



Generalized Marcinkiewicz Operators associated to Twisted Surfaces

Mohammed Ali^{1,2,*}

¹College of Integrative Studies, Abdullah Al-Salem University, Khaldiya, Kuwait

²Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan

Email: mohammed.ali@asu.edu.kw

Abstract

This paper is concerned with studying the mapping properties of generalized Marcinkiewicz integral operators along twisted surfaces. Under certain conditions on these surfaces we obtain certain L^p estimates for these operators provided that the kernel functions are in $L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. By an extrapolation argument, we prove that these operators are bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$ under very weak conditions on the kernel functions. Our results extend and improve many previously known results.

Keywords: Generalized Marcinkiewicz operator; Twisted surfaces; Rough kernels

1 Introduction

For $d > 2$, let $\mathbb{R}^d$ ($d = n$ or m) be the Euclidean space of dimension d and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the normalized Lebesgue surface measure $d\sigma_d(\cdot)$.

Let Ω be an integrable function over $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ such that

$$\Omega(rx, sy) = \Omega(x, y), \quad \forall r, s > 0 \quad (1)$$

and

$$\int_{\mathbb{S}^{n-1}} \Omega(x, y) d\sigma_n(x) = \int_{\mathbb{S}^{m-1}} \Omega(x, y) d\sigma_m(y) = 0. \quad (2)$$

For an appropriate mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, a measurable function $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and a function $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ satisfying (1)-(2), we consider the generalized Marcinkiewicz integral operator $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ along the surface $\{\Phi(u, v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m\}$ given by

$$\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}(f)(x, y) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |T_{r,s}(f)(x, y)|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda}, \quad (3)$$

where $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $\lambda > 1$, and

$$T_{r,s}(f)(x, y) = \frac{1}{rs} \int_{|v| \leq s} \int_{|u| \leq r} f((x, y) - \Phi(u, v)) \frac{\Omega(u, v) g(|u|, |v|)}{|u|^{n-1} |v|^{m-1}} du dv.$$

When $\lambda = 2$, $\Phi(u, v) \equiv I(u, v) = (u, v)$ and $g \equiv 1$, the operator $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$, denoted by $\mathcal{M}_\Omega$, is the classical Marcinkiewicz operator on product domains. Historically, the boundedness of $\mathcal{M}_\Omega$ was first studied in¹ in which the author obtained the L^2 boundedness of $\mathcal{M}_\Omega$ whenever $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. This result was later improved and extended by many authors, see,^{2,3} and,⁴ among others. Let us now recall two relevant results to our current study. The first one is the result in⁴ in which the authors proved that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$ under the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and that this condition be the weakest condition in the sense that the L^2 boundedness of $\mathcal{M}_\Omega$ will not be satisfied if this condition is replaced by any weaker condition $\Omega \in L(\log L)^\varepsilon(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $0 < \varepsilon < 1$. The second result is that under the condition $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, the author of⁵ established the L^p ($1 < p < \infty$) boundedness of $\mathcal{M}_\Omega$, and proved the optimality of this condition. Here $B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is a special class of block spaces introduced in.⁶ For a background information,⁷⁻¹⁰ development and applications¹¹⁻¹⁵ and recent studies.¹⁶⁻¹⁸

On the other hand, as a natural extension of $\mathcal{M}_\Omega$ is $\mathcal{G}_{\Phi, \Omega, g}^{(2)}$ for certain classes of functions Φ . If the classes of functions Φ are in the standard form $\Phi(u, v) = (\phi_1(u), \phi_2(v))$, where $\phi_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\phi_2 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are functions satisfy some certain conditions, the L^p boundedness of $\mathcal{G}_{\Phi, \Omega, g}^{(2)}$ has been investigated by many authors. For a sampling of past studies, readers are referred to¹⁹⁻²² and the references therein.

Recently, Al-Salman²³ introduced Marcinkiewicz operator $\mathcal{G}_{\Phi, \Omega, g}^{(2)}$ along the twisted surfaces

$$\{\Gamma(u, v) = \Phi(u, v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m\},$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ satisfies some specific conditions.

