



New Concepts of MetaStructures: Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals

Takaaki Fujita^{1,*}, Ajoy Kanti Das²

¹Independent Researcher, Tokyo, Japan

²Associate Professor, Department of Mathematics, Tripura University, Agartala-799022, Tripura, India

Emails: Takaaki.fujita060@gmail.com; ajoykantidas@gmail.com

Abstract

A *MetaStructure* is a higher-level framework that treats entire collections of structures as single objects, equipped with natural operations that preserve isomorphisms across different domains. The term “Structure” here refers broadly to mathematical systems as well as real-world models. An *Iterated MetaStructure* generalizes this idea recursively, generating successive layers in which structures of structures form deeper hierarchical meta-levels. In this work, we extend and investigate the properties of Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals through the lens of MetaStructures and Iterated MetaStructures.

Keywords: MetaStructure; Iterated MetaStructure; Algebra; Topology, Lattices; Queues; Markov Chains; Interval

1 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper.

1.1 Classical Structure

In this paper, the term *Structure* refers broadly to a mathematical system, not restricted to a single area, but encompassing domains such as Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

Definition 1.1 (Classical Structure). A *Classical Structure* \mathcal{C} is a mathematical object arising from a traditional field—for example Set Theory, Logic, Probability, Statistics, Algebra, Geometry, Graph Theory, Automata Theory, or Game Theory. Formally, it may be represented as a pair

$$\mathcal{C} = (H, \{\#^{(m)}\}_{m \in \mathcal{I}}),$$

where:

- H is a nonempty set, often called the *carrier* or *universe*.
- For each $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$, there exists an m -ary operation

$$\#^{(m)} : H^m \longrightarrow H,$$

subject to appropriate *axioms* (such as associativity, commutativity, or identity laws), which vary according to the chosen type of structure.

The collection $\{\#^{(m)} : m \in \mathcal{I}\}$ determines the *type* of \mathcal{C} . Representative examples include:

- A *Set* (S, \emptyset) , consisting solely of a carrier with distinguished elements or relations, but without operations.
- A *Logic* structure (L, \wedge, \vee, \neg) , where \wedge, \vee are binary connectives and \neg a unary connective, satisfying logical axioms.
- A *Probability* model (Ω, \mathcal{F}, P) , where $P : \mathcal{F} \rightarrow [0, 1]$ is a probability measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$.
- A *Statistical* model (X, \mathcal{A}, θ) , where θ maps data X into parameters of interest.
- *Algebraic* structures such as:
 - A *Group* $(G, *)$, with $* : G \times G \rightarrow G$ satisfying associativity, identity, and inverses.
 - A *Ring* $(R, +, \times)$, with two binary operations fulfilling ring axioms.
 - A *Vector Space* $(V, +, \cdot)$ over a field \mathbb{F} , with scalar multiplication $\cdot : \mathbb{F} \times V \rightarrow V$.
- A *Geometric* structure (X, dist) , where $\text{dist} : X \times X \rightarrow \mathbb{R}$ satisfies the axioms of a metric.
- A *Graph* (V, E) , where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ for undirected graphs, or $E \subseteq V \times V$ for directed graphs, with adjacency and incidence relations.
- An *Automaton* $(Q, \Sigma, \delta, q_0, F)$, where Q is a set of states, Σ an input alphabet, $\delta : Q \times \Sigma \rightarrow Q$ the transition function, $q_0 \in Q$ the start state, and $F \subseteq Q$ the accepting states.
- A *Game* $(N, \{A_i\}, \{u_i\})$, where N is the set of players, A_i each player's action set, and $u_i : \prod_{j \in N} A_j \rightarrow \mathbb{R}$ the payoff function for player i .

Related concepts include the *HyperStructure*^{1,2} and the *SuperHyperStructure*,³ which have also been extensively investigated in recent studies.

1.2 MetaStructure (Structure of a Structure)

Fix once and for all a single-sorted, finitary *signature*

$$\Sigma = (\text{Func}, \text{Rel}, \text{ar}_{\text{Func}}, \text{ar}_{\text{Rel}}),$$

where *Func* (resp. *Rel*) is a set of function (resp. relation) symbols and *ar* records their arities. A (single-sorted) Σ -*structure* is a tuple

$$\mathbf{C} = (H, (f^{\mathbf{C}})_{f \in \text{Func}}, (R^{\mathbf{C}})_{R \in \text{Rel}}),$$

consisting of a nonempty carrier H , together with interpretations $f^{\mathbf{C}} : H^m \rightarrow H$ for each $f \in \text{Func}$ of arity m , and relations $R^{\mathbf{C}} \subseteq H^r$ for each $R \in \text{Rel}$ of arity r . Let Str_{Σ} denote the class of all such structures.

Definition 1.2 (MetaStructure over a fixed signature). With Σ as above, a *MetaStructure* (a “structure of structures”) is a pair

$$\mathbb{M} = (U, (\Phi_{\ell})_{\ell \in \Lambda}),$$

where:

- $U \subseteq \text{Str}_{\Sigma}$ is a nonempty collection of Σ -structures (the level-0 objects);

- for each label $\ell \in \Lambda$ with *meta-arity* $k_\ell \in \mathbb{N}$, the *meta-operation*

$$\Phi_\ell : U^{k_\ell} \longrightarrow U$$

is described by uniform *constructors* acting on carriers and symbol interpretations:

$$\begin{aligned} \Gamma_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) &= H_\ell && \text{(functorially built carrier);} \\ \forall f \in \text{Func} : f^{\Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell})} &= \Lambda_\ell^f(f^{\mathbf{C}_1}, \dots, f^{\mathbf{C}_{k_\ell}}); \\ \forall R \in \text{Rel} : R^{\Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell})} &= \Xi_\ell^R(R^{\mathbf{C}_1}, \dots, R^{\mathbf{C}_{k_\ell}}), \end{aligned}$$

where the recipes Λ_ℓ^f and Ξ_ℓ^R depend only on f, R and ℓ (not on the particular representatives) and produce the output interpretations over H_ℓ .

Each Φ_ℓ is required to be *isomorphism-invariant* (natural): if $\alpha_i : \mathbf{C}_i \xrightarrow{\cong} \mathbf{D}_i$ are isomorphisms for $1 \leq i \leq k_\ell$, then there is an induced isomorphism

$$\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}) : \Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) \xrightarrow{\cong} \Phi_\ell(\mathbf{D}_1, \dots, \mathbf{D}_{k_\ell}),$$

compatible with all function and relation symbols of Σ .

1.3 Iterated MetaStructure

An *Iterated MetaStructure* is obtained by repeatedly applying a fixed *lifting constructor* U_Σ to a universe of Σ -structures (or Σ -entities), so that each step replaces “objects” by “structures made out of those objects” and thereby forms a hierarchy (*a tower of universes*). This formalizes the idea of *structures of structures* and provides a uniform way to transport meta-operations and meta-relations across levels (cf. ⁴).

We treat U_Σ as a level-raising constructor equipped with canonical embeddings, so that every element at level $r + 1$ is (by definition) a U_Σ -lift of a unique element at level r . Concretely, we *define* universes by recursion as images under U_Σ ; hence all lifted operations below are unambiguously defined on the whole universe $U^{(r)}$.

Definition 1.3 (Iterated MetaStructure of depth t). (cf. ⁴) Fix a signature Σ and a MetaStructure

$$\mathfrak{M}^{(0)} = \left(U^{(0)}, \{\odot_i^{(0)}\}_{i \in I}, \{\mathcal{S}_j^{(0)}\}_{j \in J} \right),$$

where $U^{(0)}$ is a class (universe) of Σ -objects, each $\odot_i^{(0)}$ is a meta-operation, and each $\mathcal{S}_j^{(0)}$ is a meta-relation.

Let U_Σ be a level-raising constructor. Define the *universe tower*

$$U^{(r+1)} := U_\Sigma(U^{(r)}) \quad (r \geq 0),$$

and denote the canonical embedding by

$$\iota_r : U^{(r)} \longrightarrow U^{(r+1)}, \quad \iota_r(x) := U_\Sigma(x).$$

For $0 \leq s < t$, set

$$\iota_{s \rightarrow t} := \iota_{t-1} \circ \dots \circ \iota_s = U_\Sigma^{t-s} \Big|_{U^{(s)}} : U^{(s)} \rightarrow U^{(t)}.$$

Lifting meta-operations. Assume a meta-operation at height s has the schematic typing

$$\odot_i^{(s)} : (\mathbf{E}_\Sigma^{m_i+s})^{k_i} \longrightarrow \mathcal{P}^{n_i}(\mathbf{E}_\Sigma^{n_i+s}),$$

(where \mathbf{E}_Σ^r denotes the ambient class of Σ -entities at height r). Its lift to height t is the operation

$$\odot_i^{(t)} : (\mathbf{E}_\Sigma^{m_i+t})^{k_i} \longrightarrow \mathcal{P}^{n_i}(\mathbf{E}_\Sigma^{n_i+t})$$

defined by *transport along the tower*: for all $x_1, \dots, x_{k_i} \in E_{\Sigma}^{m_i+s}$,

$$\odot_i^{(t)}\left(\mathbb{U}_{\Sigma}^{t-s}(x_1), \dots, \mathbb{U}_{\Sigma}^{t-s}(x_{k_i})\right) := \mathbb{U}_{\Sigma}^{t-s}\left(\odot_i^{(s)}(x_1, \dots, x_{k_i})\right).$$

(Equivalently, $\odot_i^{(t)} = \mathbb{U}_{\Sigma}^{t-s} \circ \odot_i^{(s)} \circ (\mathbb{U}_{\Sigma}^{t-s})^{\times k_i}$ on the relevant domain.)

Lifting meta-relations. If $\mathcal{S}_j^{(s)} \subseteq (E_{\Sigma}^{q_j+s})^{\ell_j}$ is a relation at height s , define its lift by direct image:

$$\mathcal{S}_j^{(t)} := (\mathbb{U}_{\Sigma}^{t-s})^{\times \ell_j}(\mathcal{S}_j^{(s)}) \subseteq (E_{\Sigma}^{q_j+t})^{\ell_j}.$$

Finally, the *Iterated MetaStructure of depth t* is

$$\mathfrak{M}^{(t)} := \left(U^{(t)}, \{\odot_i^{(t)}\}_{i \in I}, \{\mathcal{S}_j^{(t)}\}_{j \in J} \right).$$

We present schematic overviews of an iterated MetaStructure in Figures 1, 2, and 3.

$$U^{(0)} \xrightarrow{\mathbb{U}_{\Sigma}} U^{(1)} \xrightarrow{\mathbb{U}_{\Sigma}} U^{(2)} \xrightarrow{\mathbb{U}_{\Sigma}} \dots \xrightarrow{\mathbb{U}_{\Sigma}} U^{(t)}$$

Figure 1: Universe tower produced by iterating the lifting constructor \mathbb{U}_{Σ} .

$$\begin{array}{ccc} (E_{\Sigma}^{m_i+s})^{k_i} & \xrightarrow{\odot_i^{(s)}} & \mathcal{P}^{n_i}(E_{\Sigma}^{n_i+s}) \\ \downarrow (\mathbb{U}_{\Sigma}^{t-s})^{\times k_i} & & \downarrow \mathbb{U}_{\Sigma}^{t-s} \\ (E_{\Sigma}^{m_i+t})^{k_i} & \xrightarrow{\odot_i^{(t)}} & \mathcal{P}^{n_i}(E_{\Sigma}^{n_i+t}) \end{array}$$

Figure 2: Commutative lifting of a meta-operation from height s to height t .

$$\begin{array}{ccc} \mathcal{S}_j^{(s)} \subseteq (E_{\Sigma}^{q_j+s})^{\ell_j} & \hookrightarrow & (E_{\Sigma}^{q_j+s})^{\ell_j} \\ \downarrow (\mathbb{U}_{\Sigma}^{t-s})^{\times \ell_j} & & \downarrow (\mathbb{U}_{\Sigma}^{t-s})^{\times \ell_j} \\ \mathcal{S}_j^{(t)} \subseteq (E_{\Sigma}^{q_j+t})^{\ell_j} & \hookrightarrow & (E_{\Sigma}^{q_j+t})^{\ell_j} \end{array}$$

Figure 3: Direct-image lifting of a meta-relation.

2 MetaAlgebra: Algebra of Algebra

An algebra is a vector space over a field equipped with bilinear operations, generalizing structures like groups, rings, and fields (cf.^{5,6}). A Meta-Algebra is a MetaStructure whose objects are algebras, with meta-operations combining, transforming, or restricting them while preserving isomorphisms. An Iterated Meta-Algebra recursively applies Meta-Algebra construction, forming higher-level hierarchies of algebraic systems, generalizing and extending algebras across multiple meta-layers.

Definition 2.1 (Algebra). (cf.^{7,8}) An *algebra* over a field \mathbb{F} is a vector space A over \mathbb{F} equipped with a bilinear multiplication

$$\cdot : A \times A \longrightarrow A,$$

such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in A$,

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z), \quad z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(z \cdot y).$$

(Associativity or a unit may be imposed as additional axioms when desired, but are not required here.)

Fix a field \mathbb{F} . Consider the single-sorted, finitary *algebra signature*

$$\Sigma_{\text{alg}} = \left(\text{Func}, \text{Rel} = \emptyset, \text{ar} \right), \quad \text{Func} = \{ + : 2, 0 : 0, - : 1, m_\alpha : 1 (\alpha \in \mathbb{F}), \cdot : 2 \}.$$

A Σ_{alg} -structure is a tuple

$$\mathbf{A} = (H; +^{\mathbf{A}}, 0^{\mathbf{A}}, (-)^{\mathbf{A}}, (m_\alpha^{\mathbf{A}})_{\alpha \in \mathbb{F}}, \cdot^{\mathbf{A}})$$

with carrier $H \neq \emptyset$ and total interpretations of the symbols.

Definition 2.2 (Universe of (not-necessarily-associative) \mathbb{F} -algebras). Let $U_{\text{alg}} \subseteq \text{Str}_{\Sigma_{\text{alg}}}$ consist of those \mathbf{A} for which:

(A1) $(H, +, 0, -, (m_\alpha)_{\alpha \in \mathbb{F}})$ is a (left) \mathbb{F} -vector space;

(A2) $\cdot : H \times H \rightarrow H$ is bilinear, i.e., for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{F}$,

$$(\alpha x + \beta y) \cdot z = \alpha(x \cdot z) + \beta(y \cdot z), \quad z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(z \cdot y).$$

Remark 2.3 (Classical algebras as Σ_{alg} -structures). Every usual \mathbb{F} -algebra $(A, +, 0, -, (m_\alpha), \cdot)$ (associative or not; unital or not) yields $\mathbf{A} \in U_{\text{alg}}$ by taking $H := A$ and interpreting symbols as the given operations. If one wishes to restrict to associative (resp. unital) algebras, add the identities $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (resp. $\exists 1 \in H : 1 \cdot x = x = x \cdot 1$) as axioms on U_{alg} ; all constructions below preserve these properties.

