Binomial-Based Attribute Sampling: Formulating Multiple-Dependent State Plans Based On The Extended Odd Exponential Distribution For Truncated Life Tests

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Abstract

This paper develops a Multiple Dependent State Sampling Plan (MDSSP) based on the Extended Odd Exponential Generalized Exponential (EOEGE– E) distribution for truncated life tests. The proposed methodology determines the optimal sampling plan by minimizing the required sample size while satisfying predetermined producer's and consumer's risk requirements. An optimization algorithm is developed to obtain the optimal plan parameters for different combinations of the termination ratio, quality ratio, and consumer's risk. A comprehensive numerical study is conducted to investigate the effects of the design parameters on the optimal sampling plans and their operating characteristics. The results show that the proposed methodology produces feasible and stable sampling plans over a wide range of design set- tings. In general, the required sample size decreases as the quality ratio and termination ratio increase, whereas more stringent consumer protection re- quires larger sample sizes. The proposed methodology is illustrated using a real COVID–19 mortality dataset. The EOEGE–E distribution is first fit- ted to the data using the maximum likelihood method, and goodness-of-fit analyses confirm its suitability for modeling the observed lifetime data. The fitted distribution is then employed to construct the proposed MDSSP. Comparative studies demonstrate that the proposed sampling plan consistently re- quires fewer inspected units than both the conventional Single Sampling Plan (SSP) and the Weibull-based MDSSP while maintaining the prescribed pro- ducer's and consumer's risk requirements. The proposed MDSSP provides an efficient and economical inspection procedure for truncated life testing and represents a practical alternative for reliability analysis and industrial quality control.

Keywords: Acceptance sampling; Multiple dependent state sampling plan; Truncated life test; Extended Odd Exponential Generalized Exponential distribution; Operating characteristic function; Consumer's risk; Producer's risk

1. Introduction

Acceptance sampling is one of the most widely adopted statistical quality control techniques for making decisions on submitted production lots without conducting complete inspection [11]. Instead of examining every manufactured item, a decision regarding lot acceptance or rejection is made based on the quality characteristics observed in a representative sample. This approach substantially reduces inspection cost, testing time, and operational effort while maintaining an acceptable balance between producer's and consumer's risks. Since the pioneering work of [20], acceptance sampling has become an indispensable component of industrial quality assurance and has found extensive applications in manufacturing, electronics, pharmaceuticals, biomedical engineering, and reliability assessment [12].

For highly reliable or expensive products, destructive life testing is often impractical because it requires waiting until all test units fail. Consequently, truncated life tests have become an attractive alternative by terminating the experiment at a predetermined time and making acceptance decisions based only on the observed failures before the truncation time.

DOI: https://doi.org/10.54216/IJNS.270244

This strategy considerably shortens the testing period and lowers inspection costs while still providing reliable information regarding product lifetime characteristics. The fundamental theory of truncated life testing was established by [22] and [26], and has subsequently been extended to numerous lifetime distributions and industrial applications.

Over the past two decades, acceptance sampling plans based on truncated life tests have been developed for a wide variety of lifetime distributions in order to accommodate different failure mechanisms encountered in practice. Representative examples include the Log- Logistic distribution [29-30], the Rayleigh distribution [16], the Generalized Rayleigh distribution [37], the Marshall–Olkin extended Lomax distribution [35], the Generalized Birnbaum–Saunders distribution [17], the Birnbaum–Saunders distribution [32], the Burr Type XII distribution [33], the Gompertz distribution [25], the Transmuted Inverse Rayleigh distribution [5], the three-parameter Lindley distribution [6], the Rama distribution [7], and the two-parameter Quasi Shanker distribution [8]. More recently, acceptance sampling procedures have also been investigated using the Three-Parameter Inverted Topp– Leone distribution [33] and the Zeghdoudi distribution [9], and the Shanker distribution [10], reflecting the continuous interest in developing sampling plans under increasingly flexible lifetime models.