The L^p boundedness of the generalized Marcinkiewicz operator $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ has been investigated by many authors. For instance, the author of²⁴ confirmed L^p boundedness of $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ for all $p \in (1, \infty)$ provided that $\Phi \equiv I$, $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^n \times \mathbb{S}^{m-1})$ with $\lambda > 1$. This result was extended and improved in.²⁵ Precisely, the authors showed that the estimate

$$\left\| \mathcal{G}_{\Phi, \Omega, 1}^{(\lambda)}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \|f\|_{\dot{F}_p^{\lambda, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)}$$

holds for $1 < p < \infty$ whenever $\Phi \equiv I$, $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B_q^{(0, \frac{2}{\lambda}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and $\lambda > 1$. Thereafter, L^p boundedness of $\mathcal{G}_{\Phi, \Omega, 1}^{(\lambda)}$ was investigated under various conditions on Ω , g and the mapping Φ is in the standard form $\Phi(u, v) = (\phi_1(u), \phi_2(v))$ as can be seen in,²⁶⁻²⁸ among others.

Here, $\dot{F}_p^{\lambda, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$ denotes the space of homogeneous Triebel-Lizorkin functions which is defined as follows: For $\lambda, p \in (1, \infty)$ and $\vec{\tau} = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$, the space $\dot{F}_p^{\lambda, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the collection of tempered distributions f on $\mathbb{R}^n \times \mathbb{R}^m$ satisfying

$$\|f\|_{\dot{F}_p^{\lambda, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)} = \left\| \left(\sum_{j, k \in \mathbb{Z}} 2^{(j\tau_1 + k\tau_2)\lambda} |(\phi_j \otimes \varphi_k) * f|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} < \infty,$$

where $\widehat{\phi_j}(u) = \frac{1}{2^{jn}} \phi(2^{-j}u)$, $\widehat{\phi_k}(v) = \frac{1}{2^{km}} \varphi(2^{-k}v)$ and $\phi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^m)$ are radial mappings satisfying:

- (i) $0 \leq \phi, \varphi \leq 1$,
- (ii) $\text{supp}(\phi) \subseteq \{u : |u| \in [\frac{1}{2}, 2]\}$, $\text{supp}(\varphi) \subseteq \{v : |v| \in [\frac{1}{2}, 2]\}$,
- (iii) A positive constant C exists such that for all $|u|, |v| \in [\frac{3}{5}, \frac{5}{3}]$, $\phi(u) \geq C$ and $\varphi(v) \geq C$,
- (iv) $\sum_{j \in \mathbb{Z}} \phi(u/2^j) = 1$ if $u \neq 0$ and $\sum_{k \in \mathbb{Z}} \varphi(v/2^k) = 1$ if $v \neq 0$.

The next properties were proved in.²⁴

- (i) The Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $\dot{F}_p^{\lambda, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$,
- (ii) When $\lambda = 2$ and $\vec{\tau} = \vec{0}$, we have $\dot{F}_p^{2, \vec{0}}(\mathbb{R}^n \times \mathbb{R}^m) = L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$,
- (iii) If $1 < \lambda_1 \leq \lambda_2 < \infty$, then $\dot{F}_p^{\lambda_1, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq \dot{F}_p^{\lambda_2, \vec{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$.

In light of the results in²³ regarding the L^p boundedness of Marcinkiewicz integral $\mathcal{G}_{\Phi, \Omega, g}^{(2)}$ along twisted surfaces and of the results in²⁴⁻²⁸ regarding the L^p boundedness of generalized Marcinkiewicz integral $\mathcal{G}_{\Gamma, \Omega, g}^{(\lambda)}$ along surfaces of standard form, the following question arises naturally:

Question. Is the generalized Marcinkiewicz integral $\mathcal{G}_{\Theta, \Omega, g}^{(\lambda)}$ bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p \in (1, \infty)$ along certain classes of twisted surfaces $\{\Gamma(u, v) = \Phi(u, v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m\}$?

We shall answer the above question in affirmative as can be described in the following results.

Theorem 1.1. Suppose that $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q \in (1, 2]$ satisfies (1)-(2). Suppose that $\Phi(x, y) = (Q(|y|)x, P(|x|)y)$, where $Q, P : \mathbb{R} \rightarrow \mathbb{R}$ are polynomials. Then the inequality

$$\left\| \mathcal{G}_{\Theta, \Omega, g}^{(\lambda)}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_p \|g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \|\Omega\|_{L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})} \left(\frac{1}{q-1} \right)^{2/\lambda} \|f\|_{\dot{F}_p^{\lambda, \vec{0}}(\mathbb{R}^n \times \mathbb{R}^m)} \quad (4)$$

holds for all $p \in (1, \infty)$ if Φ satisfies one of the following conditions:

- (a) $Q(0) = 0 = P(0)$ and $\lim_{r \rightarrow 0} \frac{Q(r)}{r} = 0 = \lim_{r \rightarrow 0} \frac{P(r)}{r}$.
- (b) $\deg(Q) + \deg(P) = 1$.
- (c) $\deg(Q) = \deg(P) = 1$, $Q(0) \neq 0$, and $P(0) \neq 0$.
- (d) $Q(r) = r$ and $\deg(P) > 1$ with $\lim_{s \rightarrow 0} \frac{P(s) - P(0)}{s} = 0$ or $P(s) = s$ and $\deg(Q) > 1$ with $\lim_{r \rightarrow 0} \frac{Q(r) - Q(0)}{r} = 0$.