Definition 2.4 (Meta-Algebra). A *Meta-Algebra* is a MetaStructure (Definition 1.2) over Σ_{alg}

$$\mathbb{M}_{\text{alg}} = (U_{\text{alg}}, (\Phi_\ell)_{\ell \in \Lambda}),$$

whose meta-operations are given uniformly by the following standard algebraic constructions (each specified by carrier- and symbol-constructors as in Definition 1.2):

(DS) Direct-sum algebra. For $\mathbf{A} = (A, \dots)$, $\mathbf{B} = (B, \dots) \in U_{\text{alg}}$ define

$$\Phi_{\oplus}(\mathbf{A}, \mathbf{B}) := (A \oplus B, +_{\oplus}, 0_{\oplus}, -_{\oplus}, (m_\alpha^{\oplus}), \cdot_{\oplus}),$$

on the carrier $A \oplus B := A \times B$ with

$$(u_1, v_1) +_{\oplus} (u_2, v_2) = (u_1 + u_2, v_1 + v_2), \quad 0_{\oplus} = (0, 0), \quad -(u, v) = (-u, -v), \quad m_\alpha^{\oplus}(u, v) = (\alpha u, \alpha v), \\ (u_1, v_1) \cdot_{\oplus} (u_2, v_2) := (u_1 \cdot u_2, v_1 \cdot v_2).$$

(TENS) Tensor-product algebra.

$$\Phi_{\otimes}(\mathbf{A}, \mathbf{B}) := (A \otimes_{\mathbb{F}} B, +, 0, -, (m_\alpha), \cdot_{\otimes}),$$

where the carrier is the tensor product vector space, and the multiplication is determined on simple tensors by

$$(a \otimes b) \cdot_{\otimes} (a' \otimes b') := (a \cdot a') \otimes (b \cdot b'),$$

and extended bilinearly to $A \otimes B$.

(OP) Opposite algebra.

$$\Phi_{\text{op}}(\mathbf{A}) := (A, +, 0, -, (m_\alpha), \cdot^{\text{op}}), \quad x \cdot^{\text{op}} y := y \cdot x.$$

Example 2.5 (Meta-Algebra in practice: block-diagonal coupling of independent subsystems). Let $\mathbf{A}_1 = (M_2(\mathbb{R}), +, \cdot)$ and $\mathbf{A}_2 = (M_3(\mathbb{R}), +, \cdot)$ be matrix algebras over \mathbb{R} representing two decoupled linear subsystems of state dimensions 2 and 3 (e.g., a thermal loop and a vibration loop in a device).

Applying the Meta-Algebra direct-sum constructor (DS) from Definition 2.4 gives

$$\Phi_{\oplus}(\mathbf{A}_1, \mathbf{A}_2) = (M_2(\mathbb{R}) \oplus M_3(\mathbb{R}), +_{\oplus}, \cdot_{\oplus}),$$

with $(A_1, B_1) \cdot_{\oplus} (A_2, B_2) = (A_1 A_2, B_1 B_2)$. The map

$$\iota : M_2(\mathbb{R}) \oplus M_3(\mathbb{R}) \longrightarrow M_5(\mathbb{R}), \quad \iota(A, B) = \text{diag}(A, B),$$

is an injective algebra homomorphism, identifying the direct sum with the block-diagonal subalgebra of $M_5(\mathbb{R})$. *Real-life interpretation.* The combined plant obtained by Φ_{\oplus} acts on the stacked state $(x_{\text{therm}}, x_{\text{vib}}) \in \mathbb{R}^2 \times \mathbb{R}^3$, and block-diagonal multiplication encodes that the two subsystems operate independently while being represented within a single algebraic object.

Theorem 2.6 (Meta-Algebra is a MetaStructure and generalizes algebras). $\mathbb{M}_{\text{alg}} = (U_{\text{alg}}, (\Phi_\ell))$ in Definition 2.4 is a MetaStructure over Σ_{alg} . Moreover, every classical \mathbb{F} -algebra appears as an object of U_{alg} ; hence Meta-Algebra generalizes algebras.

Proof. Uniform constructors and closure. (DS): The carrier $A \oplus B$ is a vector space with componentwise operations; \cdot_\oplus is bilinear in each argument since each component product is bilinear. Thus $\Phi_\oplus(\mathbf{A}, \mathbf{B}) \in U_{\text{alg}}$. If A, B are associative or unital, the direct-sum product preserves these properties componentwise.

(TENS): The tensor product carrier is a vector space. The rule $(a \otimes b) \cdot_\otimes (a' \otimes b') = (a \cdot a') \otimes (b \cdot b')$ respects the bilinear relations that define $A \otimes B$, hence extends uniquely to a bilinear map $(-)\cdot_\otimes (-) : (A \otimes B) \times (A \otimes B) \rightarrow A \otimes B$. Therefore $\Phi_\otimes(\mathbf{A}, \mathbf{B}) \in U_{\text{alg}}$. If A, B are associative (resp. unital), then

$$((a \otimes b) \cdot_\otimes (a' \otimes b')) \cdot_\otimes (a'' \otimes b'') = ((aa')a'') \otimes ((bb')b'')$$

equals

$$(a \otimes b) \cdot_\otimes ((a' \otimes b') \cdot_\otimes (a'' \otimes b'')) = a(a'a'') \otimes b(b'b'')$$

by associativity in A, B ; similarly $1_A \otimes 1_B$ is a unit when units exist.

(OP): (\cdot^{op}) is bilinear since

$$(\alpha x + \beta y) \cdot^{\text{op}} z = z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(z \cdot y) = \alpha(x \cdot^{\text{op}} z) + \beta(y \cdot^{\text{op}} z),$$

and similarly in the right argument; hence $\Phi_{\text{op}}(\mathbf{A}) \in U_{\text{alg}}$. Associativity and unitality are preserved under reversal.

Naturality (isomorphism invariance). Let $\alpha : \mathbf{A} \xrightarrow{\cong} \mathbf{A}'$ and $\beta : \mathbf{B} \xrightarrow{\cong} \mathbf{B}'$ be Σ_{alg} -isomorphisms (i.e., linear bijections preserving $+, 0, -, m_\alpha$ and \cdot).

(DS): Define $\alpha \oplus \beta : A \oplus B \rightarrow A' \oplus B'$ by $(\alpha \oplus \beta)(u, v) := (\alpha u, \beta v)$. It preserves the vector-space structure componentwise and

$$\begin{aligned} (\alpha \oplus \beta)((u_1, v_1) \cdot_\oplus (u_2, v_2)) &= (\alpha(u_1 \cdot u_2), \beta(v_1 \cdot v_2)) = (\alpha u_1 \cdot \alpha u_2, \beta v_1 \cdot \beta v_2) \\ &= (\alpha u_1, \beta v_1) \cdot_\oplus (\alpha u_2, \beta v_2), \end{aligned}$$

so it is a Σ_{alg} -isomorphism.

(TENS): There is a unique linear isomorphism $\alpha \otimes \beta : A \otimes B \rightarrow A' \otimes B'$ with $(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \otimes \beta(b)$. Then

$$\begin{aligned} (\alpha \otimes \beta)((a \otimes b) \cdot_\otimes (a' \otimes b')) &= (\alpha(a \cdot a')) \otimes (\beta(b \cdot b')) = (\alpha a \cdot \alpha a') \otimes (\beta b \cdot \beta b') \\ &= (\alpha a \otimes \beta b) \cdot_\otimes (\alpha a' \otimes \beta b'), \end{aligned}$$

hence $\alpha \otimes \beta$ is multiplicative; preservation of the vector-space part is clear.

(OP): The same linear bijection $\alpha : A \rightarrow A'$ is an isomorphism $A^{\text{op}} \xrightarrow{\cong} A'^{\text{op}}$ because

$$\alpha(x \cdot^{\text{op}} y) = \alpha(y \cdot x) = \alpha(y) \cdot \alpha(x) = \alpha(x) \cdot^{\text{op}} \alpha(y).$$

Thus each Φ_ℓ is natural. The generalization statement follows from the embedding remark above. □

Proposition 2.7 (Finite-dimensional size under \oplus and \otimes). If $\dim_{\mathbb{F}} A = m$ and $\dim_{\mathbb{F}} B = n$ (finite), then

$$\dim(A \oplus B) = m + n, \quad \dim(A \otimes_{\mathbb{F}} B) = mn.$$

Proof. Choose bases $\{e_i\}_{i=1}^m$ of A , $\{f_j\}_{j=1}^n$ of B . Then $\{(e_i, 0)\}_i \cup \{(0, f_j)\}_j$ is a basis of $A \oplus B$, and $\{e_i \otimes f_j\}_{i,j}$ is a basis of $A \otimes B$. □

Definition 2.8 (Iterated Meta-Algebra of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Algebra of depth t* is an Iterated MetaStructure over Σ_{alg} ,

$$\mathfrak{M}_{\text{alg}}^{(t)} = (U_{\text{alg}}^{(t)}, (\odot_{\ell}^{(t)})_{\ell \in \Lambda}),$$

obtained by repeatedly applying the lifting functor $U_{\Sigma_{\text{alg}}}$ to \mathbb{M}_{alg} . Concretely, if $s < t$ and a base meta-operation Φ_{ℓ} has meta-arity k_{ℓ} , its lift

$$\Phi_{\ell}^{\uparrow} : (U_{\text{alg}}^{(t)})^{k_{\ell}} \longrightarrow U_{\text{alg}}^{(t)}$$

acts on representatives by

$$\Phi_{\ell}^{\uparrow}(U_{\Sigma_{\text{alg}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{alg}}}^{t-s}(X_{k_{\ell}})) := U_{\Sigma_{\text{alg}}}^{t-s}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})),$$

and analogously on symbol interpretations. We write $\odot_{\oplus}^{(t)}, \odot_{\otimes}^{(t)}, \odot_{\text{op}}^{(t)}$ for the lifts of (DS), (TENS), (OP).

Example 2.9 (Iterated Meta-Algebra: tensor-then-sum for a coupled-plus-monitor model). Consider the local (dual-number) algebras

$$A = \mathbb{R}[x]/(x^2), \quad B = \mathbb{R}[y]/(y^2),$$

and a 2×2 matrix algebra $C = M_2(\mathbb{R})$. First apply the tensor-product constructor (TENS):

$$\Phi_{\otimes}(A, B) \cong \mathbb{R}[x, y]/(x^2, y^2),$$

where multiplication is determined by $x^2 = y^2 = 0$ and $xy = yx$. Next, apply the direct-sum constructor (DS) with C :

$$\Phi_{\oplus}(\Phi_{\otimes}(A, B), C) = (\mathbb{R}[x, y]/(x^2, y^2) \oplus M_2(\mathbb{R}), +_{\oplus}, \cdot_{\oplus}).$$

Real-life interpretation. The algebra $\mathbb{R}[x, y]/(x^2, y^2)$ captures first-order (infinitesimal) interactions between two small displacement/perturbation axes of a planar mechanism (no second-order terms), while the $M_2(\mathbb{R})$ block models an independent 2-state monitoring or actuation loop (e.g., a fast stabilizer). The iterated Meta-Algebra construction “couples then aggregates”: it first builds the joint infinitesimal interaction algebra of the two axes via Φ_{\otimes} , and then packs it together with the monitor via Φ_{\oplus} into one structured algebraic object, preserving the intended separations and interactions.

Theorem 2.10 (Iterated Meta-Algebra is an Iterated MetaStructure and generalizes Meta-Algebra). For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{alg}}^{(t)}$ of Definition 2.8 is an Iterated MetaStructure. Moreover, for $s < t$ the canonical embedding

$$\iota_{s \rightarrow t} : \mathfrak{M}_{\text{alg}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{alg}}^{(t)}, \quad X \longmapsto U_{\Sigma_{\text{alg}}}^{t-s}(X),$$

preserves all lifted meta-operations; in particular, $\mathfrak{M}_{\text{alg}}^{(0)} = \mathbb{M}_{\text{alg}}$ embeds into $\mathfrak{M}_{\text{alg}}^{(t)}$, so Iterated Meta-Algebra generalizes Meta-Algebra.

Proof. Each lifted constructor is obtained by post-composing the base constructors

$$(\Gamma_{\ell}, +^{\Phi_{\ell}}, 0^{\Phi_{\ell}}, (-)^{\Phi_{\ell}}, (m_{\alpha}^{\Phi_{\ell}}), (\cdot)^{\Phi_{\ell}})$$

with $U_{\Sigma_{\text{alg}}}^{t-s}$, hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are isomorphisms at level s , then

$$\begin{aligned} &U_{\Sigma_{\text{alg}}}^{t-s}(\alpha_i) : \\ &U_{\Sigma_{\text{alg}}}^{t-s}(X_i) \xrightarrow{\cong} \\ &U_{\Sigma_{\text{alg}}}^{t-s}(Y_i) \end{aligned}$$

are isomorphisms at level t , and

$$\Phi_{\ell}^{\uparrow}(U^{t-s}\alpha_1, \dots, U^{t-s}\alpha_{k_{\ell}}) = U^{t-s}(\Phi_{\ell}(\alpha_1, \dots, \alpha_{k_{\ell}})),$$

which is an isomorphism since each base Φ_{ℓ} is natural by Theorem 2.6. The operation-preservation of $\iota_{s \rightarrow t}$ is the same identity with X_i in place of α_i . Taking $s = 0$ gives the claimed embedding of \mathbb{M}_{alg} into every $\mathfrak{M}_{\text{alg}}^{(t)}$. □

3 MetaTopology: Topology of Topology

A topology is a collection of open sets on a set, defining continuity, convergence, and neighborhood structures in mathematical spaces.⁹⁻¹¹ A Meta-Topology is a MetaStructure where objects are topological spaces, with meta-operations combining, restricting, or transforming topologies while preserving isomorphisms. An Iterated Meta-Topology recursively applies Meta-Topology, creating higher-level hierarchies of topological systems, generalizing topologies across successive meta-level structural layers.

Definition 3.1 (Topology). (cf.^{12,13}) A topology on a nonempty set X is a family $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset, X \in \mathcal{T}$,
2. If $\{U_i\}_{i \in I} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$,
3. If $U_1, \dots, U_n \in \mathcal{T}$, then $\bigcap_{j=1}^n U_j \in \mathcal{T}$.

The pair (X, \mathcal{T}) is called a *topological space*.

We work single-sorted and encode points, open sets, and membership using unary/binary relations together with function symbols for finite unions/intersections.