The increasing demand for economical and efficient inspection procedures has motivated the development of more advanced acceptance sampling strategies. Among these, the Multiple Dependent State Sampling Plan (MDSSP) has received considerable attention because it incorporates information from previously inspected lots into the current acceptance decision. Unlike the conventional Single Sampling Plan (SSP), which relies solely on the current sample, the MDSSP exploits historical inspection results to improve decision accuracy and reduce the required sample size. The concept originated from the multiple deferred state sampling inspection scheme introduced by [39], which demonstrated that incorporating information from preceding lots could substantially improve inspection efficiency.

Since then, numerous extensions of the MDSSP have been proposed under different quality characteristics and sampling environments. For example, [14] developed multiple dependent state variable sampling plans incorporating process loss considerations, whereas [40] designed an MDSSP based on the process coefficient of variation. Extensions to fuzzy environments and deferred-state mechanisms were subsequently investigated by [3] and [18], demonstrating the flexibility of dependent-state sampling concepts in practical quality control applications.

The application of MDSSP to truncated life tests has become an active research area in recent years owing to its ability to provide reliable decisions while reducing inspection cost and testing time. [4] proposed a group MDSSP under the Weibull distribution, while [13] developed generalized MDSSPs for several lifetime distributions under truncated life testing. Economic aspects of MDSSP design were later investigated by [15], highlighting the practical importance of minimizing inspection cost without sacrificing the desired protection levels for producers and consumers. Recent years have witnessed a rapid expansion in the development of MDSSPs based on increasingly flexible lifetime distributions.

[8] developed an MDSSP using the MLL3 distribution for pandemic mortality data, demonstrating the practical applicability of flexible lifetime models in reliability-based acceptance sampling. Similarly, [24] constructed an MDSSP using the Exponentiated Weibull distribution for COVID–19 mortality data, while [38] proposed adjustable multiple dependent- state sampling plans based on process capability indices. [1], [31], and [28] further demonstrated the applicability of MDSSPs in reliability analysis, medical studies, and generalized lifetime models. More recently, [36], [34], [27], and [19] proposed new MDSSP designs under several flexible lifetime distributions, confirming that the development of efficient dependent-state sampling procedures remains an active and rapidly evolving research direction.

Despite these significant developments, the performance of any acceptance sampling plan based on truncated life tests depends critically on the choice of the underlying lifetime distribution. In practical reliability studies, lifetime data frequently exhibit complex hazard rate patterns and varying degrees of skewness that cannot always be adequately represented by classical models such as the exponential or Weibull distributions. Consequently, there has been growing interest in developing more flexible lifetime distributions capable of accommodating a wider range of data characteristics while improving estimation accuracy and decision-making performance.

In response to this need, numerous generalized lifetime distributions have recently been proposed in the statistical literature. For example, [21] introduced a new extension of the exponential distribution and demonstrated its effectiveness for modeling COVID–19 mortality data. Subsequently, several flexible models have been developed for engineering and biomedical applications, including the New Explanation of Lomax distribution [10], the Modified Log–Logistic distribution [2], and the Extended Odd Exponential Generalized Exponential (EOEGE–E) distribution proposed by [23]. Owing to its flexible probability density, cumulative distribution, and hazard rate functions, the

DOI: https://doi.org/10.54216/IJNS.270244

EOEGE–E distribution is capable of modeling diverse lifetime behaviors encountered in reliability and survival studies.

Although considerable progress has been achieved in developing MDSSPs under various lifetime distributions, most existing studies have focused on classical or moderately flexible models. To the best of the authors' knowledge, no study has yet investigated the construction of a Multiple Dependent State Sampling Plan based on the EOEGE–E distribution under truncated life tests. Consequently, the potential advantages of this recently developed lifetime model for improving inspection efficiency and reducing the required sample size have not yet been explored.

Motivated by this research gap, the present study develops a new Multiple Dependent State Sampling Plan based on the EOEGE–E distribution for truncated life tests. The proposed methodology determines the optimal sampling parameters by minimizing the required sample size while simultaneously satisfying the prescribed producer's and consumer's risk requirements. Extensive numerical investigations are conducted to examine the effects of the design parameters on the optimal sampling plans. Furthermore, comprehensive comparisons with the conventional Single Sampling Plan (SSP) and the Weibull-based Multiple Depen- dent State Sampling Plan (MDSSP) are conducted to demonstrate the efficiency and practical advantages of the proposed methodology.

