An Introduction to the Algebraic Structure of Type-1 Neutrosophic-Set Theory

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Abstract

This article presents a focused investigation of type-1 neutrosophic sets, derived from classical sets by introducing an indeterminacy component, I. type-1 neutrosophic sets generalize classical set theory by incorporating four-valued logic, which was generated by Boolean logic in our work. This work will appear in the future. As we know, a neutrosophic set is based on a many-valued logic defined by three independent membership functions: truth, indeterminacy, and falsehood. This work systematically re-examines and consolidates foundational research conducted between 2024 and 2025, isolating type-1 structures from the broader frameworks of type-2 and type-3 neutrosophic sets for clearer axiomatic and theoretical development. We establish core concepts, terminology, operations, and properties specific to type-1 neutrosophic sets, constructing and analyzing the type-1 neutrosophic Cartesian product. In addition, we introduce and investigate the properties of type-1 neutrosophic ordered pairs and their corresponding products. This foundation formally defines type-1 neutrosophic relations and neutrosophic partially ordered relations, establishing their core properties. Furthermore, the article explores type-1 neutrosophic functions, detailing their various types, including injective, surjective, and bijective functions and their respective properties. A significant focus is placed on invertible neutrosophic functions, where we examine the conditions for invertibility and prove key related theorems.By focusing exclusively on type-1, we aim to create a more dynamic and effective foundation for application across diverse neutrosophic fields, including neutrosophic algebra, number theory, and logic. This focused approach is intended to open new research pathways within the neutrosophic sciences.

Keywords: Type-1 Neutrosophic Set and their properties; Operations on Type-1 Neutrosophic Set and Their Properties; Cartesian Product of Type-1; Neutrosophic Relations of Type-1; Neutrosophic Functions of Type-1; Invertible Netrosophic of Type-1

1 introduction

Neutrosophic science, pioneered by Smarandache nearly three decades ago, arose as a broad generalization of intuitionistic set theory and logic. At its core lies the triadic membership framework of truth (T), indeterminacy (I), and falsehood (F).[15–18,20] This foundational paradigm inspired the construction of neutrosophic algebraic structures—such as groups and rings—developed in collaboration with other scholars.[11,12] Within this framework, the type-1 Neutrosophic Set (T1NS) represents a specialized branch of Neutrosophic Set (NS) theory. NS itself extends both Intuitionistic and Fuzzy Sets, being grounded in Neutrosophic Logic and characterized by three independent membership degrees: (T), (I), and (F).[15–18,20] This independence distinguishes NS from Intuitionistic Sets, which are constrained by $(T + F \leq 1)$,[10,11] and from Fuzzy Sets, which rely on a single membership value.[21] For comparative analyses, see.[7,8] The algebraic foundation of neutrosophic theory employs neutrosophic numbers of the form $N = a + bI$, where I is an indeterminate literal satisfying

$I^2 = I$ and $0 \cdot I = 0$, as established in.[12,13] Building on this foundation, our formal development of Type-1 Neutrosophic Set Theory (Sections 2 and 3) follows the principles outlined in.[1–3] Specifically, we define the type-1 neutrosophic Cartesian product, examine its properties, and employ it to construct neutrosophic relations and partially ordered relations. We then extend the framework to neutrosophic functions, exploring their classifications including bijective functions and related theorems—as well as invertible neutrosophic functions and their associated properties. Philosophically, the notion of indeterminacy (I) reflects the recognition of uncertainty and partial knowledge inherent in reality, demanding nuanced judgement. This principle is rooted in philosophical neutrality, which asserts that between any two opposites ⟨A⟩ and ⟨Anti-A⟩ lies a continuum of neutralities ⟨Neut-A⟩.[15–18,20] The evolution of this idea follows Zadeh's introduction of the fuzzy set[21] and Atanassov's formulation of the Intuitionistic Fuzzy Set.[10,11] For further sources in the literature on set theory, see, for example,.[14,19]

2 Basic Concepts of Algebraic Structure of Types-1 Neutrosophic-Sets

In this section, we introduce the ideas of type-1 neutrosophic set theory based on classical set theory and establish the basic terminology and notation for type-1 neutrosophic sets, such as type-1 neutrosophic sets generated by any classical set supply the indeterminacy concept. Type-1 neutrosophic set usage: When some information is not the whole truth nor fully false, but contains unknown information or a lack of knowledge. The following terminologies, like type-1 universal neutrosophic set, type-1 emptying neutrosophic set, type-1 neutrosophic subset, type-1 complement neutrosophic set, type-1 neutrosophic complement, type-1 family neutrosophic set, with their properties, theorems, and examples.

Definition 2.1. [1] [Type-1 Neutrosophic Set] Let A be a classical set, then:

$$A[I] = \{a_1 + a_2 I : a_1, a_2 \in A\}$$

is called a neutrosophic set of type-1, and denoted by $A[I]$ which generated by A and indeterminacy I. Each element of $A[I]$ is called a neutrosophic element.

Remark 2.2. From now on, we will use the terminology " neutrosophic set" for abbreviation, instead of a" type-1 neutrosophic set", if no confusion, throughout this article.

Example 2.3. ref1 Let A be a classical set, given by $A = \{1, 2, 3\}$. Then, the neutrospohic-set $A[I]$ is given by:

$$A[I] = \{1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}.$$

Example 2.4. [1] Let $A = \{0, 1, 2\}$ be a classical set. Then, the $A[I]$ is given by:

$$A[I] = \{0, 1, 2I, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$$

Definition 2.5. [1] Let U be a universal classical set, then:

$$U[I] = \{u_1 + u_2 I : u_1, u_2 \in U\}$$

is called a universal neutrosophic set generated by U and indeterminacy I.

Definition 2.6. [1] Let ϕ be an empty classical set, then:

$$\phi[I] = \{u_1 + u_2 I : u_1, u_2 \in \phi\} = \phi$$

is called a empty neutrosophic set generated by ϕ and indeterminacy I.

Definition 2.7. [1] Let $A[I]$ and $B[I]$ be two neutrosophic sets.generated by A, B, and indeterminacy I respectively, then the neutrosophic set equality is defined by:

$$A[I] = B[I] \Longleftrightarrow A = B.$$

Definition 2.8. [1] [Type-1 Neutrosophic Subset] Let $A[I]$ and $B[I]$ be two neutrosophic sets. Then we said that $A[I]$ is a neutrosophic subset of $B[I]$, if:

$$A[I] \subseteq B[I] \iff A[I]\big(\forall a\big)\big(a \in A[I] \Longrightarrow a \in B[I]\big).$$

Theorem 2.9. [1] *If $A \subseteq B$, then $A[I] \subseteq B[I]$.*

Proof. Suppose that $A \subseteq B$. Assume that $a \in A[I] \Longrightarrow \exists\, a_1, a_2 \in A$, such that $a = a_1 + a_2 I$, and indeterminacy $I \Longrightarrow a \in B[I]$. Therefore, $A[I] \subseteq B[I]$. □

Theorem 2.10. [1] *Let $A[I]$ and $B[I]$ be two neutrosophic sets. Then:*

$$A[I] = B[I] \iff (A[I] \subset B[I]) \wedge (B[I] \subset A[I]).$$

Proof. Suppose that $A[I] = B[I] \iff A = B \iff (]ASB) \wedge (B \subseteq A)$
$\iff (A[I] \subseteq B[I]) \wedge (B[I] \subseteq A[I]$. □

Theorem 2.11. [1] *Let $A[I]$, $B[I]$ and $C[I]$ be three type-1 neutrosophic sets. Then:*

1. *$A[I] = A[I]$ (Reflexivity of type-1 neutrosophic equality).*
2. *If $A[I] = B[I] \Longrightarrow B[I] = A[I]$ (Symmetry of type-1 neutrosophic equality).*
3. *If $(A[I] = B[I]) \wedge (B[I] = C[I]) \Longrightarrow A[I] = C[I]$ (Transitivity of type-1 neutrosophic equality).*

Proof.
1. Suppose that $A[I] = A[I] \iff A = A \iff (A \subset A) \wedge (A \subset A)$
$\iff (A[I] \subset A[I]) \wedge (A[I] \subset A[I])$.
2. Presume that. $(A[I] = B[I] \Longrightarrow A = B \Longrightarrow (A \subseteq B) \wedge (B \subseteq A)$
$\Longrightarrow (B \subseteq A) \wedge (A \subseteq B) \Longrightarrow (B[I] \subseteq A[I]) \wedge (ASBN) \Longrightarrow B[I] = A[I]$.
3. Assume that. $(A[I] = B[I]) \wedge (B[I] = C[I])$

$$\because A[I] = B[I] \Longrightarrow A = B$$

$$\because B[I] = C[I] \Longrightarrow B = C$$

$\Longrightarrow A = C \Longrightarrow A[I] = C[I]$.

□

Remark 2.12. We note that the relation of equality for a type-1 neutrosophic set is atype-1 neutrosophic set is a neutrosophic transitive relation.

Theorem 2.13. [1] *Let $\phi[I]$ be a type-1 empty neutrospohic set, then:*

1. *$\phi[I] \subset A[I]$, where $A[I]$ is any arbitrary type-1 neutrosophic set.*
2. *The type-1 empty neutrospohic set $\phi[I]$ is unique.*

Proof.
1. Suppose that $a \in A[I]$. Then the premise: $a \in \phi[I] \Longrightarrow \exists\, a_1, a_2 \in \phi$ such that $a = a_1 + a_2 I$, and indeterminacy I. But ϕ is an empty set, therefore there are no elements such as $a_1, a_2 \in \phi$ satisfy the property. $a = a_1 + a_2 I$, and consequently. Premise is false $\forall a$. Therefore, the statement $\big(\forall a\big)\big(a \in \phi[I] \Longrightarrow a \in A[I]$ is true. Thus $\phi[I] \subset A[I]$.
2. To prove that uniqueness. Suppose that $\phi_1[I]$ and $\phi_2[I]$ are two type-1 empty neutrosophic sets, then by part-1, We have:
$$(\phi_1[I] \subset \phi_2[I]) \wedge (\phi_2[I] \subset \phi_1[I]) \Longrightarrow \phi_1[I] = \phi_2[I].$$

□

Definition 2.14. [1] [Type -1 Neutrosophic Complement] Let $A[I]$ be a type-1. neutrosophic set. Then, the type-1 neutrosophic complement set is denoted by $A^c[I]$ and defined by:

$$A^c[I] = \{a : a \notin A[I] \wedge a \in U[I]\} = \{a : a \notin A \wedge a \in U\}.$$

Example 2.15. [1] Let $U = \{a, b, c, d, e\}$ and $A = \{a, b, d\}$ be two classical sets. If $a \notin A$, implies that $a \in A = \{c, e\}$. Then we have,

$$A^c[I] = \{c + cI, c + eI, e + cI, e + eI\}$$

. Where

$$U[I] = \{a + aI, a + bI, a + dI, b + aI, b + bI, b + dI, d + aI, d + bI, d + dI\}$$

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Definition 2.16. [1] [Type-1 Neutrosophic Power Set] Let $A[I]$ be a type-1 neutrosophic set. Then, the type-1 neutrosophic power-set is defined by:

$$P\big(A[I]\big) = \{E[I] \colon E[I] \subseteq A[I]\}.$$

Example 2.17. [1] Let $U = \{a, b, c, d, e\}$ and $A = \{a, b, d\}$ be two classical sets. Then: $x \in A$ implies that $x \in A^c = \{c, e\}$. So, we have,

$$A^c[I] = \{c + cI, c + eI, e + cI, e + eI\}$$

And

$$U[I] = \{a + aI, a + bI, a + dI, b + aI, b + bI, b + dI, d + aI, d + bId + dI\}.$$

Theorem 2.18. [1] *Let $A[I]$ be a type-1 neutrosophic set. Then:*

1. $A^{c^c}[I] = A[I]$,
2. $U^c[I] = \phi[I]$, *and*
3. $\phi^c[I] = U[I]$ *, where A is any arbitrary classical set and U is any arbitrary universal classical set.*

Proof. 1. Suppose that $U[I]$ is any arbitrary universal neutrosophic set, where U is any arbitrary universal classical set such that $A[I] \subset U[I]$, when
$A \subset U$. To show that $A^{c^c}[I] = A[I]$. Suppose that $x \in A^{c^c}[I]$
$\Longrightarrow x \in A^c[I] \Longrightarrow x \in A[I] \Longrightarrow A^{c^c}[I] \subset A[I]$.
Conversely, assume that. $x \in A[I] \Longrightarrow x \in A^c[I] \Longrightarrow x \in A^{c^c}[I]$
$limpA[I] \subset A^{c^c}[I]$. Therefore, $A^{c^c}[I] = A[I]$.

2. Suppose that. $\phi^c[I] \neq U[I]$
$\Longrightarrow \exists\, x \in \phi^c[I] \wedge \exists\, x \in U[I] \Longrightarrow \exists\, x \in \phi[I] \wedge \exists\, x \in U[I]$
$\Longrightarrow \exists\, x \in \phi \wedge \exists\, x \in U \Longrightarrow \exists\, x \in (\phi \wedge U) = \exists\, x \in \phi$. But ϕ is an empty set. Therefore, $\phi^c[I] = U[I]$.

3. By the same argument.

□

Definition 2.19. [2,3] Let $U[I]$ be any type-1 neutrosophic universal set and,
$\mathbb{I} = \{1, 2, 3, \dots\}$. Define a type-1 neutrosophic set by:

$$\mathcal{F} = \{A_\alpha[I] \mid\ A_\alpha[I] \subseteq U_\alpha[I], \alpha \in \mathbb{I}\}.$$

Where:

i. $\mathcal{F}$ is called a type-1 neutrosophic family of neutrosophic sets,

ii. $\mathbb{I}$ is called an indexing set for the type-1 neutrosophic family.

Definition 2.20. [2,3] Let $\Psi = \{A_\alpha[I] \mid\ \alpha \in \mathbb{I}\}$ be family of type-1 neutrosophic subsets of $A_\alpha[I]$.we said that Ψ is a type-1 neutrosophic partition of $A[I]$, if satisfies the following conditions:

i. $A_\alpha[I] \neq \phi[I], \forall \alpha \in \mathbb{I}$,

ii. For each $A_\alpha[I]$ and $B_\beta[I]$, then either
$A_\alpha[I] = B_\beta[I]$ or $A_\alpha[I] \cap B_\beta[I] = \phi[I]$.

iii. $A[I] = \cup A_\alpha[I].\forall \alpha \in \mathbb{I}$.

Example 2.21. [2,3] Consider the following type-1 neutrosophic:

$$A[I] = \{(1 + I, 1 + 2I, 1 + 3I, 2 + 1I, 2 + 2I, 2 + 3I, 3 + 1I, 3 + 2I, 3 + 3I\}$$

With the following neutrosophic subsets of $A[I]$

$$A_1[I] = \{1 + I, 1 + 2I, 1 + 3I\},$$

$$A_2[I] = \{2 + 1I, 2 + 2I, 2 + 3I\},$$

And

$$A_3[I] = \{3 + I, 3 + 2I, 3 + 3I\}.$$

We note that, $\Psi = \{A_1[I], A_2[I], A_3[I]\}$ is a neutrosophic partition of $A[I]$, because

$$A[I] = A_1[I] \cup A_2[I] \cup A_3[I],$$

$$A_1[I] \cap A_2[I] = \phi[I],$$
$$A_1[I] \cap A_3[I] = \phi[I],$$

And

$$A_2[I] \cap A_3[I] = \phi[I].$$

While the following neutrosophic subsets of $A[I]$

$$A_1[I] = \{1 + 1I, 2 + 2I, 3 + 3I\},$$

$$A_2[I] = \{2 + 1I, 2 + 2I, 2 + 3I\},$$

And

$$A_3[I] = \{3 + 1I, 2 + 1I, 3 + 2I\}.$$

are not neutrosophic partition of $A[I]$.

Definition 2.22. [2,3] Let $A_\alpha[I]), \alpha \in \mathbb{I}$, where $\mathbb{I} = \{1, 2, 3, \dots, n\}$ be a sequence of finite type-1 neutrosophic sets. Define the finite general type-1 neutrosophic union as follows:

$$\bigcup_{\alpha \in \mathbb{I}} A_\alpha[I] = A_{\alpha_1}[I] \cup A_{\alpha_2}[I] \cup A_{\alpha_3}[I] \cup \dots A_{\alpha_n}[I].$$

Definition 2.23. [2,3] Let $A_\alpha[I]), \alpha \in \mathbb{I}$, where $\mathbb{I} = \{1, 2, 3, \dots\}$ of countable infinite type-1 neutrosophic sets. Define the countable infinite neutrosophic union as follows:

$$\bigcup_{\alpha \in \mathbb{I}} A_\alpha[I] = A_{\alpha_1}[I] \cup A_{\alpha_2}[I] \cup A_{\alpha_3}[I] \cup \dots A_{\alpha_n}[I] \cup \dots.$$

Definition 2.24. [2,3] Let $\Psi = \{A_\alpha \in \mathbb{I}[I] : \alpha \in \mathbb{I}\}$ be a type-1 neutrosophic family of indexed sets. Define arbitrary type-1 neutrosophic union as follows:

$$\bigcup_{\alpha \in \mathbb{I}} A_\alpha[I] = \{x : \exists\, \alpha \in \mathbb{I} \ni x \in A_\alpha \in \mathbb{I}[I]\}.$$

3 Operations on Type-1 Neutrosophic Sets and Their Properties

One of the most interesting and useful facts about classes is that under the operations of union, intersection, and complementation, they satisfy certain algebraic laws from which we can develop an algebra of classes. In this section, we will see the fundamental operations, such as type-1 neutrosophic union, type-1 neutrosophic intersection, type-1 neutrosophic difference, and type-1 symmetric difference, along with their properties and examples.