Fix the (single-sorted, finitary) signature

$$\Sigma_{\text{top}} = \left(\text{Func}, \text{Rel}, \text{ar} \right),$$

where

$$\text{Rel} = \{ \text{Pts}, \text{Op}, \text{Inc} \}, \quad \text{ar}(\text{Pts}) = \text{ar}(\text{Op}) = 1, \quad \text{ar}(\text{Inc}) = 2,$$

and

$$\text{Func} = \{ \text{cup} : 2, \text{cap} : 2, \text{empty} : 0, \text{whole} : 0 \}.$$

A Σ_{top} -structure is a tuple

$$\mathbf{X} = (H; \text{Pts}^{\mathbf{X}}, \text{Op}^{\mathbf{X}}, \text{Inc}^{\mathbf{X}}; \text{cup}^{\mathbf{X}}, \text{cap}^{\mathbf{X}}, \text{empty}^{\mathbf{X}}, \text{whole}^{\mathbf{X}})$$

with $H \neq \emptyset$, interpreting relations and function symbols over H .

Definition 3.2 (Universe of finite topological spaces). Let $U_{\text{top}} \subseteq \text{Str}_{\Sigma_{\text{top}}}$ be the class of all \mathbf{X} satisfying:

(T1) **Typing/finite:** Pts is finite, nonempty. Elements u with $\text{Op}(u)$ are *open-set tokens*. If $\text{Inc}(x, U)$ then $\text{Pts}(x)$ and $\text{Op}(U)$.

(T2) **Topological constants:** $\text{Op}(\text{empty}), \text{Op}(\text{whole})$ and

$$\forall x (\text{Pts}(x) \Rightarrow \neg \text{Inc}(x, \text{empty}) \wedge \text{Inc}(x, \text{whole})).$$

(T3) **Closure and membership laws:** For all open tokens U, V and all points x ,

$$\text{Op}(\text{cup}(U, V)), \quad \text{Op}(\text{cap}(U, V)),$$

$$\text{Inc}(x, \text{cup}(U, V)) \iff \text{Inc}(x, U) \vee \text{Inc}(x, V),$$

$$\text{Inc}(x, \text{cap}(U, V)) \iff \text{Inc}(x, U) \wedge \text{Inc}(x, V).$$

(T4) **Extensionality of open tokens:** If $\text{Op}(U), \text{Op}(V)$ and $\forall x \in \text{Pts} : \text{Inc}(x, U) \iff \text{Inc}(x, V)$, then $U = V$.

Thus each $\mathbf{X} \in U_{\text{top}}$ encodes a finite topological space (X, \mathcal{T}) with $X := \{x \in H \mid \text{Pts}(x)\}$ and $\mathcal{T} := \{U \in H \mid \text{Op}(U)\}$, where membership is given by Inc and finite unions/intersections are realized by cup , cap .

Remark 3.3 (Embedding a classical finite topology). Given a finite topological space (X, \mathcal{T}) , let the carrier be $H := \{0\} \times X \cup \{1\} \times \mathcal{T}$. Declare

$$\text{Pts}(0, x), \quad \text{Op}(1, U), \quad \text{Inc}((0, x), (1, U)) \iff x \in U,$$

$$\text{empty} = (1, \emptyset), \quad \text{whole} = (1, X), \quad \text{cup}((1, U), (1, V)) = (1, U \cup V), \quad \text{cap}((1, U), (1, V)) = (1, U \cap V).$$

Then $\mathbf{X} \in U_{\text{top}}$.

Definition 3.4 (Meta-Topology). A *Meta-Topology* is a *MetaStructure* (Definition 1.2) over Σ_{top}

$$\mathbb{M}_{\text{top}} = (U_{\text{top}}, (\Phi_\ell)_{\ell \in \Lambda}),$$

whose meta-operations are specified uniformly as follows (each by carrier- and symbol-constructors).

(COPROD) Topological disjoint sum. For $\mathbf{X}_i = (H_i; \text{Pts}_i, \text{Op}_i, \text{Inc}_i; \text{cup}_i, \text{cap}_i, \text{empty}_i, \text{whole}_i) \in U_{\text{top}}$ ($i = 1, 2$) define

$$\Phi_\sqcup(\mathbf{X}_1, \mathbf{X}_2) := (H; \text{Pts}, \text{Op}, \text{Inc}; \text{cup}, \text{cap}, \text{empty}, \text{whole})$$

on the tagged carrier

$$H := \{0\} \times H_1 \cup \{1\} \times H_2,$$

with

$$\text{Pts}(t, u) \iff (t = 0 \wedge \text{Pts}_1(u)) \vee (t = 1 \wedge \text{Pts}_2(u)),$$

$$\text{Op}(t, U) \iff (t = 0 \wedge \text{Op}_1(U)) \vee (t = 1 \wedge \text{Op}_2(U)),$$

$$\text{Inc}((t, x), (t, U)) \iff \begin{cases} \text{Inc}_1(x, U), & t = 0, \\ \text{Inc}_2(x, U), & t = 1, \end{cases} \quad (\text{and never across different tags}),$$

$$\text{empty} := (0, \text{empty}_1) \sqcup (1, \text{empty}_2), \quad \text{whole} := (0, \text{whole}_1) \sqcup (1, \text{whole}_2),$$

$$\text{cup}((t, U), (t, V)) := (t, \text{cup}_t(U, V)), \quad \text{cap}((t, U), (t, V)) := (t, \text{cap}_t(U, V)),$$

and cup , cap are undefined across mismatched tags (i.e., their values are chosen in the same tag, which is well-defined because Op never mixes tags).

(RES) Subspace topology on a subset $S \subseteq \text{Pts}$. Let $\mathbf{X} \in U_{\text{top}}$ and $S \subseteq H$ with $S \subseteq \text{Pts}$. Define

$$\Phi_{\text{res}(S)}(\mathbf{X}) := (H'; \text{Pts}', \text{Op}', \text{Inc}'; \text{cup}', \text{cap}', \text{empty}', \text{whole}')$$

on the tagged carrier $H' := \{0\} \times S \cup \{1\} \times \text{Op}$ with

$$\text{Pts}'(0, x) \iff x \in S, \quad \text{Op}'(1, U) \iff \text{Op}(U),$$

$$\text{Inc}'((0, x), (1, U)) \iff (x \in S) \wedge \text{Inc}(x, U),$$

$$\text{empty}' := (1, \text{empty}), \quad \text{whole}' := (1, \text{whole}),$$

$$\text{cup}'((1, U), (1, V)) := (1, \text{cup}(U, V)), \quad \text{cap}'((1, U), (1, V)) := (1, \text{cap}(U, V)).$$

(Thus open tokens are of the form $U \cap S$, represented extensionally by membership.)

(PUSH) Relabeling points via a bijection $f : \text{Pts} \xrightarrow{\cong} K$. For $\mathbf{X} \in U_{\text{top}}$ and a bijection f from the point set to a finite set K , define

$$\Phi_{\text{push}(f)}(\mathbf{X}) := (\widehat{H}; \widehat{\text{Pts}}, \widehat{\text{Op}}, \widehat{\text{Inc}}; \widehat{\text{cup}}, \widehat{\text{cap}}, \widehat{\text{empty}}, \widehat{\text{whole}})$$

on $\widehat{H} := \{0\} \times K \cup \{1\} \times \text{Op}$ with

$$\widehat{\text{Pts}}(0, k), \quad \widehat{\text{Op}}(1, U) \iff \text{Op}(U),$$

$$\widehat{\text{Inc}}((0, k), (1, U)) \iff \text{Inc}(f^{-1}(k), U),$$

$$\widehat{\text{empty}} := (1, \text{empty}), \quad \widehat{\text{whole}} := (1, \text{whole}),$$

$$\widehat{\text{cup}}((1, U), (1, V)) = (1, \text{cup}(U, V)),$$

$$\widehat{\text{cap}}((1, U), (1, V)) = (1, \text{cap}(U, V)).$$

Example 3.5 (Meta-Topology in practice: disjointly gluing two finite spaces). Let (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) be finite topological spaces,

$$X_1 = \{a, b, c\}, \quad \mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X_1\}, \quad X_2 = \{u, v\}, \quad \mathcal{T}_2 = \{\emptyset, \{u\}, X_2\}.$$

Via the encoding of Remark 3.3, regard them as

$$\mathbf{X}_1, \mathbf{X}_2 \in U_{\text{top}}$$

. Applying the Meta-Topology (COPROD) constructor of Definition 3.4 yields the disjoint sum

$$\Phi_{\sqcup}(\mathbf{X}_1, \mathbf{X}_2) \in U_{\text{top}},$$

whose underlying set is the tagged union $X_1 \sqcup X_2 = \{0\} \times X_1 \cup \{1\} \times X_2$, and whose open subsets are precisely $U \sqcup V := \{0\} \times U \cup \{1\} \times V$ with $U \in \mathcal{T}_1, V \in \mathcal{T}_2$. *Interpretation.* One may view (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) as two independent “rooms” (finite sensor patches or network cells). The Meta-Topology disjoint sum glues them into a single space while preserving the topology on each component and forbidding cross-component membership.

Theorem 3.6 (Meta-Topology is a MetaStructure and generalizes topologies). $\mathbb{M}_{\text{top}} = (U_{\text{top}}, (\Phi_\ell))$ of Definition 3.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite topological space embeds as an object of U_{top} (Remark 3.3); hence Meta-Topology generalizes topology.

Proof. Uniform constructors and closure. For (COPROD), typing is immediate from tags. empty, whole are open in each tag and satisfy the membership clauses tagwise. Closure under cup, cap and their membership laws hold componentwise by the input axioms (T3). Extensionality (T4) holds tagwise since tokens never mix tags and the membership characterization is preserved.

For (RES), Pts' is finite/nonempty whenever S is nonempty. The induced open tokens correspond to sets $U \cap S$; empty', whole' satisfy (T2) relative to S ; cup', cap' obey the membership laws since, for all $x \in S$,

$$x \in (U \cup V) \cap S \Leftrightarrow (x \in U \cap S) \vee (x \in V \cap S), \quad x \in (U \cap V) \cap S \Leftrightarrow (x \in U \cap S) \wedge (x \in V \cap S).$$

Extensionality follows from pointwise membership on S .

For (PUSH), finiteness and typing are preserved. The membership laws and constants transport along f by definition, so (T2)–(T3) hold verbatim, and (T4) follows since equality of memberships after pullback by f^{-1} implies equality of tokens in the source, hence in the target.

Thus each output lies in U_{top} .

Naturality (isomorphism invariance). An isomorphism of Σ_{top} -structures is a bijection $\alpha : H \rightarrow H'$ preserving Pts, Op, Inc and the function symbols. If $\alpha_i : \mathbf{X}_i \rightarrow \mathbf{X}'_i$ are isomorphisms, then for (COPROD) the map $\alpha_{\sqcup}(t, u) := (t, \alpha_t(u))$ (with $\alpha_0 := \alpha_1, \alpha_1 := \alpha_2$) preserves all relations/functions componentwise. For (RES), restricting α to the tagged carrier respects S and transports the induced structures. For (PUSH), the given bijection f yields the relabeling, and any isomorphism of the source induces the same relabeling in the target via conjugation, preserving interpretations. Hence each Φ_ℓ is natural. The generalization claim is exactly Remark 3.3. \square

Proposition 3.7 (Behavior of opens under META operations). Let \mathbf{X}_i encode (X_i, \mathcal{T}_i) .

- (i) Disjoint sum: $\Phi_{\sqcup}(\mathbf{X}_1, \mathbf{X}_2)$ encodes $(X_1 \sqcup X_2, \{U_1 \sqcup U_2 \mid U_i \in \mathcal{T}_i\})$.
- (ii) Subspace: $\Phi_{\text{res}(S)}(\mathbf{X})$ encodes the subspace $(S, \{U \cap S \mid U \in \mathcal{T}\})$.
- (iii) Relabeling: $\Phi_{\text{push}(f)}(\mathbf{X})$ encodes a space homeomorphic to (X, \mathcal{T}) via f .

Proof. All three statements follow immediately from the membership clauses defining Inc in each construction. \square

Definition 3.8 (Iterated Meta-Topology of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Topology of depth t* is an Iterated MetaStructure over Σ_{top} ,

$$\mathfrak{M}_{\text{top}}^{(t)} = (U_{\text{top}}^{(t)}, (\odot_{\ell}^{(t)})_{\ell \in \Lambda}),$$

obtained by repeatedly applying the lifting functor $U_{\Sigma_{\text{top}}}$ to \mathbb{M}_{top} . Concretely, if $s < t$ and Φ_{ℓ} has meta-arity k_{ℓ} , its lift

$$\Phi_{\ell}^{\uparrow} : (U_{\text{top}}^{(t)})^{k_{\ell}} \longrightarrow U_{\text{top}}^{(t)}$$

is defined on representatives by

$$\Phi_{\ell}^{\uparrow}(U_{\Sigma_{\text{top}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{top}}}^{t-s}(X_{k_{\ell}})) := U_{\Sigma_{\text{top}}}^{t-s}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})),$$

and similarly for all symbol interpretations. We write $\odot_{\sqcup}^{(t)}$, $\odot_{\text{res}(S)}^{(t)}$, $\odot_{\text{push}(f)}^{(t)}$ for the lifts of (COPROD), (RES), (PUSH).

Example 3.9 (Iterated Meta-Topology (depth 2): restrict–relabel–glue pipeline). Start with two finite spaces

$$(X, \mathcal{T}) = (\{a, b, c\}, \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}), \quad (Y, \mathcal{S}) = (\{p, q, r\}, \{\emptyset, \{p\}, \{p, q\}, \{p, q, r\}\}),$$

encoded as $\mathbf{X}, \mathbf{Y} \in U_{\text{top}}$ (Remark 3.3). Perform the following meta-operations (each one an instance of a lifted constructor):

Step 1 (RES: subspace). Let $S = \{a, b\} \subseteq X$. Apply $\Phi_{\text{res}(S)}$ to obtain the subspace topology $\mathbf{X}_{|S} := \Phi_{\text{res}(S)}(\mathbf{X})$; classically $\mathcal{T}_{|S} = \{\emptyset, \{a\}, \{a, b\}\}$ on S .

Step 2 (PUSH: relabel points). Fix the bijection $f : S \xrightarrow{\cong} \{1, 2\}$ with $f(a) = 1, f(b) = 2$. Apply $\Phi_{\text{push}(f)}$ to get $\widehat{\mathbf{X}} := \Phi_{\text{push}(f)}(\mathbf{X}_{|S})$, whose underlying set is $\{1, 2\}$ and whose open sets are $f[U]$ for $U \in \mathcal{T}_{|S}$.