The remainder of this paper is organized as follows. Section 2 introduces the EOEGE–E distribution and summarizes its main statistical properties. Section 3 presents the proposed Multiple Dependent State Sampling Plan together with its design methodology and opti- mization procedure. Numerical results and comparative analyses are reported in Section 4. Section 5 illustrates the practical implementation of the proposed methodology using a real COVID–19 mortality dataset. Finally, Section 6 concludes the paper and outlines possible directions for future research.

2. The Extended Odd Exponential Generalized Exponential Distribution

The proposed Multiple Dependent State Sampling Plan (MDSSP) is developed under the Extended Odd Exponential Generalized Exponential (EOEGE–E) distribution introduced by [23]. This lifetime model extends the generalized exponential family through the odd exponential generator, providing a flexible distribution capable of accommodating a broad spectrum of lifetime characteristics encountered in engineering reliability, industrial quality control, biomedical investigations, and survival analysis.

The flexibility of the EOEGE–E distribution arises from the combined effect of its shape parameters, which enable the model to represent various distributional forms and hazard rate behaviours while maintaining analytically tractable expressions for its principal reliability measures. Consequently, the distribution constitutes an attractive probabilistic model for acceptance sampling plans constructed under truncated life tests, where the probability of failure before a predetermined termination time plays a central role in the design procedure.

Since the proposed MDSSP is formulated through the probability that an experimental unit fails before the censoring time, only the distributional characteristics required for developing the sampling plan are presented in this section. These include the cumulative distribution function, probability density function, survival function, hazard rate function, and quantile function. The resulting expressions serve as the mathematical foundation for deriving the failure probability, probability of lot acceptance, and operating characteristic function pre- sented in Section 3

Let T denote a non-negative random variable following the EOEGE–E distribution with parameter vector

$$\Theta = (\alpha, \theta, \lambda, \gamma), \qquad (1)$$

Where

$$\alpha > 0, \qquad \theta > 0, \qquad \lambda > 0, \qquad \gamma > 0.$$

Throughout this paper, the notation $T \sim \text{EOEGE--E}(\alpha, \theta, \lambda, \gamma)$ is adopted.

2.1. *Distributional Properties*.

The construction of the proposed MDSSP requires an explicit expression for the probability that a test unit fails before a predetermined termination time. Consequently, the cumulative distribution function (CDF) of the underlying lifetime model constitutes the principal quantity from which all subsequent results are derived.

According to [23], the cumulative distribution function of the EOEGE–E distribution is given by

$$F(t; \alpha, \theta, \lambda, \gamma) = \left[1 - \exp\left\{-\lambda\left(\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right) - 1\right)\right\}\right]^{\theta}, \qquad t > 0, \qquad (2)$$

DOI: https://doi.org/10.54216/IJNS.270244

where $\alpha, \theta, \lambda, \gamma$ are positive model parameters governing the scale and shape of the distribution.

Equation (2) completely characterizes the probability structure of the EOEGE-E model. In particular, it provides the probability that the lifetime of a randomly selected unit does not exceed the specified time point t, that is,

$$F(t) = P(T \leq t).$$

Since acceptance decisions under truncated life tests depend directly on the probability of observing failures before a fixed termination time, the cumulative distribution function will serve as the basis for deriving the failure probability developed in Section 3.

Differentiating Equation (2) with respect to t yields the corresponding probability density function

$$f(t;\alpha,\theta,\lambda,\gamma)=\frac{d}{dt}F(t;\alpha,\theta,\lambda,\gamma) = \alpha\theta\lambda t^{\gamma-1}\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)\exp\left[\lambda\left(\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)-1\right)\right]\times \left[1-\exp\left\{-\lambda\left(\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)-1\right)\right\}\right]^{\theta-1}, t>0. \quad (3)$$

The density function in Equation (3) describes the instantaneous distribution of lifetimes and serves as the fundamental building block for obtaining other reliability characteristics, including the survival and hazard rate functions. Owing to the interaction among the four model parameters, the EOEGE–E distribution is capable of exhibiting considerable flexibility in modelling positively skewed lifetime data encountered in practical reliability studies.