The constant C_p may depend on the degrees of Q and P , but it is independent of their coefficients.

The conclusion comes from Theorem 1.1 along with Yano's extrapolation argument (see^{29,30}) lead to the following:

Theorem 1.2. Suppose that g and Φ are given as in Theorem 1.1 and that Ω belongs either to $L(\log L)^{2/\lambda}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ or to $B_q^{(0, \frac{2}{\lambda}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then the operator $\mathcal{G}_{\Theta, \Omega, g}^{(\lambda)}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$.

Remarks

- (1) For the case $\lambda = 2$, $\Phi \equiv I$ and $g \equiv 1$, L^2 boundedness of $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ was obtained in¹ under the assumption $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneq L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Thus, the findings in Theorem 1.2 are fundamental improvement and generalization to the results in¹.
- (2) For the special case $\lambda = 2$, $\Phi \equiv I$, $g \equiv 1$, and $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, Theorem 1.2 confirms that $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$. Therefore, the main results in² are improved here.

- (3) Our results extend the results in³ in which the L^p ($1 < p < \infty$) boundedness of $\mathcal{G}_{\Omega,1}^{(2)}$ is obtained whenever $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneq L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$.
- (4) For the special case $\lambda = 2$, $\Phi \equiv I$ and $g \equiv 1$, the assumptions on Ω are optimal, see.^{4,5}
- (5) In Theorem 1.2, the L^p boundedness of $\mathcal{G}_{\Phi,\Omega,g}^{(\lambda)}$ was established for the full range of $p \in (1, \infty)$.
- (6) The main result in²³ is acquired directly from Theorem 1.2 by taking $\lambda = 2$.
- (7) For the special case $\Phi \equiv I$, our results lead to the main results in.^{24,25}

2 Preliminary Lemmas

In this section, we give some subsidiary results which are needed to prove our main results. Let

$$Q(t) = \sum_{i=2}^M a_i t^i \quad \text{and} \quad P(t) = \sum_{k=2}^N b_k t^k.$$

Set $Q_1(t) = P_1(t) = 0$, and for $1 < \mu \leq M$ and $1 < v \leq N$, let

$$Q_\mu(t) = \sum_{i=2}^\mu a_i t^i \quad \text{and} \quad P_v(t) = \sum_{k=2}^v b_k t^k.$$

For $1 \leq \mu \leq M$, $1 \leq v \leq N$, and $\theta \geq 2$, we define the family of measures $\Upsilon_{\Omega,g,r,s}^{(\mu,v)} := \{\Upsilon_{r,s}^{(\mu,v)} : r, s \in \mathbb{R}_+\}$ and its corresponding maximal operators $(\Upsilon_g^{(\mu,v)})^*$ and $M_{g,\theta}$ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\iint_{\mathbb{R}^n \times \mathbb{R}^m} f d\Upsilon_{r,s}^{(\mu,v)} = \frac{1}{sr} \int_{\frac{1}{2}s \leq |v| \leq s} \int_{\frac{1}{2}r \leq |u| \leq r} f(Q_\mu(|v|)u, P_v(|u|)v) \frac{\Omega(u,v)g(|u|,|v|)}{|u|^{n-1}|v|^{m-1}} du dv,$$

$$(\Upsilon_g^{(\mu,v)})^*(f) = \sup_{r,s \in \mathbb{R}_+} |\Upsilon_{r,s}^{(\mu,v)} * f|,$$

and

$$M_{g,\theta}(f) = \sup_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} |\Upsilon_{r,s}^{(\mu,v)} * f| \frac{dr ds}{rs},$$

where $|\Upsilon_{r,s}^{(\mu,v)}|$ is defined similarly as $\Upsilon_{r,s}^{(\mu,v)}$ but with replacing g by $|g|$ and Ω by $|\Omega|$.

The next two lemmas due to.²³

Lemma 2.1. Suppose that $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q \in (1, 2]$ and $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the estimates

$$\|(\Upsilon_g^{(\mu,v)})^*(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} \quad (5)$$

and

$$\|M_{g,\theta}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} (\ln \theta)^2 \quad (6)$$

hold for all $p \in (1, \infty)$, where $C_{p,\Omega} = C_p \|g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \|\Omega\|_{L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})}$.