Step 3 (COPROD: disjoint sum). Glue the relabeled subspace with \mathbf{Y} by

$$\mathbf{Z} := \Phi_{\sqcup}(\widehat{\mathbf{X}}, \mathbf{Y}) \in U_{\text{top}}.$$

Classically, \mathbf{Z} is the disjoint union topology on $\{1, 2\} \sqcup \{p, q, r\}$ with opens $U' \sqcup V$ where $U' \in \{\emptyset, \{1\}, \{1, 2\}\}$ and $V \in \mathcal{S}$.

Interpretation. This “restrict–relabel–glue” pipeline models a typical workflow: crop a region of interest from one finite space (Step 1), rename its points to a new index set (Step 2), then assemble it with another space without creating artificial adjacency between components (Step 3). As an instance of Iterated Meta-Topology, the output is obtained by composing meta-operations, exactly as prescribed by the lifted constructors.

Theorem 3.10 (Iterated Meta-Topology is an Iterated MetaStructure and generalizes Meta-Topology). For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{top}}^{(t)}$ of Definition 3.8 is an Iterated MetaStructure. Moreover, for $s < t$ the canonical embedding

$$\iota_{s \rightarrow t} : \mathfrak{M}_{\text{top}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{top}}^{(t)}, \quad X \longmapsto U_{\Sigma_{\text{top}}}^{t-s}(X),$$

preserves all lifted meta-operations; in particular, $\mathfrak{M}_{\text{top}}^{(0)} = \mathbb{M}_{\text{top}}$ embeds into $\mathfrak{M}_{\text{top}}^{(t)}$, so Iterated Meta-Topology generalizes Meta-Topology.

Proof. The lifted constructors are obtained by post-composing the base constructors

$$(\Gamma_{\ell}; \text{cup}^{\Phi_{\ell}}, \text{cap}^{\Phi_{\ell}}, \text{empty}^{\Phi_{\ell}}, \text{whole}^{\Phi_{\ell}}; \text{Pts}^{\Phi_{\ell}}, \text{Op}^{\Phi_{\ell}}, \text{Inc}^{\Phi_{\ell}})$$

with $U_{\Sigma_{\text{top}}}^{t-s}$; hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are isomorphisms at level s , then

$$U_{\Sigma_{\text{top}}}^{t-s}(\alpha_i) : U_{\Sigma_{\text{top}}}^{t-s}(X_i) \xrightarrow{\cong} U_{\Sigma_{\text{top}}}^{t-s}(Y_i)$$

are isomorphisms at level t , and

$$\Phi_{\ell}^{\uparrow}(U^{t-s}\alpha_1, \dots, U^{t-s}\alpha_{k_{\ell}}) = U^{t-s}(\Phi_{\ell}(\alpha_1, \dots, \alpha_{k_{\ell}})),$$

which is an isomorphism since each base Φ_{ℓ} is natural by Theorem 3.6. Operation preservation of $\iota_{s \rightarrow t}$ is the same identity with X_i in place of α_i . Taking $s = 0$ yields the claimed embedding. \square

4 MetaLattice: Lattice of Lattice

A lattice is a partially ordered set where every pair of elements has a unique meet (infimum) and join (supremum).^{14,15} A Meta-Lattice is a MetaStructure treating lattices as objects, with meta-operations combining, modifying, or transforming them while preserving isomorphisms. An Iterated Meta-Lattice recursively applies Meta-Lattice construction, forming hierarchical layers of lattices of lattices, generalizing classical lattice theory across meta-levels.

Definition 4.1 (Lattice). (cf.¹⁶⁻¹⁸) A *lattice* is a partially ordered set (L, \leq) in which every pair of elements $x, y \in L$ has both a least upper bound (join) $x \vee y$ and a greatest lower bound (meet) $x \wedge y$. Formally, for all $x, y \in L$,

$$x \vee y = \inf\{z \in L \mid x \leq z, y \leq z\}, \quad x \wedge y = \sup\{z \in L \mid z \leq x, z \leq y\}.$$

Fix the single-sorted, finitary signature

$$\Sigma_{\text{lat}} = (\text{Func}, \text{Rel}, \text{ar}), \quad \text{Func} = \{\text{join} : 2, \text{meet} : 2\}, \quad \text{Rel} = \{\text{El} : 1, \text{Leq} : 2\}.$$

A Σ_{lat} -structure is a tuple

$$\mathbf{L} = (H; \text{El}^{\mathbf{L}}, \text{Leq}^{\mathbf{L}}; \text{join}^{\mathbf{L}}, \text{meet}^{\mathbf{L}}),$$

with carrier $H \neq \emptyset$.

Definition 4.2 (Universe of finite lattices). Let

$$U_{\text{lat}} \subseteq \text{Str}_{\Sigma_{\text{lat}}}$$

be the class of \mathbf{L} satisfying:

- (L1) **Typing/finite:** El is finite, nonempty; if $\text{Leq}(x, y)$ then $\text{El}(x), \text{El}(y)$; and $\text{join}, \text{meet} : \text{El} \times \text{El} \rightarrow \text{El}$ are total.
- (L2) **Lattice laws:** For all $x, y, z \in \text{El}$, writing $x \vee y := \text{join}(x, y)$ and $x \wedge y := \text{meet}(x, y)$, the four lattice axiom families hold: commutativity, associativity, idempotence, and absorption (as displayed above).
- (L3) **Order compatibility:** Leq is a partial order on El and

$$\text{Leq}(x, y) \iff x \wedge y = x \iff x \vee y = y \quad (x, y \in \text{El}).$$

Then each $\mathbf{L} \in U_{\text{lat}}$ canonically encodes a finite lattice (L, \vee, \wedge, \leq) with $L = \{x \in H \mid \text{El}(x)\}$, \vee, \wedge as above, and \leq given by Leq.

Remark 4.3 (Embedding a classical finite lattice). Given a finite lattice (L, \vee, \wedge, \leq) , let $H := L$,

$$\text{El}(x) \iff x \in L, \quad \text{Leq}(x, y) \iff x \leq y, \quad \text{join}(x, y) := x \vee y, \quad \text{meet}(x, y) := x \wedge y.$$

Then $\mathbf{L} \in U_{\text{lat}}$ by the standard lattice axioms.

Definition 4.4 (Meta-Lattice). A *Meta-Lattice* is a MetaStructure (Definition 1.2) over Σ_{lat} ,

$$\mathbb{M}_{\text{lat}} = (U_{\text{lat}}, (\Phi_{\ell})_{\ell \in \Lambda}),$$

whose meta-operations are given uniformly as follows (each by carrier- and symbol-constructors).

(PROD) Direct product. For $\mathbf{L}_i \in U_{\text{lat}}$ ($i = 1, 2$), define

$$\Phi_{\times}(\mathbf{L}_1, \mathbf{L}_2) := (H; \text{El}, \text{Leq}; \text{join}, \text{meet})$$

on $H := \text{El}_1 \times \text{El}_2$ with

$$\text{El}((x_1, x_2)), \quad \text{Leq}(((x_1, x_2), (y_1, y_2))) \iff \text{Leq}_1(x_1, y_1) \wedge \text{Leq}_2(x_2, y_2),$$

$$\text{join}((x_1, x_2), (y_1, y_2)) := (x_1 \vee_1 y_1, x_2 \vee_2 y_2), \quad \text{meet}((x_1, x_2), (y_1, y_2)) := (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2).$$

(RES) Sublattice restriction by a closed subset. Let $\mathbf{L} \in U_{\text{lat}}$ and $S \subseteq \text{El}$ be nonempty and closed under \vee, \wedge . Define

$$\Phi_{\text{res}(S)}(\mathbf{L}) := (H'; \text{El}', \text{Leq}'; \text{join}', \text{meet}')$$

on $H' := S$ with the obvious restrictions of $\text{El}, \text{Leq}, \text{join}, \text{meet}$ to S .

(PUSH) Relabeling along a bijection $f : \text{El} \xrightarrow{\cong} K$. For $\mathbf{L} \in U_{\text{lat}}$ and f a bijection, set

$$\Phi_{\text{push}(f)}(\mathbf{L}) = (K; \widehat{\text{El}}, \widehat{\text{Leq}}; \widehat{\text{join}}, \widehat{\text{meet}})$$

with $\widehat{\text{El}}(k) \iff k \in K$ and

$$\widehat{\text{Leq}}(f(x), f(y)) \iff \text{Leq}(x, y), \quad \widehat{\text{join}}(f(x), f(y)) := f(\text{join}(x, y)), \quad \widehat{\text{meet}}(f(x), f(y)) := f(\text{meet}(x, y)).$$

(DUAL) Order dual. For $\mathbf{L} \in U_{\text{lat}}$, let

$$\Phi_{\text{dual}}(\mathbf{L}) := (H; \text{El}, \text{Leq}^{\text{op}}; \text{join}^{\text{op}}, \text{meet}^{\text{op}}),$$

where $\text{Leq}^{\text{op}}(x, y) \iff \text{Leq}(y, x)$, and $\text{join}^{\text{op}} := \text{meet}$, $\text{meet}^{\text{op}} := \text{join}$.

Example 4.5 (Meta-Lattice in practice: combining permissions and urgency). Consider two finite lattices:

(A) *Permissions as a Boolean lattice.* Let $L_{\text{perm}} := \mathcal{P}(\{r, w\}) = \{\emptyset, \{r\}, \{w\}, \{r, w\}\}$ with $\text{join } U \vee V := U \cup V$, $\text{meet } U \wedge V := U \cap V$, and order $U \leq V \iff U \subseteq V$.

(B) *Urgency as a chain.* Let $L_{\text{urg}} := \{\text{lo} < \text{med} < \text{hi}\}$ with $\text{join } u \vee v := \max\{u, v\}$ and $\text{meet } u \wedge v := \min\{u, v\}$.

By the Meta-Lattice product constructor (Definition 4.4, (PROD)), their direct product is the lattice

$$L_{\text{perm}} \times L_{\text{urg}}$$

with componentwise order and operations. Intuitively, (U, u) records an access level $U \subseteq \{r, w\}$ paired with an urgency $u \in \{\text{lo}, \text{med}, \text{hi}\}$. For instance,

$$(\{r\}, \text{lo}) \vee (\{w\}, \text{med}) = (\{r, w\}, \text{med}), \quad (\{r\}, \text{lo}) \wedge (\{w\}, \text{med}) = (\emptyset, \text{lo}).$$

This Meta-Lattice object encodes policy aggregation: joins combine permissions and take the more urgent level; meets intersect permissions and take the less urgent level.

Theorem 4.6 (Meta-Lattice is a MetaStructure and generalizes lattices). $\mathbb{M}_{\text{lat}} = (U_{\text{lat}}, (\Phi_\ell))$ of Definition 4.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite lattice (L, \vee, \wedge, \leq) embeds as an object of U_{lat} (Remark 4.3); hence Meta-Lattice generalizes lattices.

Proof. Uniform constructors and closure. (PROD): Finiteness and typing are clear. The operations are componentwise, hence (L2) holds by checking each lattice law coordinatewise. The order is product order, a partial order, and

$$(x_1, x_2) \vee (y_1, y_2) = (y_1, y_2) \iff (x_1 \vee_1 y_1 = y_1) \wedge (x_2 \vee_2 y_2 = y_2) \iff \text{Leq}_1(x_1, y_1) \wedge \text{Leq}_2(x_2, y_2),$$

so (L3) holds (similarly with \wedge).

(RES): Closure of S under \vee, \wedge makes the restricted operations total on S and preserves the lattice laws. Leq' is a partial order and the characterizations in (L3) remain valid since joins/meets in S coincide with those computed in \mathbf{L} by closure.

(PUSH): Transport of structure along a bijection preserves finiteness, lattice laws, and the equivalences in (L3) by definition.

(DUAL): Swapping \vee and \wedge and reversing \leq preserves (L2); (L3) holds because

$$x \leq_{\text{op}} y \iff y \leq x \iff x \vee y = y \iff x \wedge_{\text{op}} y = x,$$

and similarly with \vee_{op} .

Thus each output lies in U_{lat} .

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{L} \rightarrow \mathbf{L}'$ is a bijection $\alpha : H \rightarrow H'$ with α preserving El, Leq and commuting with join, meet. Given isomorphisms $\alpha_i : \mathbf{L}_i \rightarrow \mathbf{L}'_i$, define $\alpha_{\times}(x_1, x_2) := (\alpha_1 x_1, \alpha_2 x_2)$ for (PROD) and check preservation componentwise. For (RES), restrict α to S . For (PUSH), use the given f as the underlying bijection. For (DUAL), the same α intertwines the swapped symbols. Hence every Φ_{ℓ} is natural. The generalization claim is Remark 4.3. \square

Proposition 4.7 (Joins/meets under META operations). *Let \mathbf{L}_i encode $(L_i, \vee_i, \wedge_i, \leq_i)$.*

(i) Product: $\Phi_{\times}(\mathbf{L}_1, \mathbf{L}_2)$ encodes $(L_1 \times L_2, (x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_1 y_1, x_2 \vee_2 y_2), (x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2))$.

(ii) Sublattice: $\Phi_{\text{res}(S)}(\mathbf{L})$ encodes the sublattice generated by S (here, S is assumed closed).

(iii) Dual: $\Phi_{\text{dual}}(\mathbf{L})$ encodes the order-dual lattice (L, \wedge, \vee, \geq) .

Proof. Immediate from the definitions of the constructors. \square

Definition 4.8 (Iterated Meta-Lattice of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Lattice of depth t* is an Iterated MetaStructure over Σ_{lat} ,

$$\mathfrak{M}_{\text{lat}}^{(t)} = (U_{\text{lat}}^{(t)}, (\odot_{\ell}^{(t)})_{\ell \in \Lambda}),$$

obtained by repeatedly applying the lifting $U_{\Sigma_{\text{lat}}}$ to \mathbb{M}_{lat} . Concretely, if $s < t$ and Φ_{ℓ} has meta-arity k_{ℓ} , its lift

$$\Phi_{\ell}^{\uparrow} : (U_{\text{lat}}^{(t)})^{k_{\ell}} \longrightarrow U_{\text{lat}}^{(t)}$$

is defined on representatives by

$$\Phi_{\ell}^{\uparrow}(U_{\Sigma_{\text{lat}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{lat}}}^{t-s}(X_{k_{\ell}})) := U_{\Sigma_{\text{lat}}}^{t-s}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})),$$

and similarly for all symbol interpretations. We denote by $\odot_{\times}^{(t)}$, $\odot_{\text{res}(S)}^{(t)}$, $\odot_{\text{push}(f)}^{(t)}$, $\odot_{\text{dual}}^{(t)}$ the lifts of (PROD), (RES), (PUSH), (DUAL).