2.2. Reliability Characteristics.

Reliability measures constitute the theoretical basis for the proposed truncated life test since the acceptance decision depends on whether a test unit survives beyond a predetermined termination time. Among these measures, the survival function and the hazard rate function play particularly important roles in describing the reliability behaviour of the EOEGE–E distribution.

The survival function, also referred to as the reliability function, is defined as the proba- bility that a unit remains functional beyond time t, namely,

$$S(t) = P(T > t) = 1 - F(t), \quad (4)$$

where F (t) is the cumulative distribution function given in Equation (2). Therefore, the

survival function of the EOEGE–E distribution is expressed as

$$S(t;\alpha,\theta,\lambda,\gamma) = 1-\left[1-\exp\left\{-\lambda\left(\exp\left(\frac{\alpha\, t^{\gamma}}{\gamma}\right)-1\right)\right\}\right]^{\theta}, t>0. \quad (5)$$

The survival function represents the probability that an experimental unit survives beyond the inspection time. Consequently, it provides a direct measure of product reliability and forms the complement of the failure probability employed in the proposed sampling procedure.

Another important reliability measure is the hazard rate function, which describes the instantaneous failure rate of a unit that has survived up to time t. It is defined as

$$h(t) = \frac{f(t)}{S(t)}, t > 0, \quad (6)$$

where f(t) and S(t) are given in Equations (3) and (5), respectively. Hence, the hazard rate function of the EOEGE–E distribution becomes

$$h(t;\alpha,\theta,\lambda,\gamma) = \frac{\alpha\theta\lambda t^{\gamma-1}\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)\exp\left[\lambda\left(\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)-1\right)\right]\left[1-\exp\left\{-\lambda\left(\exp\left(\frac{\alpha t^{\gamma}}{\gamma}\right)-1\right)\right\}\right]^{\theta-1}}{1-\left[1-\exp\left\{-\lambda\left(\exp\left(\frac{\alpha\, t^{\gamma}}{\gamma}\right)-1\right)\right\}\right]^{\theta}}, \quad (7)$$

The flexibility of the hazard rate function is one of the principal advantages of the EOEGE– E distribution. Depending on the parameter configuration, the model can accommodate a variety of lifetime patterns frequently encountered in reliability studies, making it suitable for analysing heterogeneous failure mechanisms observed in engineering and industrial applications.

DOI: https://doi.org/10.54216/IJNS.270244

2.3. *Quantile Function and Failure Probability.*

The quantile function is an essential tool in truncated life testing since it determines the specified lifetime used for constructing the proposed sampling plan. Let t_q denote the $100qth$ percentile of the EOEGE–E distribution, where $0 < q < 1$. By solving Equation (2) for t_q , the quantile function is obtained as

$$t_q = \left[\frac{\gamma}{\alpha}\log\left(1 - \frac{\log\left(1q^{1/\theta}\right)}{\lambda}\right)\right]^{1/\gamma}, 0 < q < 1. \quad (8)$$

For simplicity of notation, define

$$C_q = \log\left(1 - \frac{\log\left(1 - q^{1/\theta}\right)}{\lambda}\right). \quad (9)$$

Hence, Equation (8) can be rewritten as

$$t_q = \left(\frac{\gamma C_q}{\alpha}\right)^{1/\gamma}, \quad (10)$$

which immediately gives

$$\frac{\alpha}{\gamma} = \frac{C_q}{t_q^{\gamma}}. \quad (11)$$

Suppose that t_q^0 denotes the specified lifetime quantile. Under the proposed truncated life test, the experiment is terminated at

$$t_0 = k\, t_q^0, k > 0, \quad (12)$$

where k is the termination ratio.