Lemma 2.2. Suppose that Ω and g are given as in Lemma 2.1. Then there exists $C > 0$ such that for $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^m$, the following inequalities hold:

$$\begin{aligned} \left\| \Upsilon_{r,s}^{(\mu,v)}(\xi, \zeta) \right\| &\leq C, \\ \left| \hat{\Upsilon}_{r,s}^{(\mu,v)}(\xi, \zeta) \right| &\leq C |rs^\mu a_\mu \xi|^{-\frac{1}{8q'(\mu+1)}} |sr^v b_v \zeta|^{-\frac{1}{8q'(\mu+1)}}, \\ \left| \hat{\Upsilon}_{r,s}^{(\mu,v)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu-1,v)}(\xi, \zeta) \right| &\leq C |rs^\mu a_\mu \xi|^{\frac{1}{8q'(\mu+1)}} |sr^\mu b_v \zeta|^{\frac{-1}{8q'(\mu+1)}}, \\ \left| \hat{\Upsilon}_{r,s}^{(\mu,v)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu,v-1)}(\xi, \zeta) \right| &\leq C |rs^\mu a_\mu \xi|^{-\frac{1}{8q'(\mu+1)}} |sr^v b_v \zeta|^{\frac{1}{8q'(\mu+1)}}, \\ \left| \hat{\Upsilon}_{r,s}^{(\mu,v-1)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu-1,v-1)}(\xi, \zeta) \right| &\leq C |rs^\mu a_\mu \xi|^{\frac{1}{4q'(\mu+1)}}, \\ \left| \hat{\Upsilon}_{r,s}^{(\mu-1,v)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu-1,v-1)}(\xi, \zeta) \right| &\leq C |sr^v b_v \zeta|^{\frac{1}{4q'(\mu+1)}}, \end{aligned}$$

and

$$\begin{aligned} \left| \hat{\Upsilon}_{r,s}^{(\mu,v)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu,v-1)}(\xi, \zeta) - \hat{\Upsilon}_{r,s}^{(\mu-1,v)}(\xi, \zeta) + \hat{\Upsilon}_{r,s}^{(\mu-1,v-1)}(\xi, \zeta) \right| \\ \leq C |rs^\mu a_\mu \xi|^{\frac{1}{4q'(\mu+1)}} |sr^v b_v \zeta|^{\frac{1}{4q'(\mu+1)}}. \end{aligned}$$

Lemma 2.3. Let g and Ω be given as in Lemma 2.1. Assume that $\{\mathcal{V}_{j,k}(\cdot, \cdot), j, k \in \mathbb{Z}\}$ is arbitrary functions on $\mathbb{R}^n \times \mathbb{R}^m$. Then there exists constant $C_{p,\Omega} > 0$ such that

$$\begin{aligned} \left\| \left(\sum_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \Upsilon_{r,s}^{(\mu,v)} * \mathcal{V}_{j,k} \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\ \leq C_{p,\Omega} (\ln \theta)^{2/\lambda} \left\| \left(\sum_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \end{aligned} \quad (7)$$

for all $p \in (1, \infty)$.

Proof. Thanks to the inequality (5), we have for all $p \in (1, \infty)$,

$$\begin{aligned} \left\| \sup_{j,k \in \mathbb{Z}} \sup_{(r,s) \in [1,\theta] \times [1,\theta]} \left| \Upsilon_{\theta^k r, \theta^j s}^{(\mu,v)} * \mathcal{V}_{j,k} \right| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} &\leq \left\| (\Upsilon_g^{(\mu,v)})^* \left(\sup_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}| \right) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\ &\leq C_{p,\Omega} \left\| \sup_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}, \end{aligned} \quad (8)$$

which leads to

$$\begin{aligned} \left\| \left\| \Upsilon_{\theta^k r, \theta^j s}^{(\mu,v)} * \mathcal{V}_{j,k} \right\|_{L^\infty([1,\theta] \times [1,\theta], \frac{dr ds}{rs})} \right\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})} \left\| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\ \leq C_{p,\Omega} \left\| \left\| \mathcal{V}_{j,k} \right\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}. \end{aligned} \quad (9)$$