Example 4.9 (Iterated Meta-Lattice (depth 2): restrict–dualize–product pipeline). Start from the lattices in Example 4.5 and apply a sequence of meta-operations (each an instance of the lifted constructors in Definition 4.8):

Step 1 (RES: sublattice restriction). Let $S := \{\emptyset, \{r\}, \{r, w\}\} \subseteq L_{\text{perm}}$. Since S is closed under \cup, \cap , the restriction

$$\Phi_{\text{res}(S)}(L_{\text{perm}})$$

is the sublattice $L_{\text{perm}}^{\text{S}}$ with elements S and the inherited \vee, \wedge .

Step 2 (DUAL: order reversal on urgency). Apply the dual constructor to the urgency chain to obtain

$$\Phi_{\text{dual}}(L_{\text{urg}}) =: L_{\text{urg}}^{\text{op}},$$

where the order is reversed (hi < med < lo), so joins/meets swap: in $L_{\text{urg}}^{\text{op}}$ the join is min and the meet is max (with respect to the original order).

Step 3 (PROD: combine the results). Form the direct product

$$\Phi_{\times} (L_{\text{perm}}^S, L_{\text{urg}}^{\text{op}}),$$

a lattice of pairs (U, \tilde{u}) with $U \in S$ and $\tilde{u} \in L_{\text{urg}}^{\text{op}}$, ordered and operated componentwise.

A concrete calculation (componentwise join) illustrates the composition:

$$(\{r\}, \text{med}) \vee (\{r, w\}, \text{lo}) = (\{r\} \cup \{r, w\}, \min(\text{med}, \text{lo})) = (\{r, w\}, \text{lo}),$$

where the second component uses the *dual* (min-as-join) on urgency. This iterated Meta-Lattice construction models a common workflow: (1) restrict attention to a closed family of permissions, (2) flip the priority convention (e.g., to express “ease” instead of “urgency”), then (3) aggregate both dimensions into a single lattice of policies.

Theorem 4.10 (Iterated Meta-Lattice is an Iterated MetaStructure and generalizes Meta-Lattice). *For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{lat}}^{(t)}$ of Definition 4.8 is an Iterated MetaStructure. Moreover, for $s < t$ the embedding*

$$\iota_{s \rightarrow t} : \mathfrak{M}_{\text{lat}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{lat}}^{(t)}, \quad X \mapsto \mathbb{U}_{\Sigma_{\text{lat}}}^{t-s}(X),$$

preserves all lifted meta-operations:

$$\Phi_{\ell}^{\uparrow}(\iota_{s \rightarrow t}(X_1), \dots, \iota_{s \rightarrow t}(X_{k_{\ell}})) = \iota_{s \rightarrow t}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})).$$

In particular, $\mathfrak{M}_{\text{lat}}^{(0)} = \mathbb{M}_{\text{lat}}$ embeds into $\mathfrak{M}_{\text{lat}}^{(t)}$, so Iterated Meta-Lattice generalizes Meta-Lattice.

Proof. Each lifted constructor is obtained by post-composing the base constructors

$$(\Gamma_{\ell}; \text{join}^{\Phi_{\ell}}, \text{meet}^{\Phi_{\ell}}; \text{El}^{\Phi_{\ell}}, \text{Leq}^{\Phi_{\ell}})$$

with $\mathbb{U}_{\Sigma_{\text{lat}}}^{t-s}$; hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level- s isomorphisms, then

$$\mathbb{U}_{\Sigma_{\text{lat}}}^{t-s}(\alpha_i) : \mathbb{U}_{\Sigma_{\text{lat}}}^{t-s}(X_i) \xrightarrow{\cong} \mathbb{U}_{\Sigma_{\text{lat}}}^{t-s}(Y_i)$$

are level- t isomorphisms, and

$$\Phi_{\ell}^{\uparrow}(\mathbb{U}^{t-s}\alpha_1, \dots, \mathbb{U}^{t-s}\alpha_{k_{\ell}}) = \mathbb{U}^{t-s}(\Phi_{\ell}(\alpha_1, \dots, \alpha_{k_{\ell}})),$$

which is an isomorphism because each base Φ_{ℓ} is natural by Theorem 4.6. Operation preservation of $\iota_{s \rightarrow t}$ is the same identity with X_i in place of α_i . Taking $s = 0$ yields the claimed embedding. \square

5 MetaQueue: Queue of Queues

A *queue* is a FIFO (first-in–first-out) data structure supporting insertion at the rear (*enqueue*) and removal at the front (*dequeue*) (cf.^{19,20}). A *Meta-Queue* is a MetaStructure whose objects are *entire queue systems* (i.e., algebras of queue states), and whose meta-operations combine, transform, restrict, or relabel such systems in a uniform and isomorphism-invariant way. An *Iterated Meta-Queue* applies the Meta-Queue construction repeatedly, producing hierarchical layers (queues of queues, or more generally, structures built from queue structures).

Definition 5.1 (FIFO queue as a transition system). (cf.^{21,22}) Fix a (typically finite) alphabet E of *elements*. A *FIFO queue system* is a triple

$$\mathbf{Q} = (Q, \text{enq}, \text{deq}),$$

where Q is a nonempty set of *queue states*, $\text{enq} : Q \times E \rightarrow Q$ is a total function (enqueue at the rear), and deq is a partial function that, when defined, removes the head:

$$\text{deq} : Q \setminus Q_{\text{emp}} \rightarrow E \times Q,$$

where $Q_{\text{emp}} \subseteq Q$ is the set of empty states. The FIFO discipline is expressed by the axioms:

(F1) **Empty/front:** for all $q \in Q_{\text{emp}}$ and $a \in E$, $\text{deq}(\text{enq}(q, a)) = (a, q)$.

(F2) **Head stability:** for all $q \notin Q_{\text{emp}}$, if $\text{deq}(q) = (a_1, q_1)$, then for all $a \in E$,

$$\text{deq}(\text{enq}(q, a)) = (a_1, \text{enq}(q_1, a)).$$

5.1 A first-order signature for queue systems

We encode queue systems as Σ_{que} -structures using a single-sorted, finitary relational signature

$$\Sigma_{\text{que}} = (\text{Func} = \emptyset, \text{Rel} = \{ \text{Queues}, \text{Elems}, \text{Emp}, \text{Enq}, \text{Deq} \}, \text{ar}),$$

with intended arities

$$\text{Queues} : 1, \quad \text{Elems} : 1, \quad \text{Emp} : 1, \quad \text{Enq} : 3, \quad \text{Deq} : 3.$$

A Σ_{que} -structure is a tuple

$$\mathbf{Q} = (H; \text{Queues}^{\mathbf{Q}}, \text{Elems}^{\mathbf{Q}}, \text{Emp}^{\mathbf{Q}}, \text{Enq}^{\mathbf{Q}}, \text{Deq}^{\mathbf{Q}}),$$

where $H \neq \emptyset$ is the carrier, $\text{Queues}, \text{Elems}, \text{Emp} \subseteq H$ are unary relations, and $\text{Enq}, \text{Deq} \subseteq H^3$ are ternary relations. Intuitively, Queues marks queue states, Elems marks elements, Emp marks empty queue states, $\text{Enq}(q, a, q')$ means “enqueue a to q yields q' ,” and $\text{Deq}(q, q', a)$ means “dequeue q yields head a and remainder q' .”

Definition 5.2 (Universe of FIFO queue systems). Let $U_{\text{que}} \subseteq \text{Str}_{\Sigma_{\text{que}}}$ be the class of all $\mathbf{Q} = (H; \text{Queues}, \text{Elems}, \text{Emp}, \text{Enq}, \text{Deq})$ satisfying the following axioms:

(Q1) **Typing and disjointness.** $\text{Queues}, \text{Elems} \subseteq H$ are nonempty and disjoint, and every tuple in Enq or Deq is well-typed:

$$\text{Enq}(q, a, q') \Rightarrow q, q' \in \text{Queues} \wedge a \in \text{Elems}, \quad \text{Deq}(q, q', a) \Rightarrow q, q' \in \text{Queues} \wedge a \in \text{Elems}.$$

Moreover, $\text{Emp} \subseteq \text{Queues}$.

(Q2) **Enqueue is total and functional.** For every $q \in \text{Queues}$ and $a \in \text{Elems}$ there exists a unique $q' \in \text{Queues}$ such that $\text{Enq}(q, a, q')$.

(Q3) **Dequeue is functional with emptiness guard.**

$$\text{Emp}(q) \Rightarrow \neg \exists a \in \text{Elems}, q' \in \text{Queues} : \text{Deq}(q, q', a),$$

and

$$\neg \text{Emp}(q) \Rightarrow \exists! (a, q') \in \text{Elems} \times \text{Queues} : \text{Deq}(q, q', a).$$

(Q4) **FIFO head stability (nonempty case).** For all $q, q_1, q_2, q' \in \text{Queues}$ and $a, a_1 \in \text{Elems}$,

$$\left(\text{Deq}(q, q_1, a_1) \wedge \text{Enq}(q_1, a, q_2) \wedge \text{Enq}(q, a, q') \right) \Longrightarrow \text{Deq}(q', q_2, a_1).$$

(Q5) **Empty/front compatibility (empty case).** For all $q, q_1 \in \text{Queues}$ and $a \in \text{Elems}$,

$$\text{Emp}(q) \wedge \text{Enq}(q, a, q_1) \Longrightarrow \text{Deq}(q_1, q, a).$$

Then every $\mathbf{Q} \in U_{\text{que}}$ encodes a FIFO queue system in the sense of Definition 5.1, where the queue-state set is Queues and the element set is Elems .

Remark 5.3 (Canonical model and finiteness). Let E be a nonempty alphabet and E^* the set of all finite strings over E . Define a Σ_{que} -structure on the carrier

$$H := \underbrace{\{0\} \times E^*}_{\text{queue states}} \cup \underbrace{\{1\} \times E}_{\text{elements}}$$

by

$$\begin{aligned} \text{Queues}(0, w) &\iff w \in E^*, & \text{Elems}(1, e) &\iff e \in E, & \text{Emp}(0, w) &\iff w = \epsilon, \\ \text{Enq}((0, w), (1, e), (0, w')) &\iff w' = w \cdot e, & \text{Deq}((0, w), (0, w'), (1, e)) &\iff w = e \cdot w'. \end{aligned}$$

Then axioms (Q1)–(Q5) hold, so this canonical FIFO queue belongs to U_{que} . If one needs *finite-state* queues, one may instead restrict to strings of length $\leq L$, yielding a finite Queues -set.

Definition 5.4 (Meta-Queue). A *Meta-Queue* is a MetaStructure (Definition 1.2) over Σ_{que} ,

$$\mathbb{M}_{\text{que}} = (U_{\text{que}}, \{\Phi_\ell\}_{\ell \in \Lambda}),$$

whose basic meta-operations are the following uniform constructors.

(PROD) Parallel product (componentwise FIFO). For $\mathbf{Q}_i = (H_i; \text{Queues}_i, \text{Elems}_i, \text{Emp}_i, \text{Enq}_i, \text{Deq}_i) \in U_{\text{que}}$ ($i = 1, 2$), define

$$\Phi_\times(\mathbf{Q}_1, \mathbf{Q}_2) := (H; \text{Queues}, \text{Elems}, \text{Emp}, \text{Enq}, \text{Deq}),$$

where the carrier is *explicitly tagged* to ensure disjointness:

$$\begin{aligned} H &:= (\{0\} \times \text{Queues}_1 \times \text{Queues}_2) \cup (\{1\} \times \text{Elems}_1 \times \text{Elems}_2), \\ \text{Queues} &:= \{0\} \times \text{Queues}_1 \times \text{Queues}_2, \\ \text{Elems} &:= \{1\} \times \text{Elems}_1 \times \text{Elems}_2, \\ \text{Emp}(0, q_1, q_2) &\iff \text{Emp}_1(q_1) \wedge \text{Emp}_2(q_2), \\ \text{Enq}((0, q_1, q_2), (1, a_1, a_2), (0, q'_1, q'_2)) &\iff \text{Enq}_1(q_1, a_1, q'_1) \wedge \text{Enq}_2(q_2, a_2, q'_2), \\ \text{Deq}((0, q_1, q_2), (0, q'_1, q'_2), (1, a_1, a_2)) &\iff \text{Deq}_1(q_1, q'_1, a_1) \wedge \text{Deq}_2(q_2, q'_2, a_2). \end{aligned}$$

(COPROD) Disjoint union (sum of independent queue systems). Define

$$\Phi_\sqcup(\mathbf{Q}_1, \mathbf{Q}_2) := (H; \text{Queues}, \text{Elems}, \text{Emp}, \text{Enq}, \text{Deq})$$

on the tagged carrier

$$H := \{0\} \times H_1 \cup \{1\} \times H_2,$$

with

$$\begin{aligned} \text{Queues} &:= (\{0\} \times \text{Queues}_1) \cup (\{1\} \times \text{Queues}_2), \\ \text{Elems} &:= (\{0\} \times \text{Elems}_1) \cup (\{1\} \times \text{Elems}_2), \\ \text{Emp}(0, x) &\iff \text{Emp}_1(x), \quad \text{Emp}(1, y) \iff \text{Emp}_2(y), \end{aligned}$$

and queue dynamics *within each tag* only:

$$\begin{aligned} \text{Enq}((0, q), (0, a), (0, q')) &\iff \text{Enq}_1(q, a, q'), & \text{Enq}((1, q), (1, a), (1, q')) &\iff \text{Enq}_2(q, a, q'), \\ \text{Deq}((0, q), (0, q'), (0, a)) &\iff \text{Deq}_1(q, q', a), & \text{Deq}((1, q), (1, q'), (1, a)) &\iff \text{Deq}_2(q, q', a), \end{aligned}$$

and mixed-tag triples never satisfy Enq or Deq.