The probability that a test unit fails before the termination time is therefore

$$p = P(T \leq t_0) = F(t_0). \quad (13)$$

Substituting Equation (12) into the cumulative distribution function in Equation (2) yields

$$p = \left[1 - \exp\left\{-\lambda\left[\exp\left(\frac{\alpha}{\gamma}\left(kt_q^0\right)^{\gamma}\right) - 1\right]\right\}\right]^{\theta}. \quad (14)$$

Using Equation (11), Equation (14) becomes

$$\mathrm{p} = \left[1 - \exp\left\{-\lambda\left[\exp\left(\mathrm{C_q}\left(\frac{\mathrm{k\, t_q^0}}{\mathrm{t_q}}\right)^{\gamma}\right) - 1\right]\right\}\right]^{\theta}. \quad (15)$$

Now, define the quality ratio as

$$r = \frac{t_q}{t_q^0}, \quad (16)$$

which implies

$$\frac{t_q^0}{t_q} = \frac{1}{r}.$$

Accordingly, Equation (15) can be written as

DOI: https://doi.org/10.54216/IJNS.270244

$$p = \left[1 - \exp\left\{-\lambda\left[\exp\left(C_q\left(\frac{k}{r}\right)^{\gamma}\right) - 1\right]\right\}\right]^{\theta}, \qquad (17)$$

where $r = t_q/t_q^0$ represents the quality ratio between the true lifetime quantile and the specified lifetime quantile. Equation (17) gives the probability that a test unit fails before the predetermined termination time and forms the basis for developing the proposed Multiple Dependent State Sampling Plan in the following section.

3. Construction of the Proposed Multiple Dependent State Sampling Plan.

The proposed Multiple Dependent State Sampling Plan (MDSSP) is developed based on the probability of failure derived in Section 2.3. The decision regarding the acceptance or rejection of a submitted lot depends not only on the quality of the current sample but also on the inspection results of the immediately preceding lots. This strategy generally provides greater flexibility than the conventional single sampling plan while maintaining the desired producer's and consumer's protection.

nder the proposed plan, a random sample of size n is selected from the submitted lot and each test unit is observed up to the predetermined termination time $t_0 = kt_q^0$. Let d denote the number of failed units observed before the termination time. The lot acceptance decision is made according to two acceptance numbers, c_1 and c_2, where $c_1 < c_2$.

The proposed MDSSP operates according to the following decision rules:

1. Accept the submitted lot immediately if $d \le c_1$.
2. Reject the submitted lot immediately if $d > c_2$.
3. If $c_1 < d \le c_2$, the decision depends on the inspection outcomes of the previous m submitted lots. The current lot is accepted only if each of the preceding m lots contains at most c_1 failures before the termination time; otherwise, the current lot is rejected

Since each test unit fails independently before the truncation time with probability p given by Equation (17), the number of failed units in the sample follows a binomial distribution with parameters n and p. Consequently, the probability of lot acceptance can be derived as presented in the following subsection.

3.1. *Probability of Lot Acceptance.*

Let $P_a(p)$ denote the probability of accepting a submitted lot under the proposed Multiple Dependent State Sampling Plan (MDSSP). Since each test unit fails independently before the predetermined termination time with probability p, the number of failed units in a sample of size n follows a binomial distribution. Hence, the probability of observing exactly d failed units is

$$P(d) = \binom{n}{d} p^d (1-p)^{n-d}, \qquad d = 0,1,\dots,n. \quad (18)$$

According to the proposed MDSSP, a submitted lot is accepted immediately whenever the number of failed units does not exceed the first acceptance number c_1. Therefore, the probability of immediate acceptance is

$$P_I = \sum_{d=0}^{c_1} \binom{n}{d} p^d (1-p)^{n-d}. \qquad (19)$$

If the number of failed units falls between the two acceptance numbers, that is,

$$c_1 < d \le c_2,$$

the current lot is accepted only if each of the preceding m lots contains at most c_1 failed units. The corresponding probability is therefore

$$P_C = \sum_{d=c_1+1}^{c_2} \binom{n}{d} p^d (1-p)^{n-d}. \qquad (20)$$

DOI: https://doi.org/10.54216/IJNS.270244

Assuming that the inspection results of successive submitted lots are independent, the probability that each of the previous m lots satisfies the acceptance requirement is $(P_I)^m$.