3.1 Type-1 Neutrosophic Union and Intersection

Definition 3.1. [2,3] [Type-1 Neutrosophic Union] Let $A[I]$, and $B[I]$ be two type-1 eutrosophic sets. The type-1 eutrosophic set union is defined by:

$$A[I] \cup B[I] = \{x : x \in A[I] \vee x \in B[I]\}$$

Remark 3.2. If $x \in \big(A[I] \cup B[I]\big) \Longleftrightarrow \exists\, x_1, x_2 \in A \vee \exists\, x_1, x_2 \in B$ such that $x = x_1 + x_2 I$, and indeterminacy $I \Longleftrightarrow \exists\, x_1, x_2 \in (A \vee B)$, and indeterminacy I.

Example 3.3. [2,3] Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ be two classical sets. Then the type-1 neutrosophic sets $A[I]$ and $B[I]$ are given by:

$$A[I] = \{1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}.$$

And

$$B[I] = \{3 + 3I, 3 + 4I, 3 + 5I, 4 + 3I, 4 + 4I, 4 + 5I, 5 + 3I, 5 + 4I, 5 + 5I\}.$$

Then the type-1 neutrosophic union is given by:

$$A[I] \cup B[I] = \left\{\begin{array}{l} 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 1 + 5I, \\ 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 2 + 5I, \\ 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 3 + 5I, \\ 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, 4 + 5I, \\ 5 + I, 5 + 2I, 5 + 3I, 5 + 4I, 5 + 5I \end{array}\right\}.$$

The following theorem presents the primary properties of the type-1 neutrosophic union.

Theorem 3.4. [1–3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider the three Type-1 neutrosophic subsets of $U[I]$ such as $A[I]$, $B[I]$, and $[I]$,respectively,then:*

1. $A[I] \subset A[I] \cup B[I]$.
2. $B[I] \subset A[I] \cup B[I]$.
3. $A[I] = A[I] \cup B[I] \Longleftrightarrow B[I] \subset A[I]$.
4. $A[I] = A[I] \cup A[I]$ *" type-1 Neutrosophic idempotent".*
5. $A[I] \cup B[I] = B[I] \cup A[I]$ *" Type-1 Neutrosophic commutative".*
6. $\big(A[I] \cup B[I]\big) \cup C[I] = A[I] \cup \big(B[I] \cup C[I]\big)$ *" Type-1 Neutrosophic associative".*

Proof.
1. Assume that $x \in A[I]$.
$\Longrightarrow \exists\, x_1, x_2 \in A \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longrightarrow \exists\, x_1, x_2 \in A \vee \exists\, x_1, x_2 \in B \ni x = x_1 + x_2 I$, indeterminacy I.
$\Longrightarrow x \in A[I] \vee x \in B[I]$.
$\Longrightarrow x \in \big(A[I] \cup B[I]\big)$. Therefore, $A[I] \subset A[I] \cup B[I]$.
2. Analogously, as in proof 1.
3. Suppose that $A[I] = A[I] \cup B[I]$ and $x \in B[I]$. By part (1), we have, $B[I] \subset A[I] \cup B[I]$.
By hypotheses, we get, $B[I] \subset A[I]$. Conversely, consider $B[I] \subset A[I]$.
Assume that $x \in \big(A[I] \cup B[I]\big)$.
$\Longrightarrow x \in A[I] \vee x \in B[I]$.
$\Longrightarrow x \in A[I] \vee x \in A[I]$.
$\Longrightarrow x \in A[I]$. Therefore, $A[I] \cup B[I] \subset A[I]$.
But $A[I] \subset A[I] \cup B[I]$. So, $A[I] \cup B[I] = A[I]$.
4. Obviously.

5. Suppose that $x \in A[I] \cup B[I]$
$\Longleftrightarrow x \in A[I] \vee x \in B[I]$
$\Longleftrightarrow \exists\, x_1, x_2 \in A \vee \exists\, x_1, x_2 \in B \ni x = x_1 + x_2 I$, and indeterminacy I
$\Longleftrightarrow \exists\, x_1, x_2 \in B \vee \exists\, x_1, x_2 \in A \ni x = x_1 + x_2 I$, and indeterminacy I
$\Longleftrightarrow x \in B[I] \vee x \in A[I]$
$\Longleftrightarrow x \in \big(A[I] \cup B[I]\big)$.

6. Assume that $\big((A[I] \cup B[I]) \cup C[I]\big)$.
$\Longleftrightarrow x \in \big(A[I] \cup B[I]\big) \cup x \in C[I]$.
$\Longleftrightarrow \big(x \in A[I] \vee x \in B[I]\big) \vee x \in C[I]$.
$\Longleftrightarrow \big(\exists\, x_1, x_2 \in A \vee \exists\, x_1, x_2 \in B\big) \vee \exists\, x_1, x_2 \in C \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longleftrightarrow \exists\, x_1, x_2 \in A \vee \big(\exists\, x_1, x_2 \in B \vee \exists\, x_1, x_2 \in C\big) \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longleftrightarrow x \in A[I] \vee \big(x \in B[I] \vee x \in C[I]\big)$.
$\Longleftrightarrow x \in A[I] \vee x \in \big(B[I] \vee C[I]\big)$.
$\Longleftrightarrow x \in \big(A[I] \cup \big(B[I] \cup C[I]\big)\big)$.

□

Definition 3.5. [2,3] [Type-1 Neutrosophic Intersection] Let $A[I]$, and $B[I]$ be two type-1 neutrosophic sets. The type-1 eutrosophic set intersection is defined by:

$$A[I] \cap B[I] = \{x : x \in A[I] \wedge x \in B[I]\}$$

Remark 3.6. If $x \in \big(A[I] \cap B[I]\big) \Longleftrightarrow x \in A[I] \wedge x \in B[I] \Longleftrightarrow \exists\, x_1, x_2 \in A \wedge \exists\, x_1, x_2 \in B$ such that $x = x_1 + x_2 I$, and indeterminacy $I \Longleftrightarrow \exists\, x_1, x_2 \in (A \wedge B)$, and indeterminacy I.

Definition 3.7. [2,3] [Type-1 Neutrosophic Disjunctive] Let $A[I]$, and $B[I]$ be two type-1 neutrosophic sets. Then $A[I]$, and $B[I]$ are called neutrosophic disjunct sets, if

$$A[I] \cap B[I] = \phi[I].$$

The following theorem gives us the main properties of type-1 neutrosophic sets intersection.

Theorem 3.8. [2,3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider the three Type-1 neutrosophic subsets of $U[I]$ such as $A[I]$, $B[I]$, and $C[I]$,respectively,then:*

1. $A[I] \cap B[I] \subset A[I]$.
2. $B[I] \cap B[I] \subset B[I]$.
3. $A[I] = A[I] \cap B[I] \Longleftrightarrow A[I] \subset B[I]$.
4. $A[I] = A[I] \cap A[I]$ *" type-1 Neutrosophic idempotent".*
5. $A[I] \cap B[I] = B[I] \cap A[I]$ *" Type-1 Neutrosophic commutative".*
6. $\big(A[I] \cap B[I]\big) \cap C[I] = A[I] \cap \big(B[I] \cup C[I]\big)$ *" Type-1 Neutrosophic associative".*

Proof. 1. Assume that $x \in \big(A[I] \cap B[I]\big)$.
$\Longrightarrow \big(x \in A[I] \wedge x \in B[I]\big)$.
$\Longrightarrow \big(\exists\, x_1, x_2 \in A\big) \wedge \big(\exists\, x_1, x_2 \in B\big) \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longrightarrow \exists\, x_1, x_2 \in A \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longrightarrow x \in A[I] \ni x = x_1 + x_2 I$, and indeterminacy I.
Therefore, $A[I] \cap B[I] \subset A[I]$.

2. Similar to the proof in the part 1.

3. Suppose that $A[I] = A[I] \cap B[I]$ and $x \in A[I]$.
   By part (1), we have, $A[I] \cap B[I] \subset B[I]$.
   By hypotheses, we get, $A[I] \subset B[I]$. Conversely, consider $A[I] \subset B[I]$.
   Assume that $x \in A[I]$.
   $\Longrightarrow \exists\, x_1, x_2 \in A \ni x_1 + x_2 I$, and indeterminacy I.
   $\Longrightarrow \exists\, x_1, x_2 \in B \ni x_1 + x_2 I$, and indeterminacy I .
   $\Longrightarrow \exists\, x_1, x_2 \in A \wedge \exists\, x_1, x_2 \in B$, and indeterminacy I.
   $\Longrightarrow x \in A[I] \wedge x \in B[I]$.
   $\Longrightarrow x \in \big(A[I] \cap B[I]\big)$.
   $\Longrightarrow A[I] \subset A[I] \subset B[I]$.
   But, $A[I] \cap B[I] \subset A[I]$.
   Therefore $A[I] \cap B[I] = A[I]$.

4. By the same techniques.

5. Suppose that $x \in A[I] \cap B[I]$.
   $\Longleftrightarrow x \in A[I] \wedge x \in B[I]$.
   $\Longleftrightarrow \exists\, x_1, x_2 \in A \wedge \exists, x_1, x_2 \in B \ni x = x_1 + x_2 I$, and indeterminacy I.
   $\Longleftrightarrow \exists\, x_1, x_2 \in B \wedge \exists\, x_1, x_2 \in A \ni x = x_1 + x_2 I$, and indeterminacy I.
   $\Longleftrightarrow x \in B[I] \wedge x \in A[I]$.
   $\Longleftrightarrow x \in B[I] \wedge A[I]\big)$.
   $\Longleftrightarrow x \in \Big(A[I] \cap B[I]\Big)$.

6. Assume that $x \in \big(\big(A[I] \cap B[I]\big) \cap C[I]\big)$.
   $\Longrightarrow x \in \big(A[I] \cap B[I]\big) \wedge x \in C[I]$.
   $\Longrightarrow \big(x \in A[I] \wedge x \in B[I]\big) \wedge x \in C[I]$.
   $\Longrightarrow \big(\exists\, x_1, x_2 A \wedge \exists\, x_1, x_2 \in B\big) \wedge \exists\, x_1, x_2 \in C \ni x = x_1 + x_2 I$, and indeterminacy I.
   $\Longrightarrow \big(\exists\, x_1, x_2 \in A\big) \wedge \big(\exists x_1, x_2 \in B \wedge \exists\, x_1, x_2 \in C\big) \ni x = x_1 + x_2 I$, and indeterminacy I.
   $\Longrightarrow x \in A[I] \wedge \big(x \in B[I] \wedge x \in C[I]\big)$.
   $\Longrightarrow x \in A[I] \wedge \big(x \in B[I] \wedge C[I]\big)$.
   $\Longrightarrow x \in \big(A[I] \wedge \big(B[I] \wedge C[I]\big)$.
   $\Longrightarrow x \in \big(A[I] \cap \big(B[I] \cap C[I]\big)$.
   Therefore, $\big(A[I] \cap B[I]\big) C[I] \subset A[I] \cap \big(B[I] \cap C[I]\big)$.
   By a similar method, we can prove that:
   $A[I] \cap \big(B[I] \wedge C[I]\big) \subset \big(A[I] \cap B[I]\big) \cap C[I]$.
   And consequently, $\big(A[I] \cap B[I]\big) \cap C[I] = A[I] \cap \big(B[I] \cap C[I]\big)$.

□

The following theorem gives us the main properties of the connection between type-1 neutrosophic union and type-1 neutrosophic intersection.

Theorem 3.9. [2,3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider four type-1 neutrosophic subsets $A[I]$, $B[I]$, $C[I]$, and $[I]$ of type-1 neutrosophic universal set $U[I]$. Then:*

1. $A[I] \cap \big(B[I] \cup C[I]\big) = \big(A[I] \cap B[I]\big) \cup \big(A[I] \cap C[I]\big)$.
2. $A[I] \cup \big(B[I] \cap C[I]\big) = \big(A[I] \cup B[I]\big) \cap \big(A[I] \cup C[I]\big)$.

Proof. 1. Consider $x \in \big(A[I] \cap \big(B[I] \cup C[I]\big)\big)$.
$\Longrightarrow x \in A[I] \wedge x \in \big(B[I] \cup C[I]\big)$.
$\Longrightarrow x \in A[I] \wedge \big(x \in B[I] \vee x \in C[I]\big)$
$\Longrightarrow \big(\exists\, x_1, x_2 \in A\big) \wedge \big(\exists\, x_1, x_2 \in B_\beta \vee \exists\, x_1, x_2 \in C\big) \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longrightarrow \big(\exists\, x_1, x_2 \in A\big) \wedge \big(\exists\, x_1, x_2 \in B)\big) \vee \big((\exists\, x_1, x_2 \in A\big) \wedge \big(\exists\, x_1, x_2 \in C)\big) \ni x = x_1 + x_2 I$, and indeterminacy I.
$\Longrightarrow \big(x \in A[I] \wedge x \in B[I]\big) \vee \big(x \in A[I] \wedge x \in C[I]\big)$.

$\Longrightarrow x \in (A[I]B[I]) \vee x \in (A[I] \wedge C[I])$,
$\Longrightarrow x \in ((A[I] \wedge B[I]) \vee (A[I] \wedge C[I]))$,
$\Longrightarrow x \in ((A[I] \cap B[I]) \cup (A[I] \cap C[I]))$.
Therefore, $x \in A[I] \cap (B[I] \cup C[I]) \subset (A[I] \cap B[I]) \cup (A[I] \cap C[I])$.
Conversely,Assume that. $x \in ((A[I] \in B[I])(A[I] \cap C[I]))$.
$\Longrightarrow x \in (A[I] \cap B[I]) \vee x \in (A[I] \in C[I])$.
$\Longrightarrow x \in A[I] \wedge x \in B[I]) \vee (x \in A[I] \wedge x \in C[I])$.
$\Longrightarrow((\exists x_1, x_2 \in A) \wedge (\exists x_1, x_2 \in B)) \vee ((\exists x_1, x_2 \in A) \wedge (\exists x_1, x_2 \in C)) \ni x = x_1 + x_2 I$.
$\Longrightarrow((\exists x_1, x_2 \in A)(\exists x_1, x_2 \in B) \vee \exists x_1, x_2 \in C)) \ni x = x_1 + x_2 I$.
$\Longrightarrow((\exists x_1, x_2 \in A) \wedge (\exists x_1, x_2 \in (B \vee C))) \ni x = x_1 + x_2 I$.
$\Longrightarrow(x \in A[I][I] \wedge (x \in (B[I] \vee C[I])))$.
$\Longrightarrow x \in (A[I] \wedge (B[I] \vee C[I]))$.
$\Longrightarrow x \in (A[I] \cap (B[I] \cup C[I]))$.
$\Longrightarrow x \in (A[I] \wedge (B[I] \vee C[I]))$.
$\Longrightarrow(A[I] \wedge B[I]) \cup (A[I] \cap C[I])A[I] \cap (B[I] \cup C[I])$.
Therefore, $A[I] \cap (B[I] \cup C[I]) = (A[I] \cap B[I]) \cup (A[I] \cap C[I])$.

□

Theorem 3.10. [2,3] *Let $A[I]$, $B[I]$, $C[I]$, and $D[I]$ be four type-1 neutrosophic sets. If $A[I] \subset B[I]$, and $C[I] \subset D[I]$. Then:*

1. $A[I] \cup C[I] \subset B[I] \cup D[I]$, *and*
2. $A[I] \cap C[I] \subset B[I] \cap D[I]$.

Proof. 1. Suppose that $A[I] \subset B[I]$, and $C[I] \subset D[I]$. Assume that $x \in (A[I] \cup C[I])$.
$\Longrightarrow \big(x \in A[I]\big) \vee \big(x \in C[I]\big)$.
$\Longrightarrow \big(x \in B[I]\big) \vee \big(x \in D[I]\big)$.
$\Longrightarrow x\big(\in \big(B[I] \cup D[I]\big)$.
$\Longrightarrow A[I] \cup C[I] \subset B[I] \cup D[I]$.

2. By a similar method.

□

The following theorem generalizes De Morgan's theorem from classical set theory to neutrosophic classical set theory.

Theorem 3.11. [2,3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider $A[I]$, and$B[I] \subset U[I]$. Then:*

1. $\big(A[I] \cup B[I]\big)^c = A^c[I] \cap B^c[I]$, *and*
2. $\big(A[I] \cap B[I]\big)^c = A^c[I] \cup B^c[I]$.