(RES) Closed sub-queue subsystem. Let $\mathbf{Q} = (H; \text{Queues}, \text{Elems}, \text{Emp}, \text{Enq}, \text{Deq}) \in U_{\text{que}}$. A pair (S_Q, S_E) with $S_Q \subseteq \text{Queues}$, $S_E \subseteq \text{Elems}$ is *admissible* if

$$\forall q \in S_Q \forall a \in S_E \exists! q' \in S_Q : \text{Enq}(q, a, q'),$$

and

$$\forall q \in S_Q \left(\neg \text{Emp}(q) \Rightarrow \exists! (a, q') \in S_E \times S_Q : \text{Deq}(q, q', a) \right).$$

For admissible (S_Q, S_E) , define the restriction

$$\Phi_{\text{res}(S_Q, S_E)}(\mathbf{Q}) := (S_Q \cup S_E; S_Q, S_E, \text{Emp}|_{S_Q}, \text{Enq}|_{S_Q \times S_E \times S_Q}, \text{Deq}|_{S_Q \times S_Q \times S_E}).$$

(PUSH) Element relabeling along a bijection $g : \text{Elems} \xrightarrow{\cong} K$. Let $\mathbf{Q} \in U_{\text{que}}$ and g be a bijection. Define

$$\Phi_{\text{push}(g)}(\mathbf{Q}) := (H'; \text{Queues}', \text{Elems}', \text{Emp}', \text{Enq}', \text{Deq}'),$$

on $H' := \text{Queues} \cup K$ by $\text{Queues}' = \text{Queues}$, $\text{Elems}' = K$, $\text{Emp}' = \text{Emp}$, and

$$\text{Enq}'(q, g(a), q') \iff \text{Enq}(q, a, q'), \quad \text{Deq}'(q, q', g(a)) \iff \text{Deq}(q, q', a).$$

Example 5.5 (Parallel product in practice: synchronized two-lane service). Let $\mathbf{Q}_{\text{reg}}, \mathbf{Q}_{\text{exp}} \in U_{\text{que}}$ model two FIFO service lanes. Their parallel product

$$\mathbf{Q}_{\text{sys}} := \Phi_{\times}(\mathbf{Q}_{\text{reg}}, \mathbf{Q}_{\text{exp}})$$

has queue states $(q_{\text{reg}}, q_{\text{exp}})$ and elements $(a_{\text{reg}}, a_{\text{exp}})$, with enqueue and dequeue acting componentwise.

For instance, starting from a state $(q_{\text{reg}}, q_{\text{exp}})$, a synchronized arrival of $(a_{\text{reg}}, a_{\text{exp}})$ produces

$$(q_{\text{reg}}, q_{\text{exp}}) \xrightarrow{\text{Enq}} (\text{enq}_{\text{reg}}(q_{\text{reg}}, a_{\text{reg}}), \text{enq}_{\text{exp}}(q_{\text{exp}}, a_{\text{exp}})),$$

and a synchronized service step removes the two heads simultaneously when both are nonempty. This illustrates how Meta-Queue operations uniformly construct multi-queue systems while preserving FIFO behavior within each component.

Theorem 5.6 (Meta-Queue is a MetaStructure). $\mathbb{M}_{\text{que}} = (U_{\text{que}}, \{\Phi_{\ell}\}_{\ell \in \Lambda})$ of Definition 5.4 is a MetaStructure (Definition 1.2). Moreover, the canonical FIFO queue of Remark 5.3 is an object of U_{que} ; hence Meta-Queue generalizes classical queues.

Proof. Closure. For **(PROD)**, each axiom (Q1)–(Q5) holds componentwise: typing is enforced by tags, Enq is total/functional because each Enq_i is, Deq is functional with emptiness guard because each Deq_i is, and FIFO axioms (Q4)–(Q5) follow by applying them in each component. For **(COPROD)**, tags isolate the two components; inside each tag the source axioms hold and mixed-tag tuples never activate Enq or Deq. For **(RES)**, admissibility guarantees that the restricted Enq remains total/functional on $S_Q \times S_E$, and that Deq remains functional on nonempty states in S_Q ; universal FIFO axioms restrict automatically. For **(PUSH)**, we only relabel elements along a bijection; all axioms are preserved by transport.

Naturality (isomorphism invariance). A Σ_{que} -isomorphism $\alpha : \mathbf{Q} \rightarrow \mathbf{Q}'$ is a bijection of carriers preserving

$$\text{Queues, Elems, Emp, Enq, Deq}$$

. Each constructor Φ_{ℓ} is defined by uniform tagging/products/restrictions/relabelings on these relations, hence sends isomorphic inputs to isomorphic outputs via the induced maps (componentwise for Φ_{\times} , tagwise for Φ_{\sqcup} , restricted for Φ_{res} , and transported by g for Φ_{push}). Therefore \mathbb{M}_{que} is a MetaStructure. Finally, Remark 5.3 embeds the standard FIFO queue into U_{que} . \square

Definition 5.7 (Iterated Meta-Queue of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Queue of depth t* is an Iterated MetaStructure over Σ_{que} ,

$$\mathfrak{M}_{\text{que}}^{(t)} = (U_{\text{que}}^{(t)}, (\Phi_{\ell}^{(t)})_{\ell \in \Lambda}),$$

obtained by iterating the lifting constructor $U_{\Sigma_{\text{que}}}$ on the base Meta-Queue \mathbb{M}_{que} . Concretely, if $s < t$ and Φ_{ℓ} has meta-arity k_{ℓ} , its lift satisfies

$$\Phi_{\ell}^{(t)}(U_{\Sigma_{\text{que}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{que}}}^{t-s}(X_{k_{\ell}})) = U_{\Sigma_{\text{que}}}^{t-s}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})),$$

and analogously for all lifted relations.

Example 5.8 (Queue of queues: round-robin print scheduler). A shared printer can be modeled as an outer FIFO queue of users, where each user u carries an inner FIFO queue $J(u)$ of jobs. Scheduling proceeds by dequeuing a user from the outer queue, serving the head job of $J(u)$, and re-enqueuing u if $J(u)$ remains nonempty. This is naturally represented at one higher meta-level as an Iterated Meta-Queue: the *elements* manipulated by the outer level are themselves queue-structured objects (inner job queues), and the lifting mechanism transports FIFO operations to this hierarchical setting.

6 Meta-Markov Chains: Markov Chain of Markov Chains

A Markov chain is a stochastic process with memoryless transitions between states, governed by a fixed probability transition matrix.²³ A Meta-Markov Chain is a MetaStructure treating Markov chains as objects, enabling meta-operations like composition, restriction, or probabilistic transformation. An Iterated Meta-Markov Chain recursively applies Meta-Markov construction, forming hierarchical layers of chains of chains, generalizing classical stochastic processes.

Definition 6.1 (Markov Chain). (cf.^{24,25}) A *Markov chain* is a discrete-time stochastic process $\{X_n\}_{n \geq 0}$ on a state space S such that

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i) =: p_{ij}.$$

The matrix $P = (p_{ij})_{i,j \in S}$ is called the *transition matrix*.

Fix the single-sorted, finitary signature

$$\Sigma_{mc} = (\text{Func} = \emptyset, \text{Rel} = \{\text{States}, \text{I}, \text{Trans}\}, \text{ar}),$$

where the intended arities are

$$\text{States} : 1, \quad \text{I} : 1, \quad \text{Trans} : 3.$$

A Σ_{mc} -structure is a tuple

$$\mathbf{M} = (H; \text{States}^{\mathbf{M}}, \text{I}^{\mathbf{M}}, \text{Trans}^{\mathbf{M}}),$$

with nonempty carrier H and relations interpreted over H .

Definition 6.2 (Universe of finite Markov chains). Let $U_{mc} \subseteq \text{Str}_{\Sigma_{mc}}$ be the class of all

$$\mathbf{M} = (H; \text{States}, \text{I}, \text{Trans})$$

satisfying the following axioms:

(M1) **Typing/finite:** States is finite and nonempty; I is a set of probability tokens equipped with a fixed map

$$\iota_M : I \longrightarrow [0, 1],$$

used only to state numeric axioms.

(M2) **Value identification:** There is a fixed identification $\iota_M : \text{I} \xrightarrow{\cong} [0, 1]$ (used to state numeric axioms).

(M3) **Total functional kernel:** Trans is the graph of a total functional map $P : \text{States} \times \text{States} \rightarrow \text{I}$: for every (x, y) there is a unique $p \in \text{I}$ with $\text{Trans}(x, y, p)$.

(M4) **Row-stochasticity:** For all $x \in \text{States}$,

$$\sum_{y \in \text{States}} \iota_{\mathbf{M}}(P(x, y)) = 1.$$

Thus each $\mathbf{M} \in U_{mc}$ canonically encodes a finite, time-homogeneous Markov chain (S, P) with $S = \text{States}$ and $P(x, y) = \iota_{\mathbf{M}} \text{Trans}(x, y, \cdot) \in [0, 1]$.

Remark 6.3 (Embedding a classical finite Markov chain). Given a finite set S and a row-stochastic matrix $P : S \times S \rightarrow [0, 1]$, set

$$H := \underbrace{\{0\} \times S}_{\text{states}} \cup \underbrace{\{1\} \times [0, 1]}_{\text{probabilities}}, \quad \text{States} := \{0\} \times S, \quad \text{I} := \{1\} \times [0, 1],$$

and put

$$\text{Trans}((0, x), (0, y), (1, p)) \iff p = P(x, y), \quad \iota_{\mathbf{M}}(1, p) := p.$$

Then $\mathbf{M} \in U_{mc}$ by (M1)–(M4).

Definition 6.4 (Meta-Markov Chain). A *Meta-Markov Chain* is a *MetaStructure* (Definition 1.2) over Σ_{mc} ,

$$\mathbb{M}_{mc} = (U_{mc}, (\Phi_{\ell})_{\ell \in \Lambda}),$$

whose meta-operations are specified uniformly as follows (each by carrier- and relation-constructors).

(PROD) Independent product of chains. For $\mathbf{M}_i = (H_i; \text{States}_i, I_i, \text{Trans}_i) \in U_{\text{mc}}$ ($i = 1, 2$) define

$$\Phi_{\otimes}(\mathbf{M}_1, \mathbf{M}_2) := (H_{\times}; \text{States}_{\times}, I_{\times}, \text{Trans}_{\times})$$

on the tagged carrier $H_{\times} := \{0\} \times (\text{States}_1 \times \text{States}_2) \cup \{1\} \times [0, 1]$ with

$$\text{States}_{\times} := \{0\} \times (\text{States}_1 \times \text{States}_2), \quad I_{\times} := \{1\} \times [0, 1],$$

and kernel

$$\text{Trans}_{\times}((0, (x_1, x_2)), (0, (y_1, y_2)), (1, p)) \iff p = \iota_{\mathbf{M}_1} P_1(x_1, y_1) \cdot \iota_{\mathbf{M}_2} P_2(x_2, y_2).$$

(MIX) Convex mixture on a common state set. If $\mathbf{M}_1, \mathbf{M}_2 \in U_{\text{mc}}$ have (up to isomorphism) the same state set S and $\lambda \in (0, 1)$, set

$$\Phi_{\text{mix}(\lambda)}(\mathbf{M}_1, \mathbf{M}_2) := (H; \text{States}, I, \text{Trans})$$

on $H := \{0\} \times S \cup \{1\} \times [0, 1]$ with the obvious colors and

$$\text{Trans}((0, x), (0, y), (1, p)) \iff p = \lambda \iota_{\mathbf{M}_1} P_1(x, y) + (1 - \lambda) \iota_{\mathbf{M}_2} P_2(x, y).$$

(RES) Restriction to a closed subset. For $\mathbf{M} \in U_{\text{mc}}$ and $S \subseteq \text{States}$ closed under P (i.e. $\sum_{y \in S} \iota_{\mathbf{M}} P(x, y) = 1$ for all $x \in S$), define

$$\Phi_{\text{res}(S)}(\mathbf{M}) := (H_S; \text{States}_S, I_S, \text{Trans}_S),$$

on $H_S := \{0\} \times S \cup \{1\} \times [0, 1]$, with

$$\text{States}_S := \{0\} \times S, \quad I_S := \{1\} \times [0, 1], \quad \text{Trans}_S((0, x), (0, y), (1, p)) \iff x, y \in S, p = \iota_{\mathbf{M}} P(x, y).$$

(PUSH) State relabeling by a bijection $f : \text{States} \rightarrow K$. For $\mathbf{M} = (H; \text{States}, I, \text{Trans})$ and any bijection f set

$$\Phi_{\text{push}(f)}(\mathbf{M}) := (\widehat{H}; \widehat{\text{States}}, \widehat{I}, \widehat{\text{Trans}}),$$

on $\widehat{H} := \{0\} \times K \cup \{1\} \times [0, 1]$, with $\widehat{\text{States}} := \{0\} \times K$, $\widehat{I} := \{1\} \times [0, 1]$, and

$$\widehat{\text{Trans}}((0, f(x)), (0, f(y)), (1, p)) \iff p = \iota_{\mathbf{M}} P(x, y).$$

(LUMP) Coarse-graining by a lumpable partition. Let $\pi : \text{States} \rightarrow C$ be a surjection (blocks are $\pi^{-1}(c)$). Assume *strong lumpability*: for all $c, c' \in C$ and any $x, x' \in \pi^{-1}(c)$,

$$\sum_{y \in \pi^{-1}(c')} \iota_{\mathbf{M}} P(x, y) = \sum_{y \in \pi^{-1}(c')} \iota_{\mathbf{M}} P(x', y).$$

Define

$$\Phi_{\text{lump}(\pi)}(\mathbf{M}) := (H_C; \text{States}_C, I_C, \text{Trans}_C),$$

on $H_C := \{0\} \times C \cup \{1\} \times [0, 1]$ with $\text{States}_C := \{0\} \times C$, $I_C := \{1\} \times [0, 1]$, and

$$\text{Trans}_C((0, c), (0, c'), (1, p)) \iff p = \sum_{y \in \pi^{-1}(c')} \iota_{\mathbf{M}} P(x, y) \quad (\text{for any } x \in \pi^{-1}(c)).$$

By lumpability the right-hand side is independent of the choice of x .

Example 6.5 (Meta-Markov Chain: joint weather-traffic dynamics). Consider two ordinary (time-homogeneous) Markov chains.

Weather on $W = \{S, R\}$ (Sunny/ Rainy) with kernel

$$P_W = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \quad \text{rows/columns ordered as } (S, R).$$

Traffic on $T = \{L, H\}$ (Light/ Heavy) with kernel

$$P_T = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}, \quad \text{rows/columns ordered as (L, H).}$$

The Meta–Markov product Φ_{\otimes} forms the joint chain $X_t = (W_t, T_t)$ on $W \times T = \{(S, L), (S, H), (R, L), (R, H)\}$ with transition kernel

$$P_{W \otimes T} = P_W \otimes P_T,$$

i.e.,

$$\Pr((w, t) \rightarrow (w', t')) = P_W(w, w') P_T(t, t').$$

For instance,

$$\Pr((S, L) \rightarrow (R, H)) = 0.2 \times 0.3 = 0.06.$$

This Meta–operation preserves the Markov property while combining two independent subsystems (weather and traffic) into a single joint stochastic dynamical model.

Theorem 6.6 (Meta-Markov Chain is a MetaStructure and generalizes Markov chains). $\mathbb{M}_{mc} = (U_{mc}, (\Phi_{\ell}))$ of Definition 6.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite Markov chain (S, P) appears (via Remark 6.3) as an object of U_{mc} ; hence Meta-Markov Chains generalize Markov chains.