As $p > 1$, duality gives that there exists $\rho \in L^{p'}(\mathbb{R}^n \times \mathbb{R}^m)$ with $\|\rho\|_{L^{p'}(\mathbb{R}^n \times \mathbb{R}^m)} \leq 1$ such that

$$\begin{aligned}
 & \left\| \sum_{j,k \in \mathbb{Z}} \int_1^\theta \int_1^\theta \left| \Upsilon_{\theta^k r, \theta^j s}^{(\mu, v)} * \mathcal{V}_{j,k} \right| \frac{dr ds}{rs} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\
 &= \iint_{\mathbb{R}^n \times \mathbb{R}^m} \sum_{j,k \in \mathbb{Z}} \int_1^\theta \int_1^\theta \left| \Upsilon_{\theta^k r, \theta^j s}^{(\mu, v)} * \mathcal{V}_{j,k} \right| \frac{dr ds}{rs} \rho(x, y) dx dy \\
 &\leq C \|g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \iint_{\mathbb{R}^n \times \mathbb{R}^m} \left(\sum_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}(x, y)| \right) (\Upsilon_g^{(\mu, v)})^*(\tilde{\rho})(-x, -y) dx dy \\
 &\leq C (\ln \theta)^2 \|g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \left\| \sum_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}| \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \left\| (\Upsilon_g^{(\mu, v)})^*(\tilde{\rho}) \right\|_{L^{p'}(\mathbb{R}^n \times \mathbb{R}^m)}, \quad (10)
 \end{aligned}$$

where $\tilde{\rho}(x, y) = \rho(-x, -y)$. Define the linear operator $\mathcal{T}$ on $\{\mathcal{V}_{j,k}\}$ by $\mathcal{T}(\mathcal{V}_{j,k}) = \Upsilon_{\theta^k r, \theta^j s} * \mathcal{V}_{j,k}$. Hence, by combining (9) with (10), we deduce

$$\begin{aligned}
 & \left\| \left(\sum_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \Upsilon_{r,s}^{(\mu, v)} * \mathcal{V}_{j,k} \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\
 &\leq C \left\| \left(\sum_{j,k \in \mathbb{Z}} \int_1^\theta \int_1^\theta \left| \Upsilon_{\theta^k r, \theta^j s}^{(\mu, v)} * \mathcal{V}_{j,k} \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\
 &\leq C_{p,\Omega} (\ln \theta)^{2/\lambda} \left\| \left(\sum_{j,k \in \mathbb{Z}} |\mathcal{V}_{j,k}|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)}
 \end{aligned}$$

for all $p \in (1, \infty)$. The proof of Lemma 2.3 is complete. $\square$

3 Proof of Theorem 1.1

We shall prove only part (i) in Theorem 1.1 since the proofs of the other parts follow by similar technique with minor modifications. Assume that $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $q \in (1, 2]$ and $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. By Minkowski's inequality we have

$$\begin{aligned}
 \mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}(f)(x, y) &= \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \sum_{j,k=0}^{\infty} \frac{1}{sr} \int_{2^{-j-1}s < |v| \leq 2^{-j}s} \int_{2^{-k-1}r < |u| \leq 2^{-k}r} \frac{\Omega(u, v)g(|u|, |v|)}{|u|^{n-1}|v|^{m-1}} \right. \right. \\
 &\quad \times \left. \left. f(x - Q_\mu(|v|)u, y - P_v(|u|)v) dudv \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \\
 &\leq \sum_{j,k=0}^{\infty} \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \frac{1}{sr} \int_{2^{-j-1}s < |v| \leq 2^{-j}s} \int_{2^{-k-1}r < |u| \leq 2^{-k}r} \frac{\Omega(u, v)g(|u|, |v|)}{|u|^{n-1}|v|^{m-1}} \right. \right. \\
 &\quad \times \left. \left. f(x - Q_\mu(|v|)u, y - P_v(|u|)v) dudv \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \\
 &\leq C \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \Upsilon_{r,s}^{(\mu, v)} * f(x, y) \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda}. \quad (11)
 \end{aligned}$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^m)$ be two schwartz functions satisfying that $\hat{\phi}(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\hat{\phi}(x) = 0$ if $|x| \geq 1$, and $\hat{\varphi}(y) = 1$ if $|y| \leq \frac{1}{2}$ and $\hat{\varphi}(y) = 0$ if $|y| \geq 1$. For $1 \leq \mu \leq M$, $1 \leq v \leq N$, and $r, s \in \mathbb{R}_+$, let $\widehat{\phi_{r,s}^{(\mu)}}(\xi) = \widehat{\phi}(\theta^{r+sv} a_\mu \xi)$ and $\widehat{\varphi_{r,s}^{(v)}}(\zeta) = \widehat{\varphi}(\theta^{s+r\mu} b_v \zeta)$, where $\theta = 2^{q'}$. Define