Proof. 1. Suppose that $x \in \big(A[I] \cup B[I]\big)^c$.
$\Longleftrightarrow x \notin \big(A[I] \cup B[I]\big)$.
$\Longleftrightarrow \big(x \notin A[I] \wedge x \notin B[I]\big)$.
$\Longleftrightarrow x_1 + x_2 I \notin A[I] \wedge x_1 + x_2 I \notin B[I]$.
$\Longleftrightarrow x_1 + x_2 I \in A^c[I] \wedge x_1 + x_2 I \in B^c[I]$.
$\Longleftrightarrow x \in A^c[I] \wedge x \in B^c[I]$.
$\Longleftrightarrow x \in \big(A^c[I] \cap B^c[I]\big)$.

2. By the same techniques.

□

Theorem 3.12. [2,3] *Let $A_\alpha[I], \alpha \in \mathbb{I}$ be a type-1 neutrosophic family of indexed set and B[I] be a type-1 neutrosophic set.*
If $B[I] \subset A_\alpha[I]), \forall \alpha \in \mathbb{I}$, then $B[I] \subset \cap A_\alpha[I]), \forall \alpha \in \mathbb{I}$

Proof. Suppose that $B[I] \subset A_\alpha[I]), \forall \alpha \in \mathbb{I}$, and assume that $x \in B[I]$, "neu-hypo".
$\Longrightarrow x \in A_\alpha[I], \forall \alpha \in \mathbb{I}\, quad$ "NP$_1$" $from$" $Neu - hypo$".
$\Longrightarrow$x$\in A_{\alpha 1}[I] \wedge x \in A_{\alpha 2}[I] \wedge x \in A_{\alpha 3}[I] \wedge \dots x \in A_{\alpha n}[I] \wedge \dots, \forall \alpha \in \mathbb{I}$, "NP$_2$" $from$" NP_1".
$\Longrightarrow$x $\in \cap A_\alpha[I], \forall \alpha \in \mathbb{I}$, "NP$_3$" $from$" NP_2".
$\Longrightarrow$B[I] $\subset \cap A_\alpha[I], \forall \alpha \in \mathbb{I}$, "NC" from "NP$_3$". □

Theorem 3.13. [2,3] *Let $A_\alpha[I], \alpha \in \mathbb{I}$ be a type-1 neutrosophic family of indexed set. Then:*

1. $\big(\cup A_\alpha[I]\big)^c = \cap\big(A_\alpha[I]\big)^c$.
2. $\big(\cap A_\alpha[I]\big)^c = \cup\big(A_\alpha[I]\big)^c$, $\forall \alpha \in \mathbb{I}$. *Where the components of complements are taken in $U[I]$.*

Proof. Assume that $x \in \big(\cup A_\alpha[I]\big)^c$, "neu-hypo".
$\Longrightarrow x \in \big(U[I] - (A_\alpha[I])\big), \forall \alpha \in \mathbb{I}$, "NP$_1$" $from$" $neu - hypo$".
$\Longrightarrow\big($x $\in U[I]\big) \wedge x \notin \big(A_\alpha[I])\big), \forall \alpha \in \mathbb{I}$, "NP$_2$" $from$" NP_1".
$\Longrightarrow$x $\notin \big(A_\alpha[I])\big), \forall \alpha \in \mathbb{I}$, "NP$_4$" $from$" NP_3".
$\Longrightarrow$x $\in \big(A_\alpha[I])\big)^c, \forall \alpha \in \mathbb{I}$, "NP$_5$" $from$" NP_4".
$\Longrightarrow$x $\in \cap\big(A_\alpha[I])\big)^c, \forall \alpha \in \mathbb{I}$, "NP$_6$" $from$" NP_5".
$\Longrightarrow\big(\cup A_\alpha[I]\big)^c \subset \cap\big(A_\alpha[I]\big)^c$, "NC$_1$" $from$" NP_6".
$Suppose that$ x$\in \cap\big(A_\alpha[I]\big)^c$, "neu-hypo".
$\Longrightarrow x \in \big(A_\alpha[I]\big)^c$, "NP$_1$" $from$" $neu - hypo$".
$\Longrightarrow$x$\notin \big(A_\alpha[I]\big)$, "NP$_2$" $from$" NP_1".
$\Longrightarrow$x$\notin \cup\big(A_\alpha[I]\big)$, "NP$_3$" $from$" NP_2".
$\Longrightarrow$x$\in \cup\big(A_\alpha[I]\big)^c$, "NP$_4$" $from$" NP_3".
$\Longrightarrow\cap\big(A_\alpha[I]\big)^c \subset \big(\cup A_\alpha[I]\big)^c$, "NC$_2$" $from$" NP_4".
Therefore, $\big(\cup A_\alpha[I]\big)^c = \cap\big(A_\alpha[I]\big)^c$.

2. By the same way. □

Theorem 3.14. [2,3] *Let $A_\alpha[I], \alpha \in \mathbb{I}$ and $B_\beta[I], \beta \in \mathbb{J}$ be two type-1 neutrosophic family of indexed sets. Then:*

1. $\big(\cup A_\alpha[I], \alpha \in \mathbb{I}\big) \cap \big(\cup B_\beta[I], \beta \in \mathbb{J}\big) = \cup_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I] \cap B_\beta[I]\big)$.
2. $\big(\cap A_\alpha[I], \alpha \in \mathbb{I}\big) \cup \big(\cap B_\beta[I], \beta \in \mathbb{J}\big) = \cap_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I] \cup B_\beta[I]\big)$.

Proof. 1. Assume that $x \in \Big(\big(\cup A_\alpha[I], \alpha \in \mathbb{I}\big) \cap \big(\cup B_\beta[I], \beta \in \mathbb{J}\big)\Big)$, "neu-hypo".
$\Longrightarrow x \in \big(\cup A_\alpha[I], \alpha \in \mathbb{I}\big) \wedge x \in \big(\cup B_\beta[I], \beta \in \mathbb{J}\big)\Big)$, "NP$_1$" $from$" $neu - hypo$".
$\Longrightarrow\big(\exists\, \alpha \in \mathbb{I} \ni x \in A_\alpha[I]\big) \wedge \big(\exists\, \beta \in \mathbb{J} \ni x \in B_\beta[I]\big)$, "NP$_2$" $from$" NP_1".
$\Longrightarrow\big(\exists\, (\alpha,\beta) \in \mathbb{I} \times \mathbb{J} \ni x \in (A_\alpha[I] \wedge B_\beta[I])\big)$, "NP$_3$" $from$" NP_2".
$\Longrightarrow\big(\exists\, (\alpha,\beta) \in \mathbb{I} \times \mathbb{J} \ni x \in (A_\alpha[I] \cap B_\beta[I])\big)$, "NP$_4$" $from$" NP_3".
$\Longrightarrow\big($ x$\in \cup_{(\alpha,\beta)\in\mathbb{I}\times\mathbb{J}}(A_\alpha[I] \cap B_\beta[I])\big)$, "NP$_5$" $from$" NP_4 ".
$\Longrightarrow\big(\cup A_\alpha[I], \alpha \in \mathbb{I}\big) \cap \big(\cup B_\beta[I], \beta \in \mathbb{J}\big) \subset \cup_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I] \cap B_\beta[I]\big)$, "NC$_1$" $from$" NP_4".
$Conversely, suppose that$ x$\in \cup_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I] \cap B_\beta[I]\big)$, "neu-hypo".
$\Longrightarrow\big(\exists\, (\alpha,\beta) \in \mathbb{I} \times \mathbb{J} \ni x \in (A_\alpha[I] \cap B_\beta[I])\big)$, "NP$_1$" $from$" $neu - hypo$".
$\Longrightarrow\big(\exists\, (\alpha,\beta) \in \mathbb{I} \times \mathbb{J} \ni x \in (A_\alpha[I] \wedge B_\beta[I])\big)$, "NP$_2$" $from$" NP_1".
$\Longrightarrow\big(\exists\, \alpha \in \mathbb{I} \ni x \in A_\alpha[I]\big) \wedge \big(\exists\, \beta \in \mathbb{J} \ni x \in B_\beta[I]\big)$, "NP$_3$" $from$" NP_2".
$\Longrightarrow$x $\in \big(\cup A_\alpha[I], \alpha \in \mathbb{I}\big) \wedge x \in \big(\cup B_\beta[I], \beta \in \mathbb{J}\big)\Big)$, "NP$_4$" $from$" NP_3".

$\Longrightarrow$x $\in \Big(\big(\cup A_\alpha[I],\alpha\in\mathbb{I}\big)\cap\big(\cup B_\beta[I],\beta\in\mathbb{J}\big)\Big)$, "NP$_5$"$from$"$NP_4$ ".
$\Longrightarrow\cup_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I]\cap B_\beta[I]\big)\subset\big(\cup A_\alpha[I],\alpha\in\mathbb{I}\big)\cap\big(\cup B_\beta[I],\beta\in\mathbb{J}\big)$, "$NC_2$"$from$"$NP_5$". Therefore,

$$\big(\cup A_\alpha[I],\alpha\in\mathbb{I}\big)\cap\big(\cup B_\beta[I],\beta\in\mathbb{J}\big)=\cup_{(\alpha,\beta)\in(\mathbb{I}\times\mathbb{J})}\big(A_\alpha[I]\cap B_\beta[I]\big).$$

□

3.2 Type-1 Neutrosophic Difference and Symmetric Difference

Definition 3.15. [2,3] Let $A[I]$ and $B[I]$ be two type-1 neutrosophic sets. Then the difference of a type-1 neutrosophic sets of $A[I]$ and $B[I]$ is defined by:

$$A[I]-B[I]=\{x:(x\in A[I])\wedge(x\in B[I])\}.$$

Remark 3.16. If $x\in A[I]-B[I]\Longleftrightarrow((\exists x_1,x_2\in A)\wedge(\exists x_1,x_2\in B))\ni x=x_1+x_2I$, and indeterminacy I. The following theorem gives us the main properties of type-1 neutrosophic difference.

Theorem 3.17. [2,3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider two type-1 neutrosophic subsets $A[I]$, and $B[I]$ of $U[I]$. I.e., $A[I]$,$B[I]\subset U[I]$. Then:*

1. $A[I]-\phi[I]=A[I]$.
2. $A[I]-B[I]\subset A[I]$.
3. $B[I]-A[I]\subset B[I]$.
4. $A[I]-B[I]=B^c[I]-A^c[I]$.
5. $A[I]-B[I]=A[I]\cap B^c[I]$.
6. $A[I]\cap B[I]=A[I]-(A[I]-B[I])$.
7. $A[I]\cup B[I]=A[I]\cup(B[I]-A[I])$.
8. $A[I]-B[I]=A[I]-(A[I]\cap B[I])$.

Proof. 1. Consider $x\in(A[I]-\phi[I])$.
$\Longleftrightarrow(x\in A[I])\wedge(x\notin\phi[I])$.
$\Longleftrightarrow(x\in A[I])\wedge(x\in\phi^c[I])$.
$\Longleftrightarrow(x\in A[I])\wedge(x\in U[I])$.
$\Longleftrightarrow(x\in A[I])$. Therefore,$A[I]-\phi[I]=A[I]$.

2. Assume that:$x\in(A[I]-B[I])$.
$\Longrightarrow(x\in A[I])\wedge(x\notin B[I])$.
$\Longrightarrow(\exists x_1,x_2\in A)\wedge(\exists x_1,x_2\notin B)\ni x=x_1+x_2I$, indeterminacy I.
$\Longrightarrow(\exists x_1,x_2\in A)\ni x=x_1+x_2I$, indeterminacy I.
$\Longrightarrow x\in A[I]$. Therefore, $A[I]-B[I]\subset A[I]$.

3. By the same argument in part 2.

4. Assume that. $x\in(A[I]-B[I])$.
$\Longleftrightarrow(x\in A[I])\wedge(x\notin B[I]$.
$\Longleftrightarrow(\exists x_1,x_2\in A)\wedge(\exists x_1,x_2)\notin B).=x_1+x_2I$, indeterminacy I.
$\Longleftrightarrow(\exists x_1,x_2\notin A^c)\wedge(\exists x_1,x_2\in B^c)\ni x=x_1+x_2I$, indeterminacy I.
$\Longleftrightarrow(x\notin A^c[I])\wedge(x\in B^c[I])\ni x=x_1+x_2I$, indeterminacy I
$\Longleftrightarrow(x\in B^c[I])\wedge(x\notin A^c[I])\ni x=x_1+x_2I$, indeterminacy I.
$\Longleftrightarrow x\in(B^c[I])-A^c[I])$. Therefore, $A[I]-B[I]=B^c[I]-A^c[I]$.

5. Suppose that $x \in (A[I] - B[I])$.
$\Longleftrightarrow (x \in A[I]) \wedge (x \notin B[I])$.
$\Longleftrightarrow (\exists\, x_1, x_2 \in A) \wedge (\exists\, x_1, x_2 \notin B \ni x = x_1 + x_2 I$, indeterminacy I.
$\Longleftrightarrow (\exists x_1, x_2 \notin A^c) \wedge (\exists x_1, x_2 \in B^c) \ni x = x_1 + x_2 I$, indeterminacy I.
$\Longleftrightarrow (x \notin A^c[I]) \wedge (x \in B^c[I]) \ni x = x_1 + x_2 I$,indeterminacy I.
$\Longleftrightarrow (x \in B^c[I]) \wedge (x \notin A^c[I]) \ni x = x_1 + x_2 I$,indeterminacy I.
$\Longleftrightarrow x \in (B^c[I]) - A^c[I])$. Therefore, $A[I] - B[I] = A[I] \cap B^c[I]$.

6. Suppose that$x \in \big((A[I] - (A[I] - B[I])\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big(x \notin (A[I] - B[I])\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big(x \notin A[I] \vee x \in B[I]\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big(x \notin A[I] \vee x \in B[I]\big)$.
$\Longleftrightarrow \big(\text{x} \in A[I]\big) \wedge 4\big(\text{ x} \in A^c[I] \vee x \in B[I]\big)$.
$\Longleftrightarrow \big(x \in A[I] \wedge x \in A^c[I]\big) \vee \big(x \in A[I] \wedge x \in B[I]\big)$,
$\Longleftrightarrow \big(F_N\big) \vee \big(x \in A[I] \wedge x \in B[I]$.
$\Longleftrightarrow \big(\text{x} \in A[I] \wedge x \in B[I]\big)$.
$\Longleftrightarrow x \in (A[I] \cap B[I]\big)$. Therefore, $A[I] \cap B[I] = A[I] - \big(A[I] - B[I]\big)$.

7. Presume that $x \in \big(A[I] \cup (B[I] - A[I])\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \vee \big(x \in (B[I] - A[I])\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \vee \big(x \in B[I] \wedge x \notin A[I]\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \vee \big(x \in B[I] \wedge x \in A^c[I]\big)$.
$\Longleftrightarrow \big(x \in A[I] \vee x \in B[I]\big) \wedge \big(x \in A[I] \wedge x \in A^c[I]\big)$.
$\Longleftrightarrow \big(x \in A[I] \vee x \in B[I]\big) \wedge \big(F_N\big)$.
$\Longleftrightarrow \big(x \in A[I] \vee x \in B[I]\big)$.
$\Longleftrightarrow x \in \big(A[I] \cup B[I]\big)$. Therefore, $A[I] \cup B[I] = A[I] \cup \big(B[I] - A[I]\big)$.

8. Assume that $x \in \big(A[I] - (A[I] \cap B[I])\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big(x \notin \big(A[I] \cap B[I]\big)\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big((x \notin A[I]\big) \vee \big(x \notin B[I]\big)\big)$.
$\Longleftrightarrow \big(x \in A[I]\big) \wedge \big((x \in A^c[I]\big) \vee \big(x \notin B[I]\big)\big)$.
$\Longleftrightarrow \big(x \in A[I] \wedge x \in A^c[I]\big) \vee \big(x \in A[I] \wedge x[I]\big)$.
$\Longleftrightarrow \big(F_N\big) \vee \big(x \in A[I] \wedge x \notin B[I]\big)$.
$\Longleftrightarrow \big(x \in A[I] \wedge x \notin B[I]\big)$.
$\Longleftrightarrow A[I] - B[I]$. Therefore, $A[I] - B[I] = A[I] - \big(A[I] \cap B[I]\big)$.