Proof. Uniform constructors and closure. For each meta-operation we check (M1)–(M4).

(PROD) $States_{\times}$ is finite/nonempty. For fixed (x_1, x_2) ,

$$\sum_{y_1, y_2} \iota P_{\times}((x_1, x_2), (y_1, y_2)) = \left(\sum_{y_1} \iota P_1(x_1, y_1) \right) \left(\sum_{y_2} \iota P_2(x_2, y_2) \right) = 1,$$

so row-stochasticity holds; total functionality is by construction.

(MIX) For each x , $\sum_y (\lambda \iota P_1(x, y) + (1 - \lambda) \iota P_2(x, y)) = \lambda + (1 - \lambda) = 1$. Values lie in $[0, 1]$ and total-ity/functionality are immediate.

(RES) Closure of S gives $\sum_{y \in S} \iota P(x, y) = 1$ for all $x \in S$, hence (M4) holds on the subcarrier; other axioms are inherited.

(PUSH) Only labels are changed; the kernel is transported verbatim, preserving (M3)–(M4).

(LUMP) By strong lumpability, for fixed c the quantity $p(c, c') := \sum_{y \in \pi^{-1}(c')} \iota P(x, y)$ does not depend on $x \in \pi^{-1}(c)$. Then

$$\sum_{c'} p(c, c') = \sum_{c'} \sum_{y \in \pi^{-1}(c')} \iota P(x, y) = \sum_{y \in States} \iota P(x, y) = 1,$$

so (M4) holds; (M1)–(M3) are clear.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{M} \rightarrow \mathbf{M}'$ is a bijection of carriers preserving States, I and Trans. For (PROD), use $\alpha_1 \times \alpha_2$ on states and the identity on the probability tag; verification is componentwise. For (MIX), the common state-set isomorphism transports kernels linearly. For (RES), restrict α to the closed subset; for (PUSH), take the given f as the state part. For (LUMP), transport the partition along the state isomorphism and use that sums over fibers are preserved. Each induced map is an isomorphism of outputs. Finally, Remark 6.3 yields the generalization claim. \square

Proposition 6.7 (Stationary distributions under META operations). Let π_i be stationary distributions of \mathbf{M}_i ($i = 1, 2$).

(i) Product: $\pi_{\times}(x_1, x_2) := \pi_1(x_1)\pi_2(x_2)$ is stationary for $\Phi_{\otimes}(\mathbf{M}_1, \mathbf{M}_2)$.

(ii) Mixture: If $\text{supp}(\pi_1) = \text{supp}(\pi_2)$, then $\pi := \lambda\pi_1 + (1 - \lambda)\pi_2$ is stationary for $\Phi_{\text{mix}(\lambda)}(\mathbf{M}_1, \mathbf{M}_2)$.

(iii) Restriction: If S is closed and π is stationary on \mathbf{M} with $\pi(S) = 1$, then $\pi|_S$ is stationary for $\Phi_{\text{res}(S)}(\mathbf{M})$.

(iv) Pushforward: If π is stationary on \mathbf{M} and f is a bijection, then $\widehat{\pi}(f(x)) := \pi(x)$ is stationary on $\Phi_{\text{push}(f)}(\mathbf{M})$.

(v) Lumping: If π is stationary and π is constant on blocks of a lumpable partition $\pi : \text{States} \rightarrow C$, then $\pi_C(c) := \sum_{x \in \pi^{-1}(c)} \pi(x)$ is stationary for $\Phi_{\text{lump}(\pi)}(\mathbf{M})$.

Proof. Routine computations using the defining identities of each meta-operation. □

Definition 6.8 (Iterated Meta-Markov Chain of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Markov Chain of depth t* is an Iterated MetaStructure over Σ_{mc} ,

$$\mathfrak{M}_{\text{mc}}^{(t)} = (U_{\text{mc}}^{(t)}, (\odot_{\ell}^{(t)})_{\ell \in \Lambda}),$$

obtained by applying the lifting functor $U_{\Sigma_{\text{mc}}}$ to \mathbb{M}_{mc} repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_{ℓ} of meta-arity k_{ℓ} ,

$$\Phi_{\ell}^{\uparrow} : (U_{\text{mc}}^{(t)})^{k_{\ell}} \longrightarrow U_{\text{mc}}^{(t)}$$

is defined on representatives by

$$\Phi_{\ell}^{\uparrow}(U_{\Sigma_{\text{mc}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{mc}}}^{t-s}(X_{k_{\ell}})) := U_{\Sigma_{\text{mc}}}^{t-s}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})),$$

and similarly for all relations. We denote by $\odot_{\otimes}^{(t)}, \odot_{\text{mix}(\lambda)}^{(t)}, \odot_{\text{res}(S)}^{(t)}, \odot_{\text{push}(f)}^{(t)}, \odot_{\text{lump}(\pi)}^{(t)}$ the respective lifts of (PROD), (MIX), (RES), (PUSH), (LUMP).

Example 6.9 (Iterated Meta-Markov Chain: Markov-modulated customer load). A store’s *environment* (Normal/Sale) evolves as a two-state Markov chain $E_t \in \{N, S\}$ with kernel

$$Q = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}, \quad \text{rows/columns (N, S)}.$$

Conditional on the environment, the *customer-load* chain $C_t \in \{\text{Low, Med, High}\}$ transitions via different kernels:

$$P^{(N)} = \begin{bmatrix} 0.75 & 0.20 & 0.05 \\ 0.20 & 0.60 & 0.20 \\ 0.10 & 0.30 & 0.60 \end{bmatrix}, \quad P^{(S)} = \begin{bmatrix} 0.50 & 0.35 & 0.15 \\ 0.15 & 0.45 & 0.40 \\ 0.05 & 0.30 & 0.65 \end{bmatrix}.$$

The *Iterated Meta-Markov* construction lifts the base Meta-operations to a two-level process on the product state space $\{N, S\} \times \{\text{Low, Med, High}\}$. With the “switch-then-move” update, the joint kernel Π is

$$\Pi((e, i), (e', j)) = Q(e, e') \cdot P^{(e')}(i, j),$$

first updating the environment $e \rightarrow e'$ via Q , then moving the load $i \rightarrow j$ according to the kernel selected by the *new* environment e' .

As an illustration,

$$\Pr((N, \text{Med}) \rightarrow (S, \text{High})) = Q(N, S) \cdot P^{(S)}(\text{Med}, \text{High}) = 0.1 \times 0.40 = 0.04.$$

This “Markov-modulated Markov chain” (a queue-of-kernels viewpoint) is the canonical real-world instance of an *Iterated Meta-Markov* structure: an outer chain over regimes and an inner chain whose dynamics are functorially selected and composed by the outer state.

Theorem 6.10 (Iterated Meta-Markov Chain is an Iterated MetaStructure and generalizes Meta-Markov). For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{mc}}^{(t)}$ of Definition 6.8 is an Iterated MetaStructure. Moreover, for $s < t$ the embedding

$$\iota_{s \rightarrow t} : \mathfrak{M}_{\text{mc}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{mc}}^{(t)}, \quad X \longmapsto U_{\Sigma_{\text{mc}}}^{t-s}(X),$$

preserves all lifted meta-operations:

$$\Phi_{\ell}^{\uparrow}(\iota_{s \rightarrow t}(X_1), \dots, \iota_{s \rightarrow t}(X_{k_{\ell}})) = \iota_{s \rightarrow t}(\Phi_{\ell}(X_1, \dots, X_{k_{\ell}})).$$

In particular, $\mathfrak{M}_{\text{mc}}^{(0)} = \mathbb{M}_{\text{mc}}$ embeds into every $\mathfrak{M}_{\text{mc}}^{(t)}$, so *Iterated Meta-Markov Chains* generalize *Meta-Markov Chains*.

Proof. Each lifted constructor is obtained by post-composing the base constructors with $U_{\Sigma_{mc}}^{t-s}$, hence uniform at all heights. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level- s isomorphisms, then

$$U_{\Sigma_{mc}}^{t-s}(\alpha_i) : U_{\Sigma_{mc}}^{t-s}(X_i) \xrightarrow{\cong} U_{\Sigma_{mc}}^{t-s}(Y_i)$$

are level- t isomorphisms, and

$$\Phi_\ell^\uparrow(U^{t-s}\alpha_1, \dots, U^{t-s}\alpha_{k_\ell}) = U^{t-s}(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})),$$

which is an isomorphism because each base Φ_ℓ is natural by Theorem 6.6. The operation-preservation for $\iota_{s \rightarrow t}$ is the same identity with X_i in place of α_i . □

7 Meta-Intervals: Interval of Intervals

An interval is a connected subset of the real line, containing all points between any two of its elements. A Meta-Interval is a MetaStructure where intervals themselves are treated as objects, allowing operations like union, intersection, and scaling. An Iterated Meta-Interval recursively applies Meta-Interval construction, forming hierarchical layers of intervals of intervals, generalizing classical interval structures. We encode intervals with unary “type” predicates and endpoint flags so that all constructions (union, restriction, time-zone shift, composition) are definable as structure-level operations and therefore remain invariant under isomorphisms, independent of concrete representations of values.

Definition 7.1 (Interval). (cf.²⁶) An *interval* in the real line \mathbb{R} is a subset $I \subseteq \mathbb{R}$ such that for all $x, y \in I$ with $x < y$, the entire segment $\{z \in \mathbb{R} \mid x < z < y\}$ is contained in I . Typical forms include:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad (a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

with half-open intervals $[a, b)$, $(a, b]$ defined analogously.

Fix the single-sorted, finitary signature

$$\begin{aligned} \Sigma_{\text{int}} &= \left(\text{Func} = \emptyset, \right. \\ \text{Rel} &= \{ \text{Vals}, \text{Bool}, \text{LeftEnd}, \text{RightEnd}, \text{ClosedL}, \text{ClosedR}, \text{Ord} \}, \\ \text{ar}(\text{Vals}) &= \text{ar}(\text{Bool}) = 1, \text{ ar}(\text{LeftEnd}) = \dots = \text{ar}(\text{Ord}) = 2 \left. \right), \end{aligned}$$

where Vals and Bool are unary “colors,” LeftEnd, RightEnd, ClosedL, ClosedR $\subseteq H \times H$ are binary relations picking the (unique) endpoints and their closed/open flags, and Ord $\subseteq H \times H$ encodes an order on values.

Definition 7.2 (Universe of encoded intervals). Let

$$U_{\text{int}} \subseteq \text{Str}_{\Sigma_{\text{int}}}$$

be the class of all

$$\mathbf{I} = (H; \text{Vals}, \text{Bool}, \text{LeftEnd}, \text{RightEnd}, \text{ClosedL}, \text{ClosedR}, \text{Ord})$$

for which:

- (I1) **Typing:** Vals and Bool are disjoint nonempty unary predicates.
- (I2) **Value identification:** There is a fixed bijection $\iota_{\mathbf{I}} : \text{Vals} \xrightarrow{\cong} S_{\mathbf{I}} \subseteq \mathbb{R}$ used only to state the axioms below; likewise a fixed bijection $\jmath_{\mathbf{I}} : \text{Bool} \xrightarrow{\cong} \{0, 1\}$.
- (I3) **Order:** Ord coincides with \leq on $S_{\mathbf{I}}$ through $\iota_{\mathbf{I}}$: for $u, v \in \text{Vals}$,

$$\text{Ord}(u, v) \iff \iota_{\mathbf{I}}(u) \leq \iota_{\mathbf{I}}(v).$$

(I4) **Unique endpoints and flags:** There exist unique $a, b \in \text{Vals}$ and unique $c_L, c_R \in \text{Bool}$ such that

$$\begin{aligned} &\text{LeftEnd}(*, a), \text{RightEnd}(*, b), \\ &\text{ClosedL}(*, c_L), \text{ClosedR}(*, c_R), \end{aligned}$$

and $\text{Ord}(a, b)$ (i.e. $\iota_{\mathbf{I}}(a) \leq \iota_{\mathbf{I}}(b)$).

We interpret \mathbf{I} as the real interval

$$\begin{aligned} \text{Int}(\mathbf{I}) := & \{ x \in \mathbb{R} \mid \iota_{\mathbf{I}}(a) < x < \iota_{\mathbf{I}}(b) \\ & \text{or } (j_{\mathbf{I}}(c_L) = 1 \wedge x = \iota_{\mathbf{I}}(a)) \\ & \text{or } (j_{\mathbf{I}}(c_R) = 1 \wedge x = \iota_{\mathbf{I}}(b)) \}. \end{aligned}$$

Remark 7.3 (Embedding classical intervals). Given any classical interval $I \subseteq \mathbb{R}$ of one of the four types $[a, b], (a, b), [a, b), (a, b]$ (with $a \leq b$), construct $\mathbf{I}_I \in U_{\text{int}}$ by taking $\text{Vals} = \{a^*, b^*\}$, $\text{Bool} = \{0^*, 1^*\}$, $\iota_{\mathbf{I}}(a^*) = a$, $\iota_{\mathbf{I}}(b^*) = b$, $j_{\mathbf{I}}(1^*) = 1$, $j_{\mathbf{I}}(0^*) = 0$, $\text{Ord}(a^*, b^*)$, and setting flags c_L, c_R so that $j_{\mathbf{I}}(c_L) = 1$ iff a is included in I , and $j_{\mathbf{I}}(c_R) = 1$ iff b is included in I . Then $\text{Int}(\mathbf{I}_I) = I$.

Definition 7.4 (Meta-Interval). A *Meta-Interval* is a *MetaStructure* (Definition 1.2) over Σ_{int} ,

$$\mathbb{M}_{\text{int}} = (U_{\text{int}}, (\Phi_{\ell})_{\ell \in \Lambda}),$$

whose meta-operations are specified uniformly as follows (each by carrier- and relation-constructors):

(AFF) Positive affine image. For $\mathbf{I} \in U_{\text{int}}$, $s > 0$ and $t \in \mathbb{R}$, define

$$\Phi_{\text{aff}(s,t)}(\mathbf{I}) \in U_{\text{int}}$$

by transporting endpoints via $x \mapsto sx + t$: if a, b, c_L, c_R are the unique data in (I4), then the new endpoints are $a' := s \cdot a + t$, $b' := s \cdot b + t$ under ι , and the new flags equal the old flags ($c'_L := c_L$, $c'_R := c_R$).

(SUM) Minkowski sum. For $\mathbf{I}_1, \mathbf{I}_2 \in U_{\text{int}}$ with data $(a_1, b_1, c_{L,1}, c_{R,1}), (a_2, b_2, c_{L,2}, c_{R,2})$, set

$$\Phi_{\oplus}(\mathbf{I}_1, \mathbf{I}_2) \in U_{\text{int}}$$

with endpoints $a_{\oplus} := a_1 + a_2$, $b_{\oplus} := b_1 + b_2$ (through ι) and flags

$$c_{L,\oplus} := c_{L,1} \wedge c_{L,2}, \quad c_{R,\oplus} := c_{R,1} \wedge c_{R,2}$$

(where \wedge is interpreted via j^{-1} as Boolean AND on $\{0, 1\}$).