$$\widehat{\sigma_{r,s}^{(\mu,v)}}(\xi, \zeta) = \widehat{\Upsilon_{r,s}^{(\mu,v)}}(\xi, \zeta) \prod_{\mu < j \leq M} \widehat{\phi_{r,s}^{(j)}}(\xi) \prod_{v < k \leq N} \widehat{\varphi_{r,s}^{(k)}}(\zeta)$$

and

$$\widehat{\Theta_{r,s}^{(\mu,v)}}(\xi, \zeta) = \widehat{\sigma_{r,s}^{(\mu,v)}}(\xi, \zeta) - \widehat{\sigma_{r,s}^{(\mu-1,v)}}(\xi, \zeta) - \widehat{\sigma_{r,s}^{(\mu,v-1)}}(\xi, \zeta) + \widehat{\sigma_{r,s}^{(\mu-1,v-1)}}(\xi, \zeta).$$

Thus, by Lemma 2.1 and Lemma 2.2, we get the following:

$$\begin{aligned} \|\Theta_{r,s}^{(\mu,v)}(\xi, \zeta)\| &\leq C, \\ \Upsilon_{r,s}^{(\mu,v)}(\xi, \zeta) &= \sum_{\mu=1}^M \sum_{v=1}^N \Theta_{r,s}^{(\mu,v)}(\xi, \zeta), \\ \|(\Theta_g^{(\mu,v)})^*(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} &\leq \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} \text{ for all } p \in (1, \infty), \\ \widehat{\Theta_{r,s}^{(\mu,v)}}(\xi, \zeta) &\leq C |\theta^r \theta^{s\mu} a_\mu \xi|^{\pm \frac{1}{sq'(\mu+1)}} |\theta^s \theta^{rv} b_v \zeta|^{\pm \frac{1}{sq'(\mu+1)}}, \end{aligned} \quad (12)$$

where $a^{\pm b} = \min\{a^b, a^{-b}\}$. Let Λ_θ be a non-negative smooth function on $\mathbb{R}_+$ satisfying the following:

$$0 \leq \Lambda_\theta(t) \leq 1, \quad \text{supp}(\Lambda_\theta) \subseteq [\frac{4}{5\theta}, \frac{5\theta}{4}], \text{ and } \int_{\mathbb{R}_+} \Lambda_\theta(t) \frac{dt}{t} = 2 \ln \theta.$$

Define the family of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ functions $\mathcal{U}_{\theta,r,s}^{(\mu,v)}$ by

$$\mathcal{U}_{\theta,r,s}^{(\mu,v)}(\xi, \zeta) = \Lambda_\theta(|\theta^r \theta^{s\mu} \xi|^2) \Lambda_\theta(|\theta^s \theta^{rv} \zeta|^2). \quad (13)$$

So, for any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we have

$$\left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \Upsilon_{r,s}^{(\mu,v)} * f(x, y) \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \leq C_{M,N} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \mathcal{I}_{l,t}(f)(x, y) dl dt, \quad (14)$$

where

$$\mathcal{I}_{l,t}(f)(x, y) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \Theta_{r,s}^{(\mu,v)} * \mathcal{U}_{\theta,r+l,s+t}^{(\mu,v)} * f(x, y) \right|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda}.$$

Let us estimate the norm of $\mathcal{I}_{l,t}(f)$. First, we estimate it for the case $p = \lambda = 2$; by utilizing Fubini's theorem, Plancherel's theorem and the inequality (12), we obtain

$$\begin{aligned} &\|\mathcal{I}_{l,t}(f)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2 \\ &\leq \sum_{j,k \in \mathbb{Z}} \iint_{E_{k,j}} \left(\int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left| \widehat{\Theta_{r,s}^{(\mu,v)}}(\xi, \zeta) \right|^2 \left| \widehat{\mathcal{U}_{\theta,r+l,s+t}^{(\mu,v)}} \right|^2 \frac{dr ds}{rs} \right) \left| \widehat{f}(\xi, \zeta) \right|^2 d\xi d\zeta \\ &\leq C_p (\ln \theta)^2 C_{p,\Omega}^2 \sum_{j,k \in \mathbb{Z}} \iint_{E_{k,j}} \left| \theta^{r+l} \theta^{(s+t)\mu} a_\mu \xi \right|^{\pm \frac{1}{sq'(\mu+1)}} \left| \theta^{s+t} \theta^{(r+l)v} b_v \zeta \right|^{\pm \frac{1}{sq'(\mu+1)}} \left| \widehat{f}(\xi, \zeta) \right|^2 d\xi d\zeta \\ &\leq C_p (\ln \theta)^2 K_{t,l}^{(\mu,v)} C_{p,\Omega}^2 \sum_{j,k \in \mathbb{Z}} \iint_{E_{n,m}} \left| \widehat{f}(\xi, \zeta) \right|^2 d\xi d\zeta \\ &\leq C_p (\ln \theta)^2 K_{t,l}^{(\mu,v)} C_{p,\Omega}^2 \|f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^m)}^2, \end{aligned} \quad (15)$$