□

Definition 3.18. [2,3] Let $A[I]$, $B[I]$ be two type-1 neutrosophic sets. Then the symmetric difference of $A[I]$, and $B[I]$ is defined by:

$$A[I] \triangle B[I] = \{x : x \in A[I] \oplus x \in B[I]\}.$$

Remark 3.19. If $x \in A[I] \triangle B[I] \Longleftrightarrow (\exists\, x_1, x_2 \in A) \oplus (\exists\, x_1, x_2 \in B) \ni x = x_1 + x_2 I$, and indeterminacy I. symbol $\oplus$ means that the exclusive or, that is $x \in A[I]$ or $x \in B[I]$. But not both. In other words,

$$A[I] \triangle B[I] = A[I] \cup B[I]) - (A[I] \cap B[I].or$$

$$A[I] \triangle B[I] = A[I] - B[I]) \cup (B[I] - A[I]).$$

Theorem 3.20. [2,3] *Let $U[I]$ be a type-1 neutrosophic universal set. Consider $A[I]$, $B[I]$, and $C[I] \subset U[I]$.Then:*

1. $A[I] \triangle \phi[I] = A[I]$.
2. $A[I] \triangle A[I] = \phi[I]$.
3. $(A[I] \triangle B[I]) = (B[I] \triangle C[I]$.
4. $\big(A[I] \triangle B[I]\big) \triangle C[I] = A[I] \triangle \big(B[I] \triangle C[I]\big)$

5. $A[I] \triangle B[I] = \phi[I] \Longleftrightarrow A[I] = B[I]$.

Proof. 1. $A[I] \triangle \phi[I] = (A[I] \cup \phi[I]) - (A[I] \cap \phi[I]) = A[I] - \phi[I] = A[I]$.

2. It is clear that by definition $A[I] \triangle A[I] = \phi[I]$.

3.

$$\begin{aligned}(A[I] \triangle B[I] &= (A[I] \cup B[I]) - (A[I] \cap B[I]) \\ &= (A[I] \triangle B[I] \\ &= (A[I] \cup B[I]) - (A[I] \cap B[I]) \\ &= (B[I] \cup A[I]) - (B[I] \cap A[I] \\ &= B[I] \triangle A[I].\end{aligned}$$

4. and 5. By the same method.

□

Theorem 3.21. [2,3] *Let $A[I]$, $B[I]$, and $C[I]$ be three type-1 neutrosophic sets. Then:*

1. $A[I] - (B[I] \cap C[I]) = (A[I] - B[I]) \cup (A[I] - C[I])$.
2. $A[I] - (B[I] \cup C[I]) = (A[I] - B[I]) \cap (A[I] - C[I])$.
3. $A[I] \cap (B[I] - C[I]) = (A[I]\, capB[I]) - (A[I] \cap C[I])$.
4. $A[I] \cup (B[I] - C[I]) = (A[I] \cup B[I]) - (A[I][I])$.

Proof. 1. Presume: $x \in (A[I] - (B[I] \cap C[I]))$ "Neu-hypo"
$\Longleftrightarrow x \in A[I] \wedge (x \notin (B[I] \cap C[I]))$ "NP$_1$"$, from"Neu-hypo$"
$\Longleftrightarrow$x$\in A[I] \wedge (x \in (B[I] \cap C[I])^c)$ "NP$_2$"$, from"NP_1$
$\Longleftrightarrow$x$\in A[I] \wedge (x \in (B^c[I] \cup C^c[I]))$ "NP$_3$"$, from"NP_2$"
$\Longleftrightarrow$x$\in A[I] \wedge (x \in B^c[I] \vee x \in C^c[I])$ "NP$_4$"$, from"NP-3$"
$\Longleftrightarrow$(x$\in A[I] \wedge x \in B^c[I]) \vee (x \in A[I] \wedge x \in C^c[I])$ "NP$_5$"$, from"NP_4$"
$\Longleftrightarrow$(x$\in A[I] \wedge x \notin B[I]) \vee (x \in \wedge x \notin C[I])A[I]$ "NP$_6$"$, from"NP_5$"
$\Longleftrightarrow$(A[I]-B[I])$\cup(A[I] - C[I])$ "NP$_7$"$, from"NP_6$"
$\therefore A[I] - (B[I] \cap C[I]) = (A[I] - B[I]) \cup (A[I] - C[I])$ Conclusion from "Neu-hypo" into " "NP$_7$".

2. By the same arguments.

3. Suppose that: $x \in ((A[I] \cap B[I]) - (A[I] \cap C[I]))$ "Neu-hypo".
$\Longleftrightarrow x \in (A[I] \cap B[I]) \wedge x \notin (A[I] \cap C[I])$ "NP$_1$"$from"Neu-hypo$".
$\Longleftrightarrow$x$\in (A[I] \cap B[I]) \wedge x \in ((A[I] \cap C[I]))^c$ "NP$_2$"$from"NP_1$".
$\Longleftrightarrow$x$\in (A[I] \cap B[I]) \wedge x \in (A^c[I] \cup C^c[I])$ "NP$_3$"$from"NP_2$".
$\Longleftrightarrow$(x$\in A[I] \wedge x \in B[I]) \wedge (x \in A^c[I] \vee x \in C^c[I])$ "NP$_4$"$from"NP_3$".
$\Longleftrightarrow$(x$\in A[I] \wedge x \in B[I]) \wedge (x \notin A[I] \vee x \notin C[I])$ "NP$_5$"$from"NP_4$".
$\Longleftrightarrow$(x$\in A[I] \wedge (x \notin A[I] \wedge x \notin C[I]))$ "NP$_6$"$from"NP_5$".
$\Longleftrightarrow$(x$\in A[I] \wedge x \notin A[I]) \vee (x \in A[I] \wedge x \notin C[I]) \wedge x \in B[I]$ "NP$_7$"$from"NP_6$".
$\Longleftrightarrow$(F$_N \vee (x \in A[I] \wedge x \notin C[I])) \wedge x \in B[I]$ "NP$_8$"$from"NP_7$".
$\Longleftrightarrow$((x$\in A[I] \wedge x \notin C[I])) \wedge x \in B[I]$ "NP$_9$"$from"NP_8$".
$\Longleftrightarrow$x$\in A[I] \wedge (x \notin C[I] \wedge x \in B[I])$ "NP$_{10}$"$from"NP_9$".
$\Longleftrightarrow$x$\in A[I] \wedge (x \in B[I] \wedge x \notin C[I])$ "NP$_{11}$"$from"NP_{10}$".
$\Longleftrightarrow$x $\in A[I] \wedge (x \in (B[I] - C[I]))$ "NP$_{12}$"$from"NP_{11}$".
$\Longleftrightarrow$x$\in (A[I] \cap (B[I] - C[I]))$ "NP$_{13}$"$from"NP_{12}$".
$\therefore A[I] \cap (B[I] - C[I]) = (A[I] \cap B[I]) - (A[I] \cap C[I])$ Conclusion from "Neu-hypo" into "NP$_{13}$".

4. It can be proof by similar method.

□

Theorem 3.22. [2,3] *Let $U[I]$ be a type-1 neutrosophic set, $A[I]$ and $B[I]$ be two type-1 neutrosophic subsets of $U[I]$ such that $A[I] \subset B[I] \subset U[I]$. Then:*

1. $B[I] - A[I] = B[I] \cap (U[I] - A[I])$*, and*
2. $U[I] - B[I] \subseteq (U[I] - A[I]$.

Proof. 1. Assume that $x \in (B[I] - A[I])$, "Neu-hypo".
$\Longrightarrow (x \in B[I]) \wedge (x \notin A[I])$, "$\text{NP}_1, from" Neu - hypo"$.
$\Longrightarrow$(x $\in B[I]$) , "$\text{NP}_2", fromNP_1"$.
$\Longrightarrow$(x $\notin A[I]$) , "$\text{NP}_3", from" NP_2"$.
$\Longrightarrow$x $\in (U[I] - A[I])$, "$\text{NP}_4", from" NP_3."$
$\Longrightarrow$(x $\in B[I]) \wedge x \in (U[I] - A[I])$, "NP $_5", from" NP"_2" and" NP"{}_"4"$.
$\Longrightarrow$x $\in (B[I] \cap (U[I] - A[I]))$, "$\text{NP}_6", from" NP_5"$.
$\Longrightarrow$B[I]-A[I] $\subset B[I] \cap (U[I] - A[I])$, " $\text{NC}_1", from" neu - hypoand" NP_6"$.
Conversely, Suppose that
x $\in (B[I] \cap (U[I] - A[I]))$, "Neu-hypo".
$\Longrightarrow (x \in B[I]) \wedge (x \in (U[I] - A[I]))$, "$\text{NP}_1", from" Neu - hypo"$
$\Longrightarrow$x $\in B[I]$, "$\text{NP}_2", from" NP_1"$.
$\Longrightarrow$x $\in (U[I] - A[I])$, "$\text{NP}_3", from" NP_1"$.
$\Longrightarrow$x $\in U[I] \wedge x \notin A[I]$, "$\text{NP}_4", from" NP_3"$.
$\Longrightarrow$x$\notin A[I]$, "$\text{NP}_5", from" NP_4"$.
$\Longrightarrow$x $\in B[I] \wedge x \notin A[I]$, "$\text{NP}_6", from" NP_2 and" NP_5"$.
$\Longrightarrow$x $\in (B[I] - A[I])$, "$\text{NP}_7", from" NP_6"$.
$\Longrightarrow$(B[I]$\cap (U[I] - A[I])) \subset (B[I] - A[I])$, "$\text{NC}_2", from" neu - hypoand" NP_7"$.
$\therefore B[I] - A[I] = B[I] \cap (U[I] - A[I])$, From "$\text{NC}_1" and" NC_2"$. □

Theorem 3.23. [2,3] *Let $A[I]$ and $B[I]$ be two type-1 neutrosophic sets. Consider $P(A[I])$ and $P(B[I])$ are type-1 neutrosophic power sets. Then:*

1. $2^{A[I]} \cap 2^{B[I]} = 2^{(A[I] \cap [I])}$.
2. $2^{A[I]} \cup 2^{B[I]} \subset 2^{(A[I] \subset B[I])}$*, for any A and B are classical sets. Furthermore, $P(A[I])$ or $2^{A[I]}$ represents the exact notation of the typ-1 neutrosophic power set.*

Proof. 1. Assume that $E[I] \in (2^{A[I]} \cap 2^{B[I]})$.
$\Longleftrightarrow (E[I] \in 2^{A[I]}) \wedge (E[I] \in 2^{B[I]})$.
$\Longleftrightarrow (E[I] A[I]) \wedge (E[I] \subseteq B[I])$.
$\Longleftrightarrow E[I] \subseteq (A[I] \cap B[I])$.
$\Longleftrightarrow E[I] \in 2^{(A[I] \cap B[I])}$.
Therefore, $2^{A[I]} \cap 2^{B[I]} = 2^{(A[I] \cap B[I])}$.

2. Suppose that $E[I] \in (2^{A[I]} \cup 2^{B[I]}$.
$\Longrightarrow (E[I] \in 2^{A[I]}) \vee (E[I] \in 2^{B[I]})$.
$\Longrightarrow (E[I] \subseteq A[I]) \vee (E[I] \subseteq B[I])$.
$\Longrightarrow E[I] \subseteq (A[I] \cup B[I])$.
$\Longrightarrow E[I] \in 2^{(A[I] \cup B[I])}$.
$\Longrightarrow 2^{A[I]} \cup 2^{B[I]} \subset 2^{(A[I] \cap B[I])}$.

□

Example 3.24. Let $A = \{a, b\}$ and $B = \{c\}$ be two classical sets. then the type-1 neutrosophic sets are given by: $A[I] = \{a + aI, a + bI, b + aI, b + bI\}$. and $B[I] = \{c + cI\}$, we see that $2^{A[I]}$ consist of:

- Type-1 neutrosophic-set of order-0: $\phi_0[I] = \{\}$.
- Type-1 neutrosophic-sets of order-1: $A_1[I] = \{a + aI\}, A_2[I] = \{a + bI\}, A_3[I] = \{b + aI\}$, and $A_4[I] = b + bI$.

- Type-1 neutrosophic-sets of order-2:
  $A_5[I] = \ a + aI, a + bI\}$,$A_6[I] = \{a+aI, b+aI\}$, $A_7[I] = \{a+aI, b+bI\}$, $A[I]_8 = \{a+bI, b+aI\}$, $A_9[I] = \{a + bI, b + bI\}$, and $A_{10}[I] = \{b + aI, b + bI\}$.
- Type-1 neutrosophic-sets of order-3:
  $A_{11}[I] = \{a+aI, a+bI, b+aI\}$, $A_{12}[I] = \{a+aI, a+bI, b+bI\}$, $A_{13}[I] = \{a+bI, b+aI, b+bI\}$, and $A_{14}[I] = \{a + bI, b + aI, b + bI\}$.
- Type-1 neutrosophic-set of order-4:
  $A_{15}[I] = \{a + aI, a + bI, b + aI, b + bI\}$. And $2^{B[I]}$ consists of:
- Type-1 neutrosophic-set of order-0:
  $\phi_0[I] = \{\}$.
- Type-1 neutrosophic-set of order-1:
  $B_1[I] = \{c + cI\}$. We note that, $2^{A[I]}\cup 2^{B[I]}$ has A neutrosophic order is equal to 18, and the neutrosophic order of $2^{(A[I]B[I])}$ is equal to 32. Therefore, $2^{A[I]} \cup 2^{B[I]} \neq 2^{(A[I]\cup B[I])}$.

4 Type-1 Neutrosophic Cartesian Product Sets and Some Type-1 Neutrosophic Relations

As intuitive thinking tells us, we can build a new object from two objects. For this reason, we are discussing in this section the concepts of type-1 neutrosophic ordered pairs, type-1 neutrosophic Cartesian products, finite type-1 neutrosophic Cartesian products, and their properties. In addition, we present new information on type-1 neutrosophic relations.

Definition 4.1. [3] Let $A[I]$ and $B[I]$ be two type-1 neutrosophic sets. Define the type-1 neutrosophic order pair for two neutrosophic elements $a \in A[I], b \in B[I]$ as following:

$$<a, b> = <a_1 + a_2I, b_1 + b_2I> = \{\{a_1, a_2I\}, \{\{a_1, a_2I\}, \{b_1, b_2\}\}\}.$$

for some $a_1, a_2 \in A, b_1, b_2 \in B$ and an indeterminacy I. The neutrosophic order pair $<a, b>$ is a type-1 neutrosophic set that are related to $A[I]$ and $B[I]$ respectively.

Definition 4.2. [3] Let $A[I]$ and $B[I]$ be two type-1 neutrosophic sets. The type-1 neutrosophic Cartesian product denoted by; $A[I] \times B[I]$, and defined by:

$$\begin{aligned} A[I] \times B[I] &= \{<a, b> : a \in A[I] \ \wedge \ b \in B[I]\} \\ &= \{<a, b> : \exists\, a_1, a_2 \in A \wedge \exists\, b_1, b_2 \in B, a = a_1 + a_2I, b = b_1 + b_2I\}. \end{aligned}$$

Where I is an indeterminacy.

Theorem 4.3. [3] *Let $A[I]$ and $B[I]$ be two type-1 neutrosophic sets, and let $<a, b>$ and $<a', b'>$ be two type-1 neutrosophic order pairs belongs to $AI] \times B[I]$. Then:*

$$<a, b> = <a', b'>\Longleftrightarrow a = a' \wedge b = b'\Longleftrightarrow(a_1 = a'_1 \wedge a_2 = a'_2) \wedge (b_1 = b'_1 \wedge b_2 = b'_2).$$

Proof. Suppose that: $\big(a_1 = a'_1 \wedge a_2 = a'_2) \wedge (b_1 = b'_1 \wedge b_2 = b'_2\big)\Longrightarrow a = a' \wedge b = b'$, Now,

$$\begin{aligned} <a, b> &= <a_1 + a_2I, b_1 + b_2I>. \\ &= \{\{a_1, a_2I\}, \{\{a_1, a_2I\}, \{b_1, b_2I\}\}\}. \\ &= \{\{a'_1, a'_2I\}, \{\{a'_1, a'_2I\}, \{b'_1, b'_2I\}\}\}. \\ &= <a', b'>. \end{aligned}$$

Conversely, assume that.$<a, b> = <a', b'>$
$\Longrightarrow <a_1 + a_2I, b_1 + b_2I> = <a'_1 + a_2I, b'_1 + b'_2I>$
$\Longrightarrow \{\{a_1, a_2I\}, \{\{a_1, a_2I, \{a_1, a_2I\}\}\} = \{\{a'_1, a'_2I\}, \{\{a'_1, a'_2I\}, \{b'_1, b'_2I\}\}\}$.
$\because \{a_1, a_2I \in \{\{a_1, a_2I\}, \{\{a_1, a_2I\}, \{b_1, b_2I\}\}\}$
$\therefore \{a_1, a_2I\} \in \{\{a'_1, a'_2I\}, \{\{a'_1, a'_2I\}, b'_1, b'_2I\}\}\}$.

