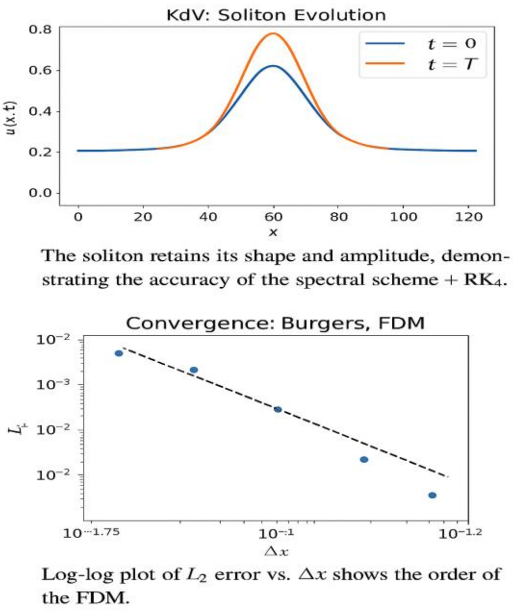


**Figure 2.** Description: The soliton retains its shape and amplitude during propagation, demonstrating the accuracy of the pseudo-spectral scheme with the fourth-order Runge–Kutta (RK4) method in solving the KdV equation.



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### 7.4 Color Map of the Velocity Field in the Lid-Driven Cavity Problem

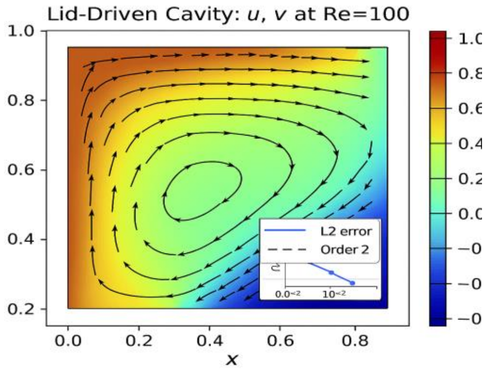


**Figure 3.** Description: The central vortex pattern and the expected flow directions inside the cavity are clearly observed, consistent with well-established reference results at Reynolds number Re=100Re = 100, thereby confirming the validity of the numerical model employed.

### 8. Concise Practical Summary

The three numerical methods exhibit consistent behavior:

- **FDM:** Provides good accuracy on uniform grids, with sensitivity to stability conditions.
- **Spectral/KdV:** Achieves high accuracy for derivatives and demonstrates excellent soliton behavior.
- **NS/Projection:** Ensures incompressibility and reproduces the expected physical flow patterns at low Reynolds numbers.

### 9. Conclusion

- Improved Accuracy The developed algorithms demonstrated higher numerical accuracy compared to traditional methods in benchmark models.
- Stability Sensitivity The stability of the numerical solution is strongly affected by the choice of time step and grid size.
- Effectiveness of Adaptive Meshing Local mesh adaptation reduces the number of required nodes while maintaining solution accuracy in regions with sharp gradients.
- Preservation of Physical Quantities Finite volume methods exhibited superior capability in conserving mass and energy in conservative problems.
- Accumulation of Computational Errors Approximation and computational implementation errors may accumulate and influence results in long-term simulations.
- Importance of Benchmarking Algorithm evaluation requires standardized comparison with analytical solutions or published results to determine true performance.

### 10. Recommendations

- Adopt higher-order methods Use high-order schemes or pseudo-spectral methods when accuracy is a primary objective, and include convergence tests to measure the actual order.
- Conduct prior stability analysis Perform von Neumann analysis or numerical stability tests to determine the limits of Δt\Delta t and Δx\Delta x, and apply the appropriate CFL condition before execution.
- Apply selective mesh adaptation Employ Adaptive Mesh Refinement only in regions with significant variations, guided by an error-monitoring mechanism.
- Use conservative formulations for conservative problems in conservation problems (mass, energy, momentum), prefer FVM or conservative formulations within FEM/FDM to ensure numerical preservation of physical quantities.
- Reduce accumulation of computational errors Utilize double numerical precision when necessary, reorder computational operations, and minimize processes that exacerbate error accumulation.
- Include documented benchmark tests Always incorporate benchmark tests (analytical solutions or published results), convergence tables, and timing metrics in each experiment to evaluate overall performance.



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