where $E_{k,j} = \{(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}^m : (|\xi|, |\zeta|) \in [\theta^{-1-k}, \theta^{1-k}] \times [\theta^{-1-j}, \theta^{1-j}]\}$ and

$$K_{t,l}^{(\mu,v)} \leq \left(\frac{5}{4}\right)^{\frac{1}{4(\mu+1)}} 2^{\pm \frac{1}{4(\mu+1)}} 2^{\pm \frac{|t+l|}{8(\mu+1)}} 2^{\pm \frac{|l+t\mu|}{8(\mu+1)}}.$$

For the other cases of p , we estimate the L^p -norm of $\mathcal{I}_{l,t}(f)$ by employing an argument as that found in.³¹ Precisely, by Lemma 2.3 and Littlewood-Paley theory, we have

$$\begin{aligned} & \|\mathcal{I}_{l,t}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\ & \leq C \left\| \left(\sum_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} |\Theta_{r,s}^{(\mu,v)} * \mathcal{U}_{\theta,r+l,s+t}^{(\mu,v)} * f|^\lambda \frac{dr ds}{rs} \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \\ & \leq (\ln \theta)^{2/\lambda} \left\| \left(\sum_{j,k \in \mathbb{Z}} |\mathcal{U}_{\theta,r+l,s+t}^{(\mu,v)} * f|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} \\ & \leq \left(\frac{1}{q-1} \right)^{2/\lambda} \|f\|_{\dot{F}_p^{\lambda, \vec{0}}(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} \end{aligned} \quad (16)$$

for all $p \in (1, \infty)$. Thus, by interpolating (15) with (16), we conclude that the estimate

$$\|\mathcal{I}_{l,t}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \left(\frac{1}{q-1} \right)^{2/\lambda} \|f\|_{\dot{F}_p^{\lambda, \vec{0}}(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} 2^{-\varepsilon(|l|+|t|)} \quad (17)$$

holds for all $p \in (1, \infty)$, where $\varepsilon \in (0, 1)$. Consequently, the inequalities (11) and (14) along with (17) lead to

$$\left\| \mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}(f) \right\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq C_{p,M,N} \left(\frac{1}{q-1} \right)^{2/\lambda} \|f\|_{\dot{F}_p^{\lambda, \vec{0}}(\mathbb{R}^n \times \mathbb{R}^m)}. \quad (18)$$

The proof of Theorem 1.1 is complete.

4 Conclusion

In this paper, we introduced the generalized Marcinkiewicz integrals $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ along certain twisted surfaces $\{\Gamma(u, v) = \Phi(u, v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m\}$. Under certain conditions on Φ , we obtained certain appropriate L^p bounds for $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ provided that $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. By using these bounds along with an extrapolation argument we proved the L^p ($1 < p < \infty$) boundedness of $\mathcal{G}_{\Phi, \Omega, g}^{(\lambda)}$ under the weak conditions $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B_q^{(0, \frac{2}{\lambda}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $\lambda, q > 1$. Several previously known results are extended and improved in this paper, see.^{1-5, 23-25}

References

- [1] Y. Ding, L^2 -boundedness of Marcinkiewicz integral with rough kernel, Hokk. Math. J. **27** (1998) 105–115.
- [2] Y. Choi, Marcinkiewicz integrals with rough homogeneous kernel of degree zero in product domains, J. Math. Anal. Appl. **261** (2001) 53–60.
- [3] J. Chen, D. Fan and Y. Ying, The method of rotation and Marcinkiewicz integrals on product domains, Stud. Math. **153** (2002) 41–58.
- [4] H. Al-Qassem, A. Al-Salman, L. Cheng, and Y. Pan, Marcinkiewicz integrals on product spaces, Studia Math. **167** (2005) 227–234.
- [5] H. Al-Qassem, Rough Marcinkiewicz integral operators on product spaces, Collec. Math. **36** (2005) 275–297.
- [6] Y. Jiang and S. Lu, A class of singular integral operators with rough kernel on product domains, Hokkaido Math. J. **25** (1995) 1–7.