Consequently, the probability of accepting a submitted lot under the proposed MDSSP is given by

$$P_a(p) = P_I + P_C(P_I)^m. \quad (21)$$

Substituting Equations (19) and (20) into Equation (21) gives

$$P_a(p) = \sum_{d=0}^{c_1} \binom{n}{d} p^d (1-p)^{n-d} + \left(\sum_{d=c_1+1}^{c_2} \binom{n}{d} p^d (1-p)^{n-d} \right) \left(\sum_{d=0}^{c_1} \binom{n}{d} p^d (1-p)^{n-d} \right)^m. \quad (22)$$

Here, the failure probability p is computed from Equation (17), which depends on the parameters of the EOEGE–E distribution together with the termination ratio k and the quality ratio $r = t_q/t_q^0$.

3.2. *Operating Characteristic Function.*

The operating characteristic (OC) function is one of the most important performance measures of an acceptance sampling plan. It describes the probability of accepting a submitted lot as a function of the product quality and provides a useful tool for assessing the discriminating ability of the proposed MDSSP.

For the proposed sampling plan, the operating characteristic function is defined by

$$OC(r) = P_a\big(p(r)\big), \quad (23)$$

where $P_a(\cdot)$ is given by Equation (22), and p(r) denotes the probability of failure obtained from Equation (17). Since the failure probability depends on the quality ratio in $r = \frac{t_q}{t_q^0} \epsilon \{2,4,6,8,10\}$, the operating characteristic function can be expressed as

$$OC(r) = \sum_{d=0}^{c_1} \binom{n}{d} p(r)^d [1-p(r)]^{n-d} + \left(\sum_{d=c_1+1}^{c_2} \binom{n}{d} p(r)^d [1-p(r)]^{n-d} \right) \times \left(\sum_{d=0}^{c_1} \binom{n}{d} p(r)^d [1-p(r)]^{n-d} \right)^m. \quad (24)$$

The OC function represents the probability of accepting a submitted lot for different quality levels. Generally, the acceptance probability increases as the quality ratio r increases, indicating that products with longer lifetimes relative to the specified lifetime are more likely to be accepted. Conversely, poorer quality products correspond to smaller values of r and therefore have lower acceptance probabilities.

The proposed MDSSP is designed by selecting the plan parameters (n, c_1, c_2, m) such that the OC function satisfies the producer's and consumer's risk requirements. The corresponding optimization procedure is presented in the next subsection.

3.3. *Design Optimization under Producer's and Consumer's Risks.*

The primary objective of the proposed Multiple Dependent State Sampling Plan is to determine the optimal plan parameters (n, c_1, c_2, m)while simultaneously satisfying the producer's and consumer's risk requirements. In practice, an effective sampling plan should provide adequate protection to both the producer and the consumer by controlling the probabilities of accepting or rejecting submitted lots at different quality levels.

Let α_p and β_c denote the producer's and consumer's risks, respectively. The producer's risk is defined as the probability of rejecting a lot whose quality is at the acceptable quality level (AQL), whereas the consumer's risk is the probability of accepting a lot whose quality is at the limiting quality level (LQL).

Let r_1 and r_2 denote the quality ratios corresponding to the AQL and LQL, respectively, where $r_1 > r_2$. The associated probabilities of failure are obtained from Equation (17) and are denoted by