$\therefore \{a_1, a_2I\} = \{a_1', a_2'I\} \vee \{a_1, a_2I\} = \{\{a_1', a_2'I\}, \{b_1', b_2'I\}\}$.
Case 1. If $\{a_1, a_2I\} = \{a_1', a_2'I\} \Longrightarrow \{\{a_1, a_2I\}, \{b_1, b_2I\}\} = \{\{a_1', a_2'I\}, \{b_1', b_2'I\}\}$.
$\Longrightarrow \{a_1, a_2I\} = \{a_1', a_2'I\} \wedge \{b_1, b_2I\} = \{b_1', b_2I\}$
$\Longrightarrow (a_1 = a_1' \wedge a_2 = a_2') \wedge (b_1 = b_1' \wedge b_2 = b_2')$
$\Longrightarrow a = a' \wedge b = b'$.
Case 2. If $\{a_1, a_2I\} = \{\{a_1', a_2'I\}, b_1', b_2'I\}$
$\Longrightarrow \{\{a_1, a_2I\}, \{b_1, b_2I\}\} = \{b_1', b_2'I\}$.
$\Longrightarrow \{a_1, a_2I\} = \{a_1', a_2'I\} \wedge \{b_1, b_2I\} = \{a_1', a_2'I\}$
$\because \{a_1, a_2I\} = \{a_1', a_2'I\}$.
$\Longrightarrow a_1 = a_1' \wedge a_2 = a_2)$.
$\because \{a_1, a_2I\} = \{a_1', a_2'I\}$.
$\Longrightarrow (b_1 = b_1' \wedge b_2 = b_2')$.
$\Longrightarrow (b_1 = a_1 \wedge b_2 = a_2)$.
$\Longrightarrow (b_1 = b_1' \wedge b_2 = b_2') \wedge (b_1 = b_1 \wedge b_2 = b_2)$.
$\Longrightarrow a = a' \wedge b = b'$. □

Example 4.4. [3] Let $A = \{a, b\}$ and $B = \{d\}$ be two classical sets. Then the neutrosophic set of type-1 is given by: $A[I] = \{a+aI, a+bI, b+aI, b+bI\}$, and $B[I] = \{d+dI\}$,and the neutrosophic Cartesian product is given by: $A[I] \times B[I] = \{<a + aI, d + dI>, <a + bI, d + dI>, <b + aI, d + dI>, <b + bI, d + dI>\}$. And

$$B[I] \times A[I] = \{<d + dI, a + aI>, <d + dI, a + bI>, <d + dI, b + aI>, <d + dI, b + bI>\}.$$

Observation. It is clear that from the definition and example, the following properties:

1. $A[I] \times B[I] \neq B[I] \times A[I]$.
2. $(A[I] \times B[I]) = (A[I]) \times (B[I])$.
3. $A[I] \times \phi[I] = \phi[I]$.

Theorem 4.5. [3] *Let$A[I]$,$B[I]$, and $C[I]$ be three type-1 neutrosophic sets.Then:*

1. $A[I] \times (B[I] \cap C[I]) = (A[I] \times B[I]) \cap (A[I] \times C[I])$.
2. $A[I] \times (B[I] \cup C[I]) = (A[I] \times B[I]) \cup (A[I] \times C[I])$.*And*
3. $A[I] \times (B[I] - C[I]) = (A[I]B[I]) - (A[I] \times C[I])$.

Proof. 1. Suppose that. $<a, b> \in \big(A[I] \times (B[I] \cap C[I])\big)$,"neu-hypo"
$\Longleftrightarrow \big(a \in A[I]\big) \wedge \big(b \in \big(B[I] \cap C[I]\big)\big)$, "NP$_1$" $from"neu - hypo"$
$\Longleftrightarrow \big(a\in A[I]\big) \wedge \big(b \in B[I] \wedge b \in C[I]\big)$, "NP$_2$" $from"NP_1"$
$\Longleftrightarrow \big(a\in A[I] \wedge b \in A[I]\big) \wedge \big(a \in A[I] \wedge b \in C[I]\big)$, "NP$_3$" $from"NP_2"$
$\Longleftrightarrow \big(a \in A[I] \wedge a \in A[I]\big) \wedge \big(b \in B[I]\big) \wedge b \in C[I]\big)$, "NP$_4$" $from"NP_3"$
$\Longleftrightarrow \big(a \in A[I] \wedge (b \in B[I]\big) \wedge (b \in A[I] \wedge b \in C[I]\big)$., "NP$_5$" $from"NP_4"$
$\Longleftrightarrow \langle a,b\rangle \in A[I] \times B[I] \wedge <a, b> \in A[I] \times C[I]$., "NP$_6$" $from"NP_5"$
$\Longleftrightarrow \langle a,b\rangle \in \big(A[I] \times B[I] \cap A[I] \times C[I]\big)$.
$\therefore$A[I]$\times(B[I] \cap C[I]) = (A[I] \times B[I]) \cap (A[I] \times C[I])$.

2. and 3. By the same argument in part 1.

□

Definition 4.6. [4] Let $C[I] = A[I] \times B[I]$ be a neutrosophic-set obtained from the Cartesian product of $A[I] \times B[I]$. A neutrosophic binary relation $\mathcal{R}$ is a neutrosophic-subset of $C[I]$.

Remark 4.7. 1. If $<a, b> \in \mathcal{R} \Longleftrightarrow a\mathcal{R}b$ or $\mathcal{R}(a) = b$, and we say that b is neutrosophic related to a or b is in a neutrosophic relation with a.

2. If $A[I] = B[I]$, then we say that $\mathcal{R}$ is a neutrosophic binary relation on $A[I]$.

Definition 4.8. [4] Let $\mathcal{R}$ be a neutrosophic relation from neutrosophic set $A[I]$ into a neutrosophic set B$[I]$. Define a neutrosophic domain of $\mathcal{R}$, written $NeuDom(\mathcal{R})$ as follows:

$$NeuDom(\mathcal{R}) = \{a \in A[I] : \exists\, b \in B[I] \ni \mathcal{R}(a) = b\}.$$

Remark 4.9. If $a \in NeuDom(\mathcal{R}) \Longleftrightarrow a \in A[I] \Longleftrightarrow .\exists\, b \in B[I] \ni \mathcal{R}(a) = b$.
$\Longleftrightarrow \exists\, a_1, a_2 \in A \wedge \exists, b_1, b_2 \in B \ni \mathcal{R}(a_1 + a_2 I) = \mathcal{R}(a_1) + \mathcal{R}(a_2 I) = \mathcal{R}(a_1) + \mathcal{R}(a_2)\mathcal{R}(I) = b_1 + b_2 I$.
For any indeterminacy I and $\mathcal{R}(I) = I$.

Definition 4.10. [4] Let $\mathcal{R}$ be a neutrosophic relation from neutrosophic set $A[I]$ into a neutrosophic set B$[I]$. Define a neutrosophic co-domain of $\mathcal{R}$, written $NeuCdom(\mathcal{R})$ as follows:

$$NeuCdom(\mathcal{R}) = \{\forall\, b \in B[I] : \exists\, a \in A[I] \ \ni < a, b > \in \mathcal{R} \subseteq B[I]\}.$$

Remark 4.11. If $b \in NeuCdom(\mathcal{R}) \Longleftrightarrow b \in B[I] \Longleftrightarrow \exists\, a \in A[I] \ni \mathcal{R}(a) = b$.
$\Longleftrightarrow \exists\, b_1, b_2 \in B \wedge \exists, a_1, a_2 \in A \ni \mathcal{R}(a_1 + a_2 I) = \mathcal{R}(a_1) + \mathcal{R}(a_2 I)$
$= \mathcal{R}(a_1) + \mathcal{R}(a_2)\mathcal{R}(I) = b_1 + b_2 I$. For any indeterminacy I and $\mathcal{R}(I) = I$.

Definition 4.12. [4] Let $A[I]$ and $B[I]$ be two neutrosophic sets. Consider $<a, b>$ and $<a', b'>$ be two neutrosophic order pairs belongs to $\mathcal{R} \subset A[I] \times B[I]$. Define a neutrosophic order relation as follows:

1. $<a, b> \, < \, <a', b> \Longleftrightarrow a < a' \wedge b < b'$

$$\Longleftrightarrow (a_1 < a_1' \wedge a_2 < a_2') \wedge (b_1 < b_1' \wedge b_2 < b_2)$$
$$\forall <a, b>, <a', b'> \in \mathcal{R}.$$

We say that the neutrosophic order pair $<a, b>$ less than the neutrosophic order pair $<a', b'>$. The dualism of $<$ is given by $>$.

2. $<a, b> \, \leq \, <a', b'> \Longleftrightarrow (a \leq a' \wedge b \leq b')$.

$$\Longleftrightarrow \big(a_1 < a_1' \vee a_1 = a_1'\big) \wedge \big(a_2 < a_2' \vee a_2 = a_2'\big)$$
$$\wedge \big(b_1 < b_1 \vee b_1 = b_1'\big) \wedge \big(b_2 < b_2' \vee b_2 = b_2'\big),$$
$$\forall <a, b>, <a', b'> \in \mathcal{R}.$$

The dualism of $\leq$ is given by $\geq$.

Example 4.13. [4] Let $A = \{1, 5\}$ and $B = \{2, 4, 6\}$ be two classical sets. Consider the classical set of relation $\mathcal{R}$ from A into B is given by:

$$\begin{aligned} \mathcal{R} &= \{<x, y> \in A \times B : x < y\} \\ &= \{<1, 2>, <1, 4>, <1, 6>, <5, 6>\}. \end{aligned} \quad .$$

Now, consider the neutrosophic set $A[I]$ and $B[I]$ are given by:
$A[I] = \{(1 + 1I, 1 + 5I, 5 + 1I, 5 + 5I)\}$, and
$B[I] = \{2 + 2I, 2 + 4I, 2 + 6I, 4 + 2I, 4 + 4I, 4 + 6I, 6 + 2I, 6 + 4I, 6 + 6I, \}$.
The neutrosophic cartesian product of $A[I] \times B[I]$ is given by:

$$A[I] \times B[I] = \left\{ \begin{array}{c} <1 + I, 2 + 2I>, \ldots <1 + I, 4 + 2I>, \ldots <1 + I, 6 + 6I>, \\ <1 + 5I, 2 + 2I>, \ldots <1 + 5I, 4 + 2I>, \ldots <1 + 5I, 6 + 6I>, \\ (<5 + 1I, 2 + 2I>, \ldots <5 + 1I, 4 + 2I>, \ldots <5 + 1I, 6 + 6I>, \\ <5 + 5I, 2 + 2I>, \ldots <5 + 5I, 4 + 2I>, \ldots <5 + 5I, 6 + 6I> \end{array} \right\}.$$

Define a neutrosophic relation $\mathcal{R}_1$ as: $\mathcal{R}_1 = \{<a, b> \in A[I] \times B[I] : a < b\}$.
$\mathcal{R}_1 = <a, b> \in A[I] \times B[I] : (a_1 + a_2 I) < (b_1 +_2 I)\}$, for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$.
$R_1 = <a, b> \in A[I] \times B[I : ((a_1 < b_1) \wedge (a_2 < b_2))\}$, for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

$$R_1 = \left\{ \begin{array}{c} <1 + I, 2 + 2I>, <1 + I, 2 + 4I>, <1 + I, 2 + 6I>, \\ <1 + I, 6 + 6I>, <1 + 5I, 2 + 6I>, <1 + 5I, 2 + 6I>, \\ <1 + 5I, 6 + 6I>, (<5 + I, 6 + 2I>, <5 + I, 6 + 4I>, \\ <5 + I, 6 + 6I>, <5 + 5I, 6 + 6I> \end{array} \right\}.$$

Then the neutrosophic domain and co-domain are represented by:
$NeuDom(R_1) = \{1 + I, 1 + 5I, 5 + 1I, 5 + 5I\}$, and
$NeuCdom(R_1) = \{2 + 2I, 2 + 4I, 2 + 6I, \ldots, 6 + 6I, 2 + 6I, 6 + 2I, 6 + 4I, 6 + 6I\}$.

Definition 4.14. [4] Let $A[I]$ be a neutrosophic-set. Consider $x, y, z \in A[I]$. Define $\preccurlyeq$ is a partially ordered type-1 neutrosophic set as follows:

1. (Neutrosophic Reflexive Axiom):

$$(\forall x \in A[I]), x \preccurlyeq x \Longleftrightarrow ((x_1 \preccurlyeq x_1) \wedge (x_2 \preccurlyeq x_2.)$$

2. (Neutrosophic Antisymmetric Axiom):

$$(\forall x, y \in A[I]), ((x \preccurlyeq y) \wedge (y \preccurlyeq x) \Longrightarrow x = y)$$

$$\Longleftrightarrow ((x_1 \preccurlyeq y_1) \wedge (x_2 \preccurlyeq y_2)) \wedge ((y_1 \preccurlyeq x_1) \wedge (y_2 \preccurlyeq x_2))$$
$$\Longrightarrow ((x_1 = y_1) \wedge (x_2 = y_2)), \forall x, y \in A[I].$$

3. (Neutrosophic Transitive Axiom):

$$(\forall x, y, andz \in A[I]), ((x \preccurlyeq y) \wedge (y \preccurlyeq z) \Longrightarrow x \preccurlyeq z)$$

$$\Longleftrightarrow ((x_1 \preccurlyeq y_1) \wedge (x_2 \preccurlyeq y_2)) \wedge ((y_1 \wedge z_1) \wedge (y_2 \preccurlyeq z_2))$$
$$\Longrightarrow ((x_1 \preccurlyeq z_1) \wedge (x_2 \preccurlyeq z_2)).$$

If the neutrosophic relation $\preccurlyeq$ is a partially neutrosophic order on $A[I]$. If the neutrosophic relation is given by $\prec$, then we said that $\prec$ is strictly neutrosophic orders of $A[I]$.

Remark 4.15. $x \preccurlyeq y$ is reading x precedes y or y dominates x.

Theorem 4.16. [4] *Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers. Then the neutrosophic-natural numbers is given by:*

$$\mathbb{N} = \left\{ \begin{array}{c} 0, I, 2I, 3I, 4I, 5I, \dots \\ 1, 1+I, 1+2I, 1+3I, \dots \\ 2, 2+I, 2+2I, 2+3I, \dots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \dots \end{array} \right\}.$$

For any $x, y \in \mathbb{N}[I]$, we define the neutrosophic order relation:

$$x \preccurlyeq y \Longleftrightarrow x \leq y \text{ "x less than or equal to y"}$$

.

$$\Longleftrightarrow \exists\, x_1, x_2, y_1, y_2 \in \mathbb{N} \ni (x_1 \leq y_1) \wedge (x_2 \leq y_2), \forall x_1, x_2, y_1, y_2 \in \mathbb{N}.$$

Then The neutrosophic relation less than or equal $\leq$ is a neutrosophic partial order relation on $\mathbb{N}[I]$.

Proof. Proof.

PN$_1$. Since, $(x_1 \leq x_1) \wedge (x_2 \leq x_2), \forall x_1, x_2 \in \mathbb{N} \Longrightarrow (x_1 \leq x_1) \wedge (x_2 I \leq x_2 I) \Longrightarrow (x_1 + x_2 I \leq x_1 + x_2 I)$ $\Longrightarrow x \leq x, \forall x \in \mathbb{N}[I]$, thus $\leq$ is a neutrosophic reflexive relation.

PN$_2$. Suppose that $(\forall x, y \in \mathbb{N}[I])$,we have,

$$((x \leq y) \wedge (y \leq x)) \leq \exists\, x_1, x_2, y_1, y_2 \in \mathbb{N},$$

and I is an indeterminacy with:

$$x = x_1 + x_2 I, y = y_1 + y_2 I \ni ((x_1 \leq y_1) \wedge (x_2 \leq y_2)) \wedge ((y_1 \leq x_1) \wedge (y_2 \leq x_2)).$$

Since,

$$((x_1 \leq y_1) \wedge (y_1 \leq x_1)) \Longrightarrow (x_1 = y_1)$$

Also, since:

$$((x_2 \leq y_2) \wedge (y_2 \leq x_2)) \Longrightarrow (x_2 = y_2)$$

we deduced that,

$$(x_1 = y_1) \wedge (x_2 = y_2)$$
$$\Longrightarrow (x_1 = y_1) \wedge (x_2 I = y_2 I)$$

$$\Longrightarrow x_1 + x_2 I = y_1 + y_2 I \Longrightarrow x = y.$$

Hence $\leq$ is a neutrosophic antisymmetric relation.

PN$_3$. Suppose that $(\forall x, y, z \in \mathbb{N}[I])$,we have:

$$((x \leq y) \wedge (y \leq z)) \Longrightarrow \exists\, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{N}.$$

And I is an indeterminacy with
$x = x_1 + x_2 I, y = y_1 + y_2 I$, and $z = z_1 + z_2 I$, such that

$$((x_1 \leq y_1) \wedge (x_2 \leq y_2)) \wedge ((y_1 \leq z_1) \wedge (y_2 \leq z_2)).$$
$$\because \quad ((x_1 \leq y_1) \wedge (y_1 \leq z_1)) \Longrightarrow (x_1 \leq z_1).$$
$$\because \quad ((x_2 \leq y_2) \wedge (y_2 \leq z_2)) \Longrightarrow (x_2 \leq z_2).$$
$$\therefore \quad (x_1 \leq z_1) \wedge (x_2 \leq z_2),$$
$$\therefore \quad (x_1 \leq z_1) \wedge (x_2 I \leq z_2 I)$$

For any indeterminacy I.
$\Longrightarrow x_1 + x_2 I \leq z_1 + z_2 I \Longrightarrow x \leq z$. Hence $\leq$ is a neutrosophic transitive relation. Thus $\mathbb{N}[I]$ is a partially order set under neutrosophic relation less than or equal $\leq$.