(INT) Intersection. For $\mathbf{I}_1, \mathbf{I}_2 \in U_{\text{int}}$, define

$$\Phi_{\cap}(\mathbf{I}_1, \mathbf{I}_2) \in U_{\text{int}}$$

with endpoints $a_{\cap} := \max\{a_1, a_2\}$, $b_{\cap} := \min\{b_1, b_2\}$ (w.r.t. Ord), and flags determined by the standard interval-intersection rule:

$$c_{L,\cap} = \begin{cases} c_{L,1}, & a_1 > a_2, \\ c_{L,2}, & a_2 > a_1, \\ c_{L,1} \wedge c_{L,2}, & a_1 = a_2, \end{cases} \quad c_{R,\cap} = \begin{cases} c_{R,1}, & b_1 < b_2, \\ c_{R,2}, & b_2 < b_1, \\ c_{R,1} \wedge c_{R,2}, & b_1 = b_2. \end{cases}$$

(When $a_{\cap} > b_{\cap}$, the resulting encoded interval is empty; this is allowed.)

(PUSH) Order-preserving relabeling. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing bijection, put

$$\Phi_{\text{push}(f)}(\mathbf{I}) \in U_{\text{int}}$$

by sending endpoints to $f(\iota(a))$, $f(\iota(b))$ (transported back through a chosen identification), and keeping flags c_L, c_R unchanged.

Example 7.5 (Meta-Interval: Coordinating meeting times with time-zone relabeling). Model calendar windows as closed real intervals on the time axis (hours, UTC). Suppose Alice is free in

$$A = [9, 12] \quad (\text{UTC}),$$

while Bob is free in

$$B = [10.5, 11.5] \quad (\text{UTC}).$$

The Meta-Interval intersection (a uniform constructor on interval objects) yields the common window

$$A \cap B = [10.5, 11.5] \quad (\text{UTC}).$$

Applying the Meta-Interval *pushforward* by the affine bijection $t \mapsto t + 9$ (relabeling times from UTC to JST) transports this interval to

$$[10.5, 11.5] + 9 = [19.5, 20.5] = [19:30, 20:30] \quad (\text{JST}).$$

Thus, the meta-operations “intersect” and “affine relabel” act on interval-objects to produce the concrete local meeting slot.

Proposition 7.6 (Endpoint laws). *Let $\mathbf{I}_i \in U_{\text{int}}$ encode $I_i = \text{Int}(\mathbf{I}_i) \subseteq \mathbb{R}$.*

- (i) $\text{Int}(\Phi_{\text{aff}(s,t)}(\mathbf{I}_1)) = s \cdot I_1 + t$ for $s > 0$.
- (ii) $\text{Int}(\Phi_{\oplus}(\mathbf{I}_1, \mathbf{I}_2)) = I_1 + I_2 := \{x_1 + x_2 \mid x_i \in I_i\}$.
- (iii) $\text{Int}(\Phi_{\cap}(\mathbf{I}_1, \mathbf{I}_2)) = I_1 \cap I_2$.
- (iv) $\text{Int}(\Phi_{\text{push}(f)}(\mathbf{I}_1)) = f[I_1]$ for strictly increasing bijection f .

Proof. Each item is the classical endpoint calculus: (i) positive affine maps preserve interval type and endpoint inclusion; (ii) Minkowski sums add endpoints and include the extremal sum iff both summands include their corresponding extremal endpoints (Boolean AND); (iii) intersections take max of left and min of right endpoints, with inclusion inherited from the interval contributing the extremal endpoint, and AND when equal; (iv) increasing bijections send intervals to intervals, preserving endpoint inclusion. All statements are immediate from the definitions through ι, j . □

Theorem 7.7 (Meta-Interval is a MetaStructure and generalizes intervals). $\mathbb{M}_{\text{int}} = (U_{\text{int}}, (\Phi_\ell))$ of Definition 7.4 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical real interval $I \subseteq \mathbb{R}$ appears (via Remark 7.3) as an object of U_{int} ; hence Meta-Interval generalizes intervals.

Proof. Uniform constructors and closure. For each meta-operation, the carrier- and relation-constructors define a new Σ_{int} -structure by specifying the two endpoints and two flags (through ι, j) and re-instating Ord as the pullback of \leq . Items (i)–(iv) of Proposition 7.6 show that the resulting object encodes a (possibly empty) interval, hence belongs to U_{int} .

Naturality. An isomorphism $\alpha : \mathbf{I} \rightarrow \mathbf{I}'$ in $\text{Str}_{\Sigma_{\text{int}}}$ is a bijection preserving colors and relations, in particular it carries the unique endpoints and flags of \mathbf{I} to those of \mathbf{I}' and respects Ord. For (AFF), (SUM), and (INT), the induced isomorphisms are obtained by transporting endpoint data componentwise (affine map, sum, max/min with flag logic); for (PUSH) they are given by the specified f . In each case the construction commutes with isomorphisms by definition, so the Φ_ℓ are natural. The generalization claim is Remark 7.3. □

Definition 7.8 (Iterated Meta-Interval of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Interval of depth t* is an Iterated MetaStructure over Σ_{int} ,

$$\mathfrak{M}_{\text{int}}^{(t)} = (U_{\text{int}}^{(t)}, (\odot_\ell^{(t)})_{\ell \in \Lambda}),$$

obtained by applying the lifting functor $U_{\Sigma_{\text{int}}}$ to \mathbb{M}_{int} repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ ,

$$\Phi_\ell^\uparrow : (U_{\text{int}}^{(t)})^{k_\ell} \longrightarrow U_{\text{int}}^{(t)}, \quad \Phi_\ell^\uparrow(U_{\Sigma_{\text{int}}}^{t-s}(X_1), \dots, U_{\Sigma_{\text{int}}}^{t-s}(X_{k_\ell})) := U_{\Sigma_{\text{int}}}^{t-s}(\Phi_\ell(X_1, \dots, X_{k_\ell})),$$

and similarly for all relations.

Example 7.9 (Iterated Meta-Interval: Day-tagged weekly scheduling with lifted intersection). Consider two days $D = \{\text{Mon}, \text{Tue}\}$. A *day-tagged interval family* is an iterated object

$$\mathcal{A} = \{ (\text{Mon}, [9, 12]), (\text{Tue}, [13, 16]) \}$$

encoding one interval per day (the inner level), and similarly an interviewer’s availability

$$\mathcal{B} = \{ (\text{Mon}, [10, 11]), (\text{Tue}, [14, 15.5]) \}.$$

The *lifted* Meta-Interval intersection \cap^\uparrow acts fiberwise over the outer day-index and intersects the inner intervals on each fiber:

$$\begin{aligned} \mathcal{A} \cap^\uparrow \mathcal{B} &= \{ (\text{Mon}, [9, 12] \cap [10, 11]), (\text{Tue}, [13, 16] \cap [14, 15.5]) \} \\ &= \{ (\text{Mon}, [10, 11]), (\text{Tue}, [14, 15.5]) \}. \end{aligned}$$

Hence the iterated structure (days \Rightarrow intervals) together with the lifted meta-operation selects concrete per-day meeting windows while preserving the outer indexing.

Theorem 7.10 (Iterated Meta-Interval is an Iterated MetaStructure and generalizes Meta-Interval). *For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{int}}^{(t)}$ of Definition 7.8 is an Iterated MetaStructure. Moreover, for $s < t$ the embedding*

$$\iota_{s \rightarrow t} : \mathfrak{M}_{\text{int}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{int}}^{(t)}, \quad X \longmapsto \mathbb{U}_{\Sigma_{\text{int}}}^{t-s}(X),$$

preserves all lifted meta-operations. In particular, $\mathfrak{M}_{\text{int}}^{(0)} = \mathbb{M}_{\text{int}}$ embeds into $\mathfrak{M}_{\text{int}}^{(t)}$; hence Iterated Meta-Interval generalizes Meta-Interval.

Proof. Each lifted constructor is obtained by post-composing the base constructors with $\mathbb{U}_{\Sigma_{\text{int}}}^{t-s}$; therefore the resulting constructors are uniform. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are isomorphisms at level s , then $\mathbb{U}_{\Sigma_{\text{int}}}^{t-s}(\alpha_i)$ are isomorphisms at level t , and

$$\Phi_\ell^\uparrow(\mathbb{U}^{t-s} \alpha_1, \dots, \mathbb{U}^{t-s} \alpha_{k_\ell}) = \mathbb{U}^{t-s}(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}))$$

is an isomorphism because each base Φ_ℓ is natural (Theorem 7.7). The operation-preservation of $\iota_{s \rightarrow t}$ is the same identity with X_i in place of α_i . □

8 Conclusion

In this work, we have extended and examined the properties of Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals through the framework of MetaStructures and Iterated MetaStructures. For future research, we aim to explore the connections of these concepts with Fuzzy Sets,²⁷ HyperFuzzy Sets,²⁸ and Double-valued Neutrosophic Set.²⁹

Funding

This research received no external funding.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to this work.

Acknowledgments

The authors thank the editors, reviewers, and colleagues for their constructive comments and questions, which improved the clarity and presentation of this manuscript. We also acknowledge the foundational contributions of the authors cited in this paper. Any remaining errors are solely the responsibility of the authors.

Data Availability

No empirical data were generated or analyzed in this theoretical study. We encourage future work that applies and evaluates the proposed concepts in practical settings.

Research Integrity

The authors confirm that this manuscript is original, has not been published previously, and is not under consideration for publication elsewhere.

Use of Computational Tools

All proofs and derivations were carried out manually. No computational software (e.g., Mathematica, Sage-Math, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This study did not involve human participants or animals and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for tasks such as English grammar checking and language polishing. These tools were not used to generate scientific results, and their use did not violate ethical standards.

Supplementary Information

No supplementary materials are associated with this paper.

Disclaimer

The ideas presented in this paper are theoretical and have not yet been validated by empirical testing. While we have made every effort to ensure accuracy and proper attribution, inadvertent errors may remain. Readers are encouraged to verify cited material independently. The views expressed are those of the authors and do not necessarily represent those of their affiliated institutions.

References

- [1] Gulay Oguz and Bijan Davvaz. Soft topological hyperstructure. *J. Intell. Fuzzy Syst.*, 40:8755–8764, 2021.
- [2] Souzana Vougioukli. Helix hyperoperation in teaching research. *Science & Philosophy*, 8(2):157–163, 2020.
- [3] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
- [4] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond, 2025.
- [5] Narupon Tacha, Phongsakon Phayapsiang, and Aiyared Iampan. Length and mean fuzzy up-subalgebras of up-algebras. *Caspian Journal of Mathematical Sciences*, 11(1):264–303, 2022.
- [6] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.
- [7] Robert Wisbauer. *Foundations of module and ring theory*. Routledge, 2018.
- [8] Bo Stenstrom. *Rings of quotients: an introduction to methods of ring theory*, volume 217. Springer Science & Business Media, 2012.
- [9] Setyaningsih Ana and P.Elav arasi. Topology. *International Journal of Mathematics Trends and Technology*, 2018.
- [10] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.
- [11] MS Burgin. Topology in nonlinear extensions of hypernumbers. *Discrete Dynamics in Nature and Society*, 2005(2):145–170, 2005.
- [12] Casper Schousboe Andreasen, Martin Ohrt Elingaard, and Niels Aage. Level set topology and shape optimization by density methods using cut elements with length scale control. *Structural and Multidisciplinary Optimization*, 62(2):685–707, 2020.
- [13] Ela Ozkan-Canbolat and Aydin Beraha. Configuration and innovation related network topology. *Journal of Innovation & Knowledge*, 1(2):91–98, 2016.
- [14] Jia Shen. Packing and covering k-chain free subsets in boolean lattices. *Discrete mathematics*, 309(13):4628–4634, 2009.
- [15] Herbert W Hamber. Quantum gravity on the lattice. *General Relativity and Gravitation*, 41:817–876, 2009.
- [16] Yang Xu, Da Ruan, KY Qin, and Jun Liu. Lattice-valued logic. *Studies in fuzziness and soft computing*, 132, 2003.
- [17] Ivan Chajda, Radomir Halas, and Jan Kuhr. *Semilattice structures*, volume 30. Heldermann Lemgo, 2007.
- [18] BBN Koguep and C Lele. On hyperlattices: congruence relations, ideals and homomorphism. *Afrika Matematika*, 30(1):101–111, 2019.
- [19] Tom Van Woensel and Nico Vandaele. Modeling traffic flows with queueing models: a review. *Asia-Pacific Journal of Operational Research*, 24(04):435–461, 2007.
- [20] Rachna Rathore. A review on study of application of queueing models in hospital sector. *International Journal for Global Academic & Scientific Research*, 1(2):1–6, 2022.
- [21] Jau-Chuan Ke, Chia-Huang Wu, and Zhe George Zhang. Recent developments in vacation queueing models: a short survey. *International Journal of Operations Research*, 7(4):3–8, 2010.
- [22] R Srinivasan. Fuzzy queueing model using dsw algorithm. *International Journal of Advanced Research in Mathematics and Applications*, 1(1):57–62, 2014.
- [23] Onésimo Hernández-Lerma and Jean B Lasserre. *Markov chains and invariant probabilities*, volume 211. Birkhäuser, 2012.
- [24] Thomas Harter. Finite-size scaling analysis of percolation in three-dimensional correlated binary markov chain random fields. *Physical Review E-Statistical, Nonlinear, and Soft Matter Physics*, 72(2):026120, 2005.
- [25] Alex S Fukunaga, DJ-H Huang, and Andrew B Kahng. Large-step markov chain variants for vlsi netlist partitioning. In *1996 IEEE International Symposium on Circuits and Systems (ISCAS)*, volume 4, pages 496–499. IEEE, 1996.
- [26] Leonard Gillman. On intervals of ordered sets. *Annals of Mathematics*, 56(3):440–459, 1952.
- [27] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
- [28] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol.*, 41:27–37, 2012.
- [29] Maha Noorwali, Raed Hatamleh, Ahmed Salem Heilat, Haitham Qawaqneh, Arif Mehmood Khattak, Aqeeda Hussain, Jamil J Hamja, and Cris L Armada. Artificial intelligence enhanced framework for complex double valued neutrosophic soft sets and emotional affinity evaluation via cotangent similarity. *European Journal of Pure and Applied Mathematics*, 18(4):7228–7228, 2025.