- [7] J. Chen, D. Fan and Y. Ying, Rough Marcinkiewicz integrals with $L(\log L)^2$ kernels, *Adv. Math.* **30** (2001) 179–181.
- [8] D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties, *Amer. J. Math.*, **119** (1997) 799–839.
- [9] W. Kim, S. Wainger, J. Wright, and S. Ziesler, Singular Integrals and Maximal Functions Associated to Surfaces of Revolution, *Bull. Lond. Math. Soc.* **28** (1996) 291–296.
- [10] J. Chen, Y. Ding and D. Fan, L^p boundedness of the rough Marcinkiewicz integral on product domains, *Chin. J. Contemp. Math.* **21** (2000) 47–54.
- [11] H. Wu and J. Xu, Rough Marcinkiewicz integrals associated to surfaces of revolution on product domains, *Acta Math. Sci.* **29** (2009) 294–304.
- [12] G. Hu, S. Lu and D. Yan, $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ boundedness of Marcinkiewicz integral on product spaces, *Sci China Ser A* **46** (2003) 75–82.
- [13] F. Liu and H. Wu, Rough Marcinkiewicz Integrals with Mixed Homogeneity on Product Spaces, *Acta Math. Sci.* **29** (2013) 231–244.
- [14] F. Liu and H. Wu, On the L^2 boundedness for the multiple Littlewood Paley functions with rough kernels, *J. Math. Anal. Appl.* **410** (2014) 403–410.
- [15] M. Ali and O. Al-Mohammed, Boundedness of a class of rough maximal functions, *J. Ineq. Appl.* **9** (2018) art. 305.
- [16] Q. Xue and J. Yu, Marcinkiewicz Integrals Associated with Schrödinger Operator on Campanato Type Spaces, *Front. Math.* **20** (2025) 1201–1237.
- [17] H. Wu and L. Wu, An extension of Marcinkiewicz integrals with rough kernels, *Acta Math. Sci.* **45** (2025) 789–808.
- [18] H. Al-Qassem and M. Ali, On the functions of Marcinkiewicz integrals along surfaces of revolution on product domains via extrapolation, *Symmetry* **15** (2023) art. 1814.
- [19] M. Ali and H. Al-Qassem, H. On certain estimates for parabolic Marcinkiewicz integrals related to surfaces of revolution on product spaces and extrapolation, *Axioms* **12** (2023) 1–16.
- [20] M. Ali and A. Al-Senjlawi, Boundedness of Marcinkiewicz integrals on product spaces and extrapolation, *Inter. J. Pure Appl. Math.* **97** (2014) 49–66.
- [21] B. Al-Azriyah and A. Al-Salman, Singular and Marcinkiewicz integral operators on product domains, *Commun. Korean Math. Soci.* **38** (2023) 401–430.
- [22] A. Al-Salman, Rough Marcinkiewicz integrals on product spaces, *Inter. Math. Forum* **2** (2007) 1119–1128.
- [23] A. Al-Salman, Marcinkiewicz integral operators along twisted surfaces, *Commun. Pure Appl. Anal.* **12** (2022) 159–181.
- [24] D. Fan and H. Wu, On the generalized Marcinkiewicz integral operators with rough kernels, *Canada Math. Bull.* **54** (2011) 100–112.
- [25] H. Al-Qassem, L. Cheng and Y. Pan, Generalized Littlewood-Paley functions on product spaces, *Turk. J. Math.* **45** (2021) 319–345.
- [26] M. Ali and H. Al-Qassem, A Class of rough generalized Marcinkiewicz integrals on product domains, *Symmetry*, **15** (2023) art. 823.
- [27] M. Ali and H. Al-Qassem, Estimates for generalized parabolic Marcinkiewicz integrals with rough kernels on product domains, *Axioms* **12** (2023) 1–10.
- [28] M. Ali and H. Al-Qassem, On rough generalized Marcinkiewicz integrals along surfaces of revolution on product spaces, *AIMS Math.* **9** (2024) 4816–4829.
- [29] S. Yano, Notes on Fourier analysis. XXIX. An extrapolation theorem, *J. Math. Soc. Japan* **3** (1951) 296–305.
- [30] S. Sato, Estimates for singular integrals and extrapolation, *Studia Math.* **192** (2009) 219–233.
- [31] J. Duoandikoetxea and J. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, *Invent Math.* **84** (1986) 541–561.