□

5 Type-1 Neutrosophic Functions and Invertible Neutrosophic Functions

This section includes neutrosophic functions on neutrosophic sets of type-1, along with their neutrosophic graphs, neutrosophic restrictions, extensions, identities, and constant functions. In addition, the concepts of one-to-one, onto, and composition of functions are addressed through theorems and examples.In addition, the properties of neutrosophic functions on some neutrosophic subsets of neutrosophic domain with the operator's union, intersection, difference, and on generalization of union and intersection.

5.1 Type-1 Neutrosophic Functions and Their Properties

Definition 5.1. [5] Let $X[I]$ and $Y[I]$ be two type-1 neutrosophic sets generated by X and Y. Assume that $f_n(I) = I$ and $f_n(xI) = f_c(x)f_n(I)$. Intuitively, we can define the neutrosophic function $f_n : X[I] \longrightarrow Y[I]$ of type-1 generated by a classical function $f_c : X \longrightarrow Y$ as follows:

$$f_n(x) = f_c(x_1) + f_n(x_2 I) = f_c(x_1) + f_c(x_2) f_n(I).$$

$\forall x \in X[I], x_1, x_2 X$, and an indeterminacy I. In other word, a correspondence from neutrosophic set $X[I]$ to a neutrosophic set $Y[I]$ is a quadruple:

$$f_n = \big(X[I], Y[I], f_n(I), \Gamma_n[I]\big).$$

Where $X[I]$ is a neutrosophic domain of $f_n, Y[I]$ is the neutrosophic co-domain of $f_n, f_n(I)$ is a neutrosophic image of indeterminacy I, and $_n[I]$ is a neutrosophic subset of $X[I] \times Y[I]$, and it's called the neutrosophic graph of f_n. The neutrosophic set:

$$NeuDom(f_n) = \{x \in XX[I] : \exists\, y \in Y[I] \ni f_n(x) = y \Longleftrightarrow (x, y) \in \Gamma_n[I] \subset X[I]\},$$

is the neutrosophic domain of f_n, and the neutrosophic set:

$$NeuCod(f_n^) = \{y \in Y[I] : \exists x \in X[I] \ni f_n^(x) = y \Longleftrightarrow (x, y)_{\in}\Gamma_n[I] \subseteq [I]\},$$

is the neutrosophic range (or neutrosophic co-domain) of f_n.

Example 5.2. [5] Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$ be two classical sets with a classical function $f_c : X \longrightarrow Y$ such that $f_c(a) = 1$, and $f_c(b) = 2$, the type-1 neutrosophic set, which are generated by X, and Y are given by:

$$X[I] = \{a + aI, a + bI, b + aI, b + bI)\}, and$$

$$Y[I] = \{1 + 1I, 1 + 2I, 1 + 3I, 2 + 1I, 2 + 2I, 2 + 3I, 3 + 1I, 3 + 2I, 3 + 3I\}.$$

The neutrosophic function f_n of type-1 is given by:

$$f_n(a + aI) = f_c(a) + f_n(aI) = f_c(a) + f_c(a)f_n^(I) = 1 + 1I.$$
$$f_n(a + bI) = f_c(a) + f_n(bI) = f_c(a) + f_c(b)f_n(I) = 1 + 2I.$$
$$f_n(b + aI) = f_c(b) + f_n(aI) = f_c(b) + f_c(a)f_n(I) = 2 + 1I.$$
$$f_n(b + bI) = f_c(b) + f_n(bI) = f_c(b) + f_c(b)f_n(I) = 2 + 2I.$$

The neutrosophic graph of a neutrosophic function f_n is shown in Figure 1.

Figure 1: The graph $\Gamma_n[I]$ of f_n

.

Definition 5.3. [5] Let $f_n : X[I] \longrightarrow Y[I]$ be the neutrosophic function of type-1 generated by a classical function $f_c : X \longrightarrow Y$ and $A[I][I]$; the function f_n considered only on $A[I]$ is called the type-1 neutrosophic restriction of f_n to $A[I]$,
written $f_n|_{A[I]}$, if:

$$f_n|_{A[I]} = f_n \cap (A[I] \times Y[I]).$$

.

Definition 5.4. [5] Consider $A[I] \subset X[I]$ with $g_n : A[I] \longrightarrow Y[I]$ is a given neutrosophic function, then:
$f_n : X[I] \longrightarrow Y[I]$ is called the type-1 neutrosophic extension function of type-1 of g_n over $X[I]$, if:

$$f_n|_{A[I]} = g_n, \forall x \in A[I].$$

Definition 5.5. [5] A neutrosophic function of type-1 as, $I_{dn} : X[I] \longrightarrow X[I]$ is called a type-1 neutrosophic identity function, if $I_{id}(x) = x, \forall x \in X[I], x_1, x_2 \in X$, and an indeterminacy I.

Definition 5.6. [5] Let $f_n : X[I] \longrightarrow Y[I]$ be the neutrosophic function of type generated by a classical function $f_c : X \longrightarrow Y$, then f_n is called a neutrosophic constant function, if there exists a neutrosophic element

$$y_0 \in Y[I] \ni f_n(x) = y_0, \forall x \in X[I].$$

Theorem 5.7. [5] *Let $f_c : X \longrightarrow Y$ be a one-to-one (injective) function, then $f_n : X[I] \longrightarrow Y[I]$ be a one-to-one type-1 neutrosophic function.*

Proof. Suppose that $f_c : X \longrightarrow Y$ is a one-to-one function, and consider $x, y \in X[I]$ such that $f_n(x) = f_n(y)$.
$\Longrightarrow f_c(x_1) + f_n(x_2I) = f_c(y_1) + f_n(y_2I)..$
$\Longrightarrow f_c(x_1) + f_c(x_2)f_n(I) = f_c(y_1) + f_c(y_2)f_n(I).$
$\Longrightarrow f_c(x_1) + f_c(x_2)I = f_c(y_1) + f_c(y_2)I.$
$\Longrightarrow (f_c(x_1) = f_c(y_1)) \wedge (f_c(x_2) = f_c(y_2))$, because f_c is a one-to-one.
$\Longrightarrow (\mathrm{x}_1 = y_1) \wedge (x_2 = y_2).$
$\Longrightarrow (x_1 + x_2I) \wedge (y_1 + y_2I).$
$\Longrightarrow x = y$. Hence f_n is a one-to-one type-1 neutrosophic function. □

Theorem 5.8. [5] *Let $f_c : X \longrightarrow Y$ be an onto (surjective) function, then:*
$f_n : X[I] \longrightarrow Y[I]$ is an onto type-1 neutrosophic function.

Proof. Suppose that $f_c : X \longrightarrow Y$ is a onto function, and consider $y \in Y[I]$.
$\Longrightarrow \exists\, y_1, y_2 \in Y$, and indeterminacy I such that $y = y_1 + y_2 I$.
$\Longrightarrow \exists\, x_1, x_2 \in X$, and indeterminacy I such that $f_c(x_1) = y_1, f_c(x_2) = y_2$, and $f_n(I) = I$.
Therefore, $f_n(x) = f_c(x_1) + f_n(x_2 I) = f_c(x_1) + f_c(x_2) f_n(I) = y_1 + y_2 I = y$.
Hence, f_n is an onto type-1 neutrosophic function. □

Theorem 5.9. [5] *Let $f_c : X \longrightarrow Y$ be a bijective (injective surjective) function, then f is a type-1 neutrosophic bijective function.*

By Theorems 5.7 and 5.9.

Theorem 5.10. [5] *Let $I_{dc} : X \longrightarrow X$ be a bijective(injective+surjective) identity function, then $I_{dn} : X[I] \longrightarrow Y[I]$ is a bijective neutrosophic identity function.*

Proof. Let $I_{dc} : X \longrightarrow X$ be a bijective identity function.
Assume that $x, y \in X[I] \ni I_{dn}(x) = I_{dn}(y)$
$\Longleftrightarrow I_{dc}(x_1) + I_{dn}(x_2 I) = I_{dc}(y_1) + I_{dn}(y_2 I)$
$\Longleftrightarrow I_{dc}(x_1) + I_{dc}(x_2) I_{dn}(I) = I_{dc}(y_1) + I_{dc}(y_2) I_{dn}(I)$.
$\Longleftrightarrow x_1 + x_2 I = y_1 + y_2 I \Longrightarrow x = y$. Hence, I_{dn} is a one-to-one neutrosophic identity. □

Definition 5.11. [5] Let f_n,$g_n : X[I] \longrightarrow Y[I]$ be two neutrosophic functions, where $f_c, g_c : X \longrightarrow Y$ be two classical functions, then f_n is neutrosophic equal to g_n , if: $f_n = g_n \Longleftrightarrow f_n(x) = g_n(x), \forall x \in X[I]$.

Definition 5.12. [5] Let $f_n : X[I] \longrightarrow Y[I]$ and $g_n : Y[I] \longrightarrow Z[I]$ be two typ-1 neutrosophic functions, where $f_c : X \longrightarrow Y$ and $g_c : Y \longrightarrow Z$ are two classical functions, the type-1 neutrosophic composite functions of f_n and g_n is defined by:

$$(g_n \circ f_n)(x) = g_n(f_n(x)), \forall x \in X[I].$$

Example 5.13. [5] Let $\mathbb{R}$ be a set of classical real numbers and $\mathbb{R}[I]$ be a set of neutrosophic real numbers of type-1. Consider two classical functions:

$$f_c, g_c : \mathbb{R} \longrightarrow \mathbb{R} \ni f_c(x) = x^2 \text{ and } g_c(x) = x + 1, \forall x \in \mathbb{R}.$$

We can be generate two neutrosophic functions: $f_n, g_n : \mathbb{R}[I] \longrightarrow \mathbb{R}[I]$ induced by f_c and g_c, respectively. Suppose that $x \in \mathbb{R}[I]$,then the neutrosophic composite is given by:

$$\begin{aligned}(g_n \circ f_n)(x) &= g_n(f_n(x)).\\ &= g_n(f_c(x_1) + f_c(x_2)I).\\ &= g_n(x_1^2 + x_2^2 I).\\ &= g_c(x_1^2) + g_c(x_2^2)I.\\ &= (x_1^2 + 1) + (x_2^2 + 1)I.\end{aligned}$$

For instance, the neutrosophic image of the neutrosophic element $2 + 3I$ is given by: $(g_n \circ f_n)(2 + 3I)) = 5 + 10I$. While,

$$\begin{aligned}(f_n \circ g_n)(x) &= f_n(g_n(x)).\\ &= f_n(g_c(x_1) + g_c(x_2)I).\\ &= f_n((x_1 + 1) + (x_2 + 1)I).\\ &= f_c(x_1 + 1)) + f_c(x_2 + 1)I.\\ &= (x_1 + 1)^2 + (x_2 + 1)^2 I.\\ &= x_1^2 + 2x_1 + 1 + (x_2^2 + 2x_2 + 1)I.\end{aligned}$$

So, the neutrosophic image of neutrosophic element $2 + 3I$ is given by:

$$(f_n \circ g_n)(2 + 3I) = (4 + 2.2 + 1) + (9 + 2.3 + 1)I = 9 + 16I.$$

We see that the composition of the neutrosophic function is not commutative. i.e. $(g_n \circ f_n)(x) \neq (f_n \circ g_n)(x)$.

Definition 5.14. [5] Let $f_n : X[I] \longrightarrow Y[I]$ be a the neutrosophic function of type-1 generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. Let $C[I]$ be a neutrosophic subset of $X[I]$ generated by $C \subset X$. Define a neutrosophic direct image of C[I] under f_n , written $f_n(C[I])$, as follows:

$$f_n(C[I]) = \{y \in Y[I] : \exists\, x \in C[I] \ni f_n(x) = y\}.$$

Theorem 5.15. [5] *Let $f_n : X[I] \longrightarrow Y[I]$ be a the type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively, and Let $C[I] \subset X[I]$ and $B[I] \subset X[I]$, if $C[I] = B[I]$, then:*
$f_n(C[I]) = f_n(B_[I])$.

Proof. Suppose that $C[I] = B[I]$,and let $y \in f_n(C_[I])$,then there exists a neutrosophic element $x \in C[I]$ such that $f_n(x) = y$. Since $C[I] = B[I]$, implies that $x \in B[I]$, hence $f_n(x)f_n(B[I])$, therefore $y \in f_n(B[I])$, and consequently, $f_n(C[I]) \subset f_n(B[I])$. By similar method we can prove the second part $f_n(B[I]) \subset f_n(C[I])$, to get the conclusion, $f_n(C[I]) = f_n(B[I])$. The converse of the theorem is not true, by the following example. ☐

Example 5.16. [5] Let $f_n : \mathbb{Z}[I] \longrightarrow \mathbb{R}[I]$ be a type-1 neutrosophic function from the neutrosophic set of integers into the neutrosophic set of real numbers defined by $f_n(x) = x_1^2 + x_2^2 I$. Consider $C[I] = \{-2 - 2I, -2 + 3I, 3 - 2I, 3 + 3I)\}$, and $B[I] = \{2 + 2I, 2 - 3I, -3 + 2I, -3 - 3I)\}$, then:

$$f_n(C[I]) = \{4 + 4I, 4 + 9I, 9 + 4I, 9 + 9I)\}.$$

And

$$f_n(B[I]) = \{4 + 4I, 4 + 9I, 9 + 4I, 9 + 9I)\}.$$

We say that $(f_n(B[I]) = f_n(C[I]))$, but $C[I] \neq B[I]$.

Theorem 5.17. [5] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function of generated by a classical function $f_c : X \longrightarrow Y$, and classical sets X and Y respectively. Let $C[I] \subset X[I]$ and $B[I] \subset X[I]$, then:*

1. *$f_n(C[I] \cup B[I]) = f_n(C[I]) \cup f_n(B[I])$,*
2. *$f_n(C[I] \cap B[I]) \subseteq f_n(C[I]) \cap f_n(B[I])$, and*
3. *$f_n(C[I]) - f_n(B[I]) \subseteq f_n(C[I] - B[I])$.*

Proof. 1. Let $y \in f_n(C[I] \cup B[I])$.
$\Longrightarrow \exists\, x \in (C[I] \cup B[I]) \ni f_n(x) = y$.
$\Longrightarrow \exists\, x \in C[I] \vee \exists\, x \in B[I] \ni f_n(x) = y$.
$\Longrightarrow (\exists\, x \in C[I] \ni f_n(x) = y) \vee (\exists\, x \in B[I] \ni f_n(x) = y)$.
$\Longrightarrow (f_n(x) \in f_n(C[I])) \vee (f_n(x) \in f_n(B[I]))$.
$\Longrightarrow (y \in f_n(C[I])) \vee (y \in f(B[I]))$.
$\Longrightarrow (y \in (f_n(C[I]) \cup f_n(B[I])$.

$$\Longrightarrow f_n(C[I] \cup B[I]) \subseteq f_n(C[I]) \cup f_n(B[I]) \tag{1}$$

Conversely, let $y \in (f_n(C[I]) \cup f_n(B[I]))$.
$\Longrightarrow$y $\in f_n(C[I]) \vee y \in f_n(B[I])$.
$\Longrightarrow (\exists y \in C[I] \ni f_n(x) = y) \vee (\exists\, z \in B[I] \ni f_n(z) = y)$.
$\Longrightarrow x \in (C[I] \cup B[I]) \ni f_n(x) = y) \vee (\exists \in z(C[I] \cup B[I]) \ni f_n(z) = y)$.
$\Longrightarrow y \in f_n(C[I] \cup B[I])$.

$$\Longrightarrow f_n(C[I] \cup B[I]) \subseteq f_n(C[I] \cup B[I]) \tag{2}$$

$\Longrightarrow f_n(C[I] \cup B[I]) = f_n(C[I]) \cup f_n(B[I])$.

2. Let $y \in f_n(C[I] \cap B[I])$.
$\Longrightarrow \exists\, x \in (C[I] \cap B[I]) \ni f_n(x) = y$.
$\Longrightarrow \exists\, x \in C[I] \wedge \exists\, x \in B[I] \ni f_n(x) = y$.
$\Longrightarrow (\exists x \in C[I] \ni f_n(x) = y) \wedge (\exists x \in B[I] \ni f_n(x) = y)$.
$\Longrightarrow (f_n(x) \in f_n(C[I])) \wedge (f_n(x) \in f_n(B[I]))$.
$\Longrightarrow (y \in f_n(C[I])) \wedge (y f_n(B[I]))$.
$\Longrightarrow (y \in (f_n(C[I])\, cap f(B[I])$.
$\Longrightarrow f_n(C[I] \cap B[I) \subseteq f_n(C[I]) \cap f_n(B[I])$.
Let $\in f_n(C[I] \cap B[I])$.
$\Longrightarrow \exists\, x]n(C[I] \cap B[I]) \in f_n(x) = y$.
$\Longrightarrow \exists x \in C[I] \wedge \exists\, x \in B[I] f_n(x) = y$.
$\Longrightarrow (\exists\, x \in C[I] \ni f_n(x) = y) \wedge (\exists\, x B[I] \ni f_n(x) = y)$.
$\Longrightarrow (f_n(x) \in f_n(C[I])) \wedge (f_n(x) \in f_n(B[I]))$.
$\Longrightarrow (y \in f_n(C[I])) \wedge (y \in f_n(B[I]))$.
$\Longrightarrow (y \in (f_n(C[I])[f_n(B[I])$.
$\Longrightarrow f_n(C[I] \cap B[I]) \subseteq f_n(C[I]) \cap f_n(B[I])$.

3. Let $y \in (f_n(C[I]) - f_n(B[I]))$.
$\Longrightarrow y \in f_n(C[I]) \wedge y notin f_n(B[I])$.
$\because y \in f_n(C[I]) \Longrightarrow (\exists\, x C[I] \ni f_n(x) = y)$.
$\because y \notin f_n(B[I]) \Longrightarrow f_n(x) \notin f_n(B[I])$.
$\Longrightarrow x \notin B[I]$.
$\Longrightarrow \exists\, x \in C[I] \wedge \exists\, x \notin B[I] \ni f_n(x) = y$.
$\Longrightarrow (\exists\, x \in (C[I] - B[I]) \ni f_n(x) = y)$.
$\Longrightarrow (f_n(x) \in f_n(C[I]) - (B[I])$.
$\Longrightarrow (y \in f_n(C[I]) - (B[I])$.
$\Longrightarrow (f_n(C[I]) - f_n(B[I])) \subseteq f_n(C[I]) - (B[I]$. The following examples illustrates that the equality in part 2 of the previous theorem does not hold.

□

Example 5.18. [5] Let f_n be a constant type-1 neutrosophic function, where

$$X[I] = \{2 + 2I, 2 + 7I, 7 + 2I, 7 + 7I\} \text{ and } Y[I] = \{4 + 4I\}.$$

Take:

$$C[I] = \{2 + 2I, 2 + 7I, 7 + 2I\}, \text{ and } B[I] = \{7 + 7I\}.$$

We have

$$C[I] \cap B[I] = \phi[I] \Longrightarrow f_n(C[I] \cap B[I]) = f_n(\phi[I]) = \phi[I].$$

$$f_n(C[I]) = f_n(\{2 + 2I, 2 + 7I, 7 + 2I\}) = 4 + 4I.$$

And,

$$f_n(B[I]) = f_n(\{7 + 7I\}) = 4 + 4I, \text{ so } f_n(C[I]) \cap f_n(B[I]) = 4 + 4I.$$

We see that $f_n(C[I]) \cap f_n(B[I]) \nsubseteq f_n(C[I] \cap B[I])$.

Example 5.19. [5] Let $f_n : \mathbb{R}[I] \longrightarrow \mathbb{R}[I]$ be a type-1 neutrosophic function from type-1 neutrosophic set of real numbers to itself such that

$$f_n(x) = f_c(x_1) + f_n(x_2 I) = f_c(x_1) + f_c(x_2) f_n(I).$$

where $f_c(x) = x^2$. Let $C[I] = \{a + bI : a, b \in C = [-2, 0]\}$ and

$$B[I] = \{c + dI : c, d \in B = [0, 2]\}$$

be two neutrosophic sets of type-1 generated by C and B, then the intersection of $C[I] \cap B[I] = 0 + 0I$, and $f_n(C[I] \cap B[I]) = f_n(0 + 0I) = 0 + 0I$, while $f_n(C[I]) = f_n([-2, -2I]) = 4 + 4I$, and $f_n(B[I]) = f_n([2, 2I]) = 4 + 4I$, hence $f_n(C[I]) \cap f_n(B[I]) = 4 + 4I \neq f_n(C[I] \cap B[I]) = 0 + 0I$.

Theorem 5.20. [5] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical one-to-one function $f_c : X \longrightarrow Y$, and classical sets X and Y respectively, $C[I] \subseteq X[I]$ and $B[I] \subset X[I]$, then:*

$$f_n(C[I] \subset B[I]) = f_n(C[I]) \cap f_n(B[I]) \Longleftrightarrow f_n : X[I] \longrightarrow Y[I]$$

is a one-to-one neutrosophic function.

Proof. Suppose that $f_n : X[I] \longrightarrow Y[I]$ is a one-to-one type-1 neutrosophic function, $C[I] \subset X[I]$ and $B[I] \subset X[I]$. Let $y \in f_n(C[I] \cap B[I])$
$\Longleftrightarrow \exists\, x \in (C[I] \cap B[I]) \ni f_n(x) = y$
$\Longleftrightarrow \exists\, x \in (C[I] \cap B[I]) \ni f_n(x) = y$
$\Longleftrightarrow (\exists\, x \in C[I] \ni f_n(x) = y) \wedge (\exists x \in B[I] \ni f_n(x) = y)$
$\Longleftrightarrow (y \in f_n(C[I])) \wedge (y \in f_n(B[I]))$.
$\Longleftrightarrow y \in (f_n(C[I]) \cap f_n(B[I]))$.
Hence $f_n(C[I] \cap B[I]) = f_n(C[I]) \cap f_n(B[I])$.
Conversely, suppose that $f_n(C[I] \cap B[I]) = f_n(C[I]) \cap f_n(B[I])$.
Where, $C[I] \subset X[I]$ and $B[I] \subset X[I]$.
To show that $f_n : X[I] \longrightarrow Y[I]$ is a one-to-one type-1 neutrosophic function.
Consider $x, z \in X[I]$, $x \neq z$ such that $f_n(x) = f_n(z) = y$. Consider $C[I] = x$, and $B[I] = z$ are two neutrosophic of type-1, we have $f_n(C[I]) = f_n(x) = y$ and $f_n(B[I]) = f_n(z) = y$. So, $f_n(C[I]) \cap f_n(B[I]) = y$, but $C[I] \cap B[I] = \phi[I]$, and $f_n(C[I] \cap B[I]) = f_n(\phi[I]) = \phi[I]$. Therefore, f_n is a one-to-one neutrosophic function. □

Theorem 5.21. [5] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively, and Let $A_\alpha[I], \alpha \in \mathbb{I} \in \mathbb{Z}^+$ be a family of neutrosophic subsets of $X[I]$, then:*

1. $f_n\big(\cup_\alpha A_\alpha[I]\big) = \cup_\alpha f_n\big(A_\alpha[I]\big)$, $\alpha \in \mathbb{I}$ *and*
2. $f_n\big(\cap_\alpha A_\alpha[I]\big) \subset \cap_\alpha f_n\big(A_\alpha[I]\big)$, $\alpha \in \mathbb{I}$.

Proof. 1. Suppose that $y \in \cup_\alpha f_n\big(A_\alpha[I]\big), \alpha \in \mathbb{I}$.
$\Longleftrightarrow \exists\, \alpha \in \mathbb{I} \ni y \in f_n\big(A_\alpha[I]\big)$.
$\Longleftrightarrow \exists\, x \in \big(A_\alpha[I]\big) \ni f_n(x) = y$.
$\Longleftrightarrow \exists\, x \in \cup_\alpha\big(A_\alpha[I]\big) \ni f_n(x) = y$.
$\Longleftrightarrow \exists\, y = f_n(x) \in f_n\big((A_\alpha[I]\big)$.

2. Suppose that $y \in f_n\big((A_\alpha[I]\big)$.
$\Longrightarrow \exists\, x \in \cap_\alpha\big(A_\alpha[I]\big) \ni f_n(x) = y$.
$\Longrightarrow \exists\, x \in \big(A_\alpha[I]\big) \ni f_n(x) = y, \forall \alpha \in \mathbb{I}$.
$\Longrightarrow y = f_n(x) \in f_n\big((A_\alpha[I]\big), \forall \alpha \in \mathbb{I}$.
$\Longrightarrow y = f_n(x) \in \cap_\alpha f_n\big((A_\alpha[I]$
$\Longrightarrow f_n\big(\cap_\alpha A_\alpha[I]\big) \subset \cap_\alpha f_n\big(A_\alpha[I]\big)$.

□

5.2 Type-1 Neutrosophic Invertible Functions and Their Properties

In this section, we investigate the invertible neutrosophic functions on the neutrosophic set of type-1 with their properties. we investigate the properties of invertible neutrosophic functions on neutrosophic sets of type-1, and we proved some theories with few examples explaining (or illustrating) the invertible of neutrosophic functions.

Definition 5.22. [6] Let $f_n : X[I] \longrightarrow Y[I]$ be a neutrosophic function of type-1.
Where, f_n generated by a classical function $f_c : X \longrightarrow Y$, $X[I]$, and $Y[I]$ are generated by a classical set X and Y. We say that f_n is a neutrosophic invertible, if $f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic function, and it's called the invertible type-1 neutrosophic function of f_n.

Remark 5.23. [6] $<x, y> \in f_n \Longleftrightarrow <y, x> \in f_n^{-1}|, and\, f_n(I) = I \Longleftrightarrow f_n^{-1}(I) = I.$

Theorem 5.24. [6] *If $f_n : X[I] \longrightarrow Y[I]$ is a type-1 neutrosophic function, then $f_n^{-1} : Y[I] \longrightarrow X[I]$. It may be a type-1 neutrosophic function or a non-neutrosophic function, and vice versa.*

Proof. By counterexample. Let $X[I] = \{(1 + 1I, 1 + 2I, 2 + 1I, 2 + 2I\}$ be a type-1 neutrosophic set, and $Y[I] = \{-1 - 1I\}$. Define:

$$f_n(1 + 1I) = f_n(1 + 2I) = f_n(2 + 1I) = f_n(2 + 2I) = -1 - 1I.$$

It is clear that $_n$ is a type-1 neutrosophic function, and it's known that the constant type-1 neutrosophic function, but $f_n^{-1} : Y[I] \longrightarrow X[I]$ is not a type-2 neutrosophic function. Also, if $f_n : X[I] \longrightarrow Y[I]$, where;

$$f_n(-1 - 1I) = 1 + 1I, f_n(-1 - 1I) = 1 + 2I, f_n(-1 - 1I) = 2 + 1I.$$

And $f_n(-1-1I) = 2+2I$. We see that f_n is not a type-1 neutrosophic function, while $f_n^{-1} : X[I] \longrightarrow Y[I]$ is a type-1 neutrosophic function. □

The following theorem gives us the necessary and sufficient condition for the existence of the inverse neutrosophic function for any neutrosophic function.

Theorem 5.25. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function, then f_n is an invertible type-1 neutrosophic function iff $f_n : X[I] \longrightarrow Y[I]$ is a type-1 neutrosophic bijective function.*

Proof. Let $f_n : X[I] \longrightarrow Y[I]$ be a invertible type-1 neutrosophic function, that is
$f_n^{-1} : Y[I] \longrightarrow X[I]$ exists. We want to show that $f_n : X[I] \longrightarrow Y[I]$ is a bijectivetype-1 neutrosophic function. Suppose that $x, z \in X[I] \ni f_n(x) = f_n(z)$. Consider $f_n(x) = f_n(z) = y$, we have $(<x, y> \in f_n) \wedge (<z, y> \in f_n)$
$\Longleftrightarrow (<y, x> \in f_n^{-1}) \wedge (<y, z> \in f_n^{-1})$, since f_n^{-1} is a type-1 neutrosophic function, implies that $y = z$. Therefore f_n a type-1 neutrosophic injective function. Second, assume that $y \in Y[I]$, and $f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic function, so $\Longrightarrow \exists\, x \in X[I] \ni <y, x> \in f_n^{-1}$
$\Longrightarrow \exists\, x \in X[I] \ni <x, y> \in f_n$
$\Longrightarrow \exists\, x \in X[I] \ni f_n(x) = y$
$\Longrightarrow f_n$ is a type-1 neutrosophic surjective function.
$\Longrightarrow f_n$ is a type-1 neutrosophic bijective (one-to-one onto) function. Conversely, Suppose that $f_n : X[I] \longrightarrow Y[I]$ is a type- neutrosophic bijective function, we need to show that $f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic inverse function. Suppose that $y \in Y[I]$, but $f_n : X[I] \longrightarrow Y[I]$ (bijective) $\exists\, x \in X[I] \ni f_n(x) = y$
$\Longrightarrow \exists\, x \in X[I] \ni <x, y> \in f_n.$
$\Longrightarrow \exists\, x \in X[I] \ni <y, x> \in f_n^{-1}$
$\Longrightarrow NeuDom(f_n^{-1}) = Y[I]$.
Now, suppose that $(<y, x> \in f_n^{-1}) \wedge (<y, z> \in f_n^{-1}$,
$\Longrightarrow (<x, y> \in f_n) \wedge (<z, y> \in f_n)$
$\Longrightarrow (f_n(x) = y) \wedge (f_n(z) = y)$
$\Longrightarrow f_n(x) = f_n(z)$
$\Longrightarrow x = z.$
Since $f_n : X[I] \Longrightarrow Y[I]$ is a type-1 neutrosophic bijective function,
$\Longrightarrow f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic function. □

Theorem 5.26. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function. If f_n is an invertible type-1 neutrosophic function, then $f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic bijective function.*

Proof. Let$f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function is invertible. Then:
$f_n^{-1} : Y[I] \longrightarrow X[I]$ is a neutrosophic function by definition 5.22. Suppose that $y, z \in Y[I] \ni f_n^{-1}(y) = f_n^{-1}(z) = x$.
$\Longrightarrow (<y, x> \in f_n^{-1}) \wedge (<z, x> \in f_n^{-1})$.
$\Longrightarrow (<x, y> \in f_n) \wedge (<x, z> \in f_n)$.
$\Longrightarrow (f_n^(x) = y) \wedge (f_n(x) = z)$.
$\Longrightarrow y = z \Longrightarrow f_n^{-1}$ is a type-1 neutrosophic injective (one-to-one) function. Also, suppose that $x \in X[I]$. Since, $f_n : X[I] \longrightarrow Y[I]$ is a type-1 neutrosophic function.
$\Longrightarrow \exists\, y \in Y[I] \ni f_n(x) = y$.
$\Longrightarrow \exists\, y \in Y[I] \ni <x, y> \in f_n$.
$\Longrightarrow \exists\, y \in Y[I] \ni <y, x> \in f_n^{-1}$.
$\Longrightarrow \exists\, y \in Y[I] \ni f_n^{-1}(y) = x$.
$\Longrightarrow f_n^{-1}$ is a type-1 neutrosophic surjective (onto) function.
$\Longrightarrow f_n^{-1}$ is a type-1 neutrosophic bijective (one-to-one +onto) function. □

Theorem 5.27. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be an invertible type-1 neutrosophic function. Then:*

1. $f_n^{-1} \circ f_n = I_{ndX}$, *and*
2. $f_n \circ f_n^{-1} = I_{ndY}$.

Proof. Suppose that $f_n : X[I] \longrightarrow Y[I]$ is an invertible type-1 neutrosophic function, then $f_n^{-1} : Y[I] \longrightarrow X[I]$ is a type-1 neutrosophic function, then
$f_n^{-1} \circ f_n : X[I] \longrightarrow X[I]$ is a type-1 neutrosophic function.
Assume that $x \in X[I] \ni f_n(x) = y$. Therefore,

$$(f_n^{-1} \circ f_n)(x) = f_n^{-1}(f_n(x)) = f_n^{-1}(y) = x.$$

In addition,$I_{ndX} : X[I] \longrightarrow X[I]$ is a type-1 neutrosophic identity function. Therefore $I_{ndX}(x) = x$, for all $x \in X[I]$.
We have, $f_n^{-1} \circ f_n(x) = I_{ndX}(x)$, for all $x \in X[I]$, that is $f_n^{-1} \circ f_n = I_{ndX}$. By Definition 5.11. □

Theorem 5.28. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ and $g_n : Y[I] \longrightarrow Z[I]$ be two one-to-one and onto type-1 neutrosophic functions. Then: $g_n \circ f_n$ is one-to-one and onto a type-1 neutrosophic function.*

Proof. Let $f_n : X[I] \longrightarrow Y[I]$ and $g_n : Y[I] \longrightarrow Z[I]$ be two one-to-one and onto type-1 neutrosophic functions. Suppose that $(g_n \circ f_n)(x) = (g_n \circ f_n)(y), \forall x, y \in X[I]$. We conclude that $\Longrightarrow (g_n(f_n(x)) = (g_(n) f_n^(y))$.
$\Longrightarrow g_n(x) = g_n(y)$.
$\Longrightarrow x = y$.
Therefore $g_n \circ f_n$ is a one-to-one.
Now consider, $z \in Z[I]$.
$\Longrightarrow \exists\, y \in Y[I] \ni g_n(y) = z$.
$\Longrightarrow \exists\, x\ inX[I] \ni f_n(x) = y$.
$\Longrightarrow \exists\, x \in X[I] \ni g_n(y) = g_n(f_n(x)) = (g_n \circ f_n)(x)$.
$\Longrightarrow g_n \circ f_n$ is an onto, thus it's an injective neutrosophic function. □

Theorem 5.29. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ and $g_n : Y[I] \longrightarrow Z[I]$ be two one-to-one and onto type-1 neutrosophic functions. Then:*

1. $(f_n^{-1})^{-1} = f_n$, *and*
2. $(g_n \circ f_n)^{-1} = f_n^{-1} \circ g_n^{-1}$.

Proof. 1. Assume that $(x, y) \in (f_n^{-1})^{-1}$.
$\Longleftrightarrow (f_n^{-1})^{-1}(x) = y$.
$\Longleftrightarrow x = f_n^{-1}(y)$.
$\Longleftrightarrow f_n(x) = y$.
$\Longleftrightarrow (x, y) \in f_n$.
Thus $(f_n^{-1})^{-1} = f_n$.

2. Let $f_n : X[I] \longrightarrow Y[I]$ and $g_n : Y[I] \longrightarrow Z[I]$ be two one-to-one and onto type-1 neutrosophic functions. Then: $g_n \circ f_n$ is one-to-one and onto by Theorem 5.9 , therefore, the inverse neutrosophic function is one-to-one and onto. Now, from the right-hand side, we can construct:

$$\begin{aligned}(f_n^{-1} \circ g_n^{-1}) \circ (g_n \circ f_n) &= f_n^{-1} \circ (g_n^{-1} \circ ((g_n \circ f_n))).\\ &= f_n^{-1} \circ ((g_n^{-1} \circ g_n) \circ f_n).\text{" by the composition associative property".}\\ &= f_n^{-1} \circ (I_{ndY} \circ f_n)\\ &= f_n^{-1} \circ f_n.\\ &= I_{ndX}.\end{aligned}$$

By a similar argument, we have:

$$\begin{aligned}(g_n \circ f_n) \circ (f_n^{-1} \circ g_n^{-1}) &= g_n \circ (f_n \circ (f_n^{-1} \circ g_n^{-1})).\\ &= g_n \circ ((f_n \circ f_n^{-1}) \circ g_n^{-1}).\\ &= g_n \circ (I_{ndY} \circ g_n^{-1})\\ &= g_n \circ g_n^{-1}\\ &= I_{ndZ}.\end{aligned}$$

Finally, to check that:

$$(g_n \circ f_n)^{-1}(z) = f_n^{-1} \circ g_n^{-1}(z).$$

By taking the right-hand side:

$$f_n^{-1} \circ g_n^{-1}(z) = f_n^{-1} g_n^{-1}(z)) = f_n^{-1}(y) = x = I_{ndX}.$$

And by taking the left-hand side:

$$(f_n^{-1} \circ g_n^{-1})(z) = f_n^{-1}(g_n^{-1}(z)) = f_n^{-1}(y) = x = I_{ndX}.$$

Moreover,

$$(g_n \circ f_n)(x) = g_n(f_n(x)) = g_n(y) = z = I_{ndZ}.$$

Thus, $(g_n \circ f_n)^{-1} = f_n^{-1} \circ g_n^{-1}$.

□

Definition 5.30. [6] Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. Let $B[I]$ be a type-1 neutrosophic subset of $Y[I]$ generated by $B \subset Y$. Define the type-1 neutrosophic image of $B[I]$ under f_n , written $f_n^{-1}(B[I])$, as follows:

$$f_n^{-1}(B[I]) = \{x \in X[I] : f_n(x) \in B[I]\}.$$

Remark 5.31. [6] If $B[I] = \{x\}$ consist of a singleton element x, the notation $f_n^{-1}(x)$ use instead of $f_n^{-1}(x)$.

Example 5.32. [?] Let $f_n : \mathbb{R}[I] \longrightarrow \mathbb{R}[I]$ be a type-1 neutrosophic function on type-1 neutrosophic set of real numbers defined by:
$f_n(x) = f_n(x_1 + x_2 I) = f_c(x_1) + f_c(x_2) f_n(I)$, where $f_c(x) = x^2 + 2$, then:

$$\begin{aligned}f_n^{-1}(11 + 11I) &= \{x \in \mathbb{R}[I] : f_n(x) = 11 + 11I\}.\\ &= \{x \in \mathbb{R}[I] : f_c(x_1) + f_c(x_2) f_n(I) = 11 + 11I\}.\\ &= \{x \in \mathbb{R}[I] : (x_1^2 + 2) + (x_2^2 + 2)I = 11 + 11I\}.\\ &= \{(3 + 3I), (-3 - 3I)\}.\end{aligned}$$

Theorem 5.33. [?] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. $C[I]Y[I]$ and $B[I]Y[I]$. if $C[I] = B[I]$, then $f_n^{-1}(C[I]) = f_n^{-1}(B[I])$.*

Proof. Suppose that $C[I] = B[I]$, and let $x \in f_n^{-1}(C[I])$.
$\Longrightarrow f_n(x) \in C[I]$.
$\Longrightarrow f_n(x) \in B[I]$.
$\Longrightarrow x \in f_n^{-1}(B[I])$.
$\Longrightarrow f_n^{-1}(C[I]) \subseteq f_n^{-1}(B[I])$.
By similar method, $f_n^{-1}(B[I]) \subseteq f_n^{-1}(C[I])$. Hence, $f_n^{-1}(C[I]) = f_n^{-1}(B[I])$. □

The converse of the theorem is not true, by the following example.

Example 5.34. [6] Let $f_n : \mathbb{R}[I] \longrightarrow \mathbb{R}[I]$ be a neutrosophic function of type-1 generated by the neutrosophic set of real numbers to itself. Defined by $f_n(x) = |x_1 + |x_2 I$. Consider $C[I] = \{x_1 + x_2 I : x_1, x_2 \in C = (1, 2)\} = (1 + 1I, 2 + 2I)$, and $B[I] = \{x_1 + x_2 I : x_1, x_2 \in B = -1, 2)\} = (-1 - 1I, 2 + 2I)$, here $C[I] \neq B[I]$, but $f_n^{-1}((1 + 1I, 2 + 2I)) = (1 + 1I, 2 + 2I)$ and $f_n^{-1}((-1 - 1I, 2 + 2I)) = (1 + 1I, 2 + 2I)$, we have $f_n^{-1}(C[I]) = f_n^{-1}(B[I])$, but $C[I] \neq B[I]$.

Theorem 5.35. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. Let $C[I] \subset Y[I]$ and $B[I] \subset Y[I]$, then:*

1. $f_n^{-1}(C[I] \cup B[I]) = f_n^{-1}(C[I]) \cup f_n^{-1}(B[I])$,
2. $f_n^{-1}(C[I] \cap B[I]) = f_n^{-1}(C[I]) \cap f_n^{-1}(B[I])$,
3. $f_n^{-1}(C[I] - B[I]) = f_n^{-1}(C[I]) - f_n^{-1}(B[I])$, *and*
4. $f_n^{-1}(C^c[I]) = (f_n^{-1}(C[I]))^c$.

Proof. 1. Let $x \in f_n^{-1}(C[I] \cup B[I])$.
$\Longleftrightarrow f_n(x) \in (C[I] \cup B[I])$.
$\Longleftrightarrow f_n(x) \in C[I] \vee f_n(x) \in B[I]$.
$\Longleftrightarrow x \in f_n^{-1}(C[I]) \vee x \in f_n^{-1}(B[I])$.
$\Longleftrightarrow x \in (f_n^{-1}(C[I]) \cup f_n^{-1}(B[I]))$.
$\Longrightarrow f_n^{-1}(C[I] \cup BI]) = f_n^{-1}(C[I]) \cup f_n^{-1}(B[I])$.

2. Let $x \in f_n^{-1}(C[I] \cap B[I])$.
$\Longleftrightarrow f_n(x) \in (C[I] \cap B[I])$.
$\Longleftrightarrow f_n(x) \in C[I] \vee f_n(x) \in B[I]$.
$\Longleftrightarrow x \in f_n^{-1}C[I]) \wedge x \in f_n^{-1}(B[I])$.
$\Longleftrightarrow x \in (f_n^i{-}1(C[I]) \cap f_n^{-1}(B[I]))$.
$\Longrightarrow f_n^{-1}(C[I] \cap B[I]) = f_n^{-1}(C[I]) \cap f_n^{-1}(B[I])$.

3. Let $x \in f_n^{-1}(C[I] - B_[I])$.
$\Longleftrightarrow f_n(x) \in (C[I] - B[I])$.
$\Longleftrightarrow f_n(x) \in C[I] \wedge f_n(x) \notin B[I]$.
$\Longleftrightarrow x \in f_n^{-1}(C[I]) \wedge x \notin f_n^{-1}(B[I])$.
$\Longleftrightarrow x \in (f_n^{-1}(C[I]) - f_n^{-1}(B[I]))$.
$\Longrightarrow f_n^{-1}(C[I] - B[I]) = f_n^{-1}(C[I]) - f_n^{-1}(B[I])$.

4. Suppose that $x \in f_n^{-1}(C^c[I]))$.
$\Longleftrightarrow f_n(x) \in (C^c[I])$.
$\Longleftrightarrow f_n(x) \notin C[I]$.
$\Longleftrightarrow xinf_n^{-1}(C[I])$.
$\Longleftrightarrow x \in (f_n^{-1}(C[I])^c$. Hence, $f_n^{-}(C[I]) = (f_n^{-1}(C[I]))^c$.

□

Definition 5.36. [6] Let $f_n : X[I] \longrightarrow Y[I]$ be a type-1 neutrosophic function generated by a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. Consider $P(X[I])$ and $P(Y[I])$ are type-1 neutrosophic power sets of $X[I]$ and $Y[I]$. A function f_n is called induce type-1 neutrosophic function, if

$$f : P(XI]) \longrightarrow P(Y[I]) \text{ by } C[I] \longrightarrow f_n(C[I]), C[I] \subset X[I].$$

And

$$f_n^{-1} : P(Y[I]) \longrightarrow X[I] \text{ by } B[I] \longrightarrow f_n^{-1}(B[I]), B[I] \subseteq Y[I].$$

Theorem 5.37. [?] *Let $f_n : X[I] \longrightarrow Y[I]$ be a neutrosophic one-to-one function of type-1 generated from a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively, then: $f_n : P(X[I]) \longrightarrow P(Y[I])$ is a neutrosophic one-to-one function.*

Proof.

Case 1. Let $X[I] \neq \phi[I]$, then $P(X[I])$ contains at least two neutrosophic elements, say:

$$C[I] \in P(X[I]) \wedge B[I] \in P(X[I]) \ni C[I] \neq B[I].$$

$$\Longrightarrow (\exists\, x \in X[I] \ni x \in C[I]) \wedge x \notin B[I]).$$

$$\Longrightarrow f_n(x) \in f_n(C[I]) \wedge f_n(x) \notin f_n(B[I]).$$

We conclude that

$$X[I] \neq \phi[I] \Longrightarrow f_n(C[I]) \neq f_n B[I]).$$

Hence

$$f_n : P(X[I]) \longrightarrow P(Y[I])$$

is a neutrosophic one-to-one function.

Case 2. Let $X[I] = \phi[I]$, that $P(X[I]) = \{\phi_1 + \phi_1 I\}$, it's obvious one-to-one function, since there is no two different elements have the same image.

□

Theorem 5.38. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be a neutrosophic one-to-one function of type-1 generated from a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. then the neutrosophic induced function:*

$$f_n^{-1} : P(Y[I]) \longrightarrow P(X[I])$$

preserves the neutrosophic elementary operations.

1. $f_n^{-1}\big(\cup_{\alpha\in\mathbb{I}}\ B_\alpha[I]\big) = \cup_{\alpha\in\mathbb{I}}\ \big(f_n^{-1}(B_\alpha[I])\big)$, *and*
2. $f_n^{-1}\big(\cap_{\alpha\in\mathbb{I}}\ (B_\alpha[I]\big) = \cap_{\alpha\in\mathbb{I}}\big(f_n^{-1}\ (B_\alpha[I])\big)$.

Proof. 1. Suppose that $x \in f_n^{-1}\big(\cup_{\alpha\in\mathbb{I}} B_\alpha[I]\big)$.

$$\Longleftrightarrow f_n(x) \in \cup_{\alpha\in\mathbb{I}} B_\alpha[I]).$$

$$\Longleftrightarrow \exists\, \alpha \in \mathbb{I}, f_n(x) \in \big(B_\alpha[I]\big).$$

$$\Longleftrightarrow \exists\, \alpha \in \mathbb{I}, x \in f_n^{-1}\big(B_\alpha[I]\big).$$

$$\Longleftrightarrow \exists\, x \in \cup_{\alpha\in\mathbb{I}}\, f_n^{-1}\big(B_\alpha[I]\big).$$

Hence, $f_n^{-1}\big(\cup_{\alpha\in\mathbb{I}}\ (B_\alpha[I]\big) = \cup_{\alpha\in\mathbb{I}}\ \big(f_n^{-1}(B_\alpha[I])\big)$.

2. Suppose that $x \in f_n^{-1}\big(\cap_{\alpha\in\mathbb{I}} B_\alpha[I]\big)$.

$$\begin{aligned}&\Longleftrightarrow f_n(x) \in \cap_{\alpha\in\mathbb{I}} B_\alpha[I]).\\&\Longleftrightarrow \forall\, \alpha \in \mathbb{I}, f_n(x) \in \big(B_\alpha[I]\big).\\&\Longleftrightarrow \forall\, \alpha \in \mathbb{I}, x \in f_n^{-1}\big(B_\alpha[I]\big).\\&\Longleftrightarrow \forall\, x \in \cap_{\alpha\in\mathbb{I}} f_n^{-1}\big(B_\alpha[I]\big).\end{aligned}$$

Hence, $f_n^{-1}\big(\cap_{\alpha\in\mathbb{I}}\ (B_\alpha[I]\big) = \cap_{\alpha\in\mathbb{I}}\ \big(f_n^{-1}(B_\alpha[I])\big)$.

□

Theorem 5.39. [6] *Let $f_n : X[I] \longrightarrow Y[I]$ be a neutrosophic one-to-one function of type-1 generated from a classical function $f_c : X \longrightarrow Y$ and classical sets X and Y respectively. Let $C[I] \subset X[I]$ and $B[I] \subseteq Y[I]$ Then:*

1. *$C[I] \subset f_n^{-1}(f_n(C[I]))$, and*
2. *$f_n(f_n^{-1}(B[I])) \subset B[I]$.*

Proof. 1. Consider $x \in C[I] \Longrightarrow f_n(x) \in f_n(C[I]) \Longrightarrow x \in f_n^{-1}(f_n(C[I]))$. Hence, $C[I] \subset f_n^{-1}(f_n(C[I]))$.

2. By the same technique.

□

Example 5.40. [6] Let $f_n : \mathbb{R}[I] \longrightarrow \mathbb{R}[I]$ be a neutrosophic function of type-1 generated by the neutrosophic set of real numbers to itself. Defined by:

$$f_n(x) = x_1^2 + x_2^2 I.$$

Consider:

$$\begin{aligned}C[I] &= \{x_1 + x_2 I : x_1, x_2 \in C = \{1, 2, 3\}\}\\&= \{1 + 1I, 1 + 2I, 1 + 3I, 2 + 1I, 2 + 2I, 2 + 3I, 3 + 1I, 3 + 2I, 3 + 3I\}.\end{aligned}$$

And

$$\begin{aligned}f_n(C[I]) &= f_n\{(1 + 1I, 1 + 2I, 1 + 3I, 2 + 1I, 2 + 2I, 2 + 3I, 3 + 1I, 3 + 2I, 3 + 3I\}.\\&= \{1 + 1I, 1 + 4I, 1 + 9I, 4 + 1I, 4 + 4I, 4 + 9I, 9 + 1I, 9 + 4I, 9 + 9I\}.\end{aligned}$$

Therefore,

$$\begin{aligned}f_n^{-1}\big(f_n(C[I])\big) &= f_n^{-1}\{1 + 1I, 1 + 4I, 1 + 9I, 4 + 1I, 4 + 4I, 4 + 9I, 9 + 1I, 9 + 4I, 9 + 9I\}.\\&= \left\{\begin{array}{c}1 + 1I, -1 - 1I, 1 + 2I, -1 - 2I, 1 + 3I, -1 - 3I,\\2 + 1I, -2 - 1I, 2 + 2I, -2 - 2I, 2 + 3I, -2 - 3I,\\3 + 1I, -3 - 1I, 3 + 2I, -3 - 2I, 3 + 3I, -3 - 3I\end{array}\right\}.\end{aligned}$$

It is clear that $f_n^{-1}(f_n(C[I])) \neq C[I]$. Also, consider

$$f_n^{-1}(-4 - 4I) = \phi[I] = \{u_1 + u_2 I : u_1, u_1 \in \phi\}.$$

Then

$$f_n(f_n^{-1}(-4 - 4I)) = f_n(\phi[I]) = \phi[I] \neq B[I] = -4 - 4I.$$

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