DOI: https://doi.org/10.54216/IJNS.270244

$$p_1 = p(r_1), \qquad (25)$$

and

$$p_2 = p(r_2). \qquad (26)$$

To satisfy the producer's risk requirement, the probability of accepting a good-quality lot should be at least $1 - \alpha_p$. Hence,

$$P_a(p_1) \geq 1 - \alpha_p. \qquad (27)$$

Similarly, to satisfy the consumer's risk requirement, the probability of accepting a poor-quality lot should not exceed β_c. Therefore,

$$P_a(p_2) \leq \beta_c. \qquad (28)$$

Among all feasible sampling plans satisfying Equations (27) and (28), the optimal plan is selected by minimizing the sample size. Accordingly, the optimization problem is formulated as

$$\min n,$$

subject to

$$P_a(p_1) \geq 1 - \alpha_p,$$

$$P_a(p_2) \leq \beta_c,$$

$$0 \leq c_1 < c_2 < n, \quad m \geq 1, n > 1.$$

The resulting optimal values of (n, c_1, c_2, m) are obtained through an exhaustive search over all feasible combinations satisfying the above constraints. The computed optimal sampling plans for different choices of the design parameters are reported in the numerical results section.

3.4. *Algorithm for Determining the Optimal Sampling Plan.*

The optimal parameters of the proposed Multiple Dependent State Sampling Plan are determined through an exhaustive search procedure. The objective is to identify the sampling plan with the minimum sample size while satisfying both the producer's and consumer's risk requirements.

The computational procedure is summarized as follows.

1. Specify the design parameters, namely the producer's risk α_p, the consumer's risk β_c, the percentile level q, the termination ratio k, and the quality ratios r_1 and r_2 corresponding to the acceptable and limiting quality levels, respectively.
2. Compute the auxiliary quantity C_q using Equation (9).
3. Evaluate the probabilities of failure p_1 and p_2 from Equation (17) by substituting $r = r_1$ and $r = r_2$, respectively.
4. Select feasible values of the plan parameters (n, c_1, c_2, m) satisfying

$0 \leq c_1 < c_2 < n, \qquad m \geq 1.$

5. Compute the probability of lot acceptance $P_a(p_1)$ and $P_a(p_2)$ using Equation (22).
6. Verify whether the following conditions are satisfied:

$P_a(p_1) \geq 1 - \alpha_p$, and $P_a(p_2) \leq \beta_c$.

7. Repeat the search over all feasible combinations of (n, c_1, c_2, m) until all admissible sampling plans are obtained.
8. Among all feasible plans, select the one having the minimum sample size n. If more than one plan possesses the same minimum sample size, the plan with the smallest acceptance numbers (c_1, c_2) is selected.

The above algorithm is implemented in the **R** statistical software to obtain the optimal sampling plans reported in the subsequent numerical study.

4. Numerical Results and Discussion.

This section presents the numerical investigation of the proposed Multiple Dependent State Sampling Plan (MDSSP) developed for the EOEGE–E distribution. The optimal plan parameters are obtained by minimizing the sample size

DOI: https://doi.org/10.54216/IJNS.270244

while satisfying the producer's and consumer's risk requirements described in Section 3.3. The numerical study examines the effects of the termination ratio, the quality ratio, and the consumer's risk on the optimal sampling plans. Furthermore, the proposed MDSSP is compared with the conventional Single Sampling Plan (SSP), followed by an illustration using a real COVID-19 mortality dataset.

4.1. *Optimal Sampling Plans.*

Tables 1–3 summarize the optimal parameters of the proposed MDSSP under different combinations of the model parameters and design specifications. Each table reports the minimum sample size n, the acceptance numbers (c_1, c_2), the dependent-state parameter m, and the corresponding probability of acceptance at the producer's quality level, $P_a(p_1)$.

The optimal sampling plans were obtained using the optimization algorithm presented in Section 3.4. For each combination of the design parameters, the selected plan satisfies the prescribed producer's and consumer's risk constraints while minimizing the required sample size.

Table 1: Optimal parameters of the proposed MDSSP for EOEGE-E with $\theta = 0.8, \lambda = 0.9$ *and* $\gamma = 3.2$

β_c	r	$k=0.5$					$k=0.7$					$k=1.00$				
		n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$
0.25	2	31	1	2	3	0.976460	13	1	2	2	0.982273	5	1	2	1	0.990961
	4	16	0	1	3	0.995038	7	0	1	2	0.996196	3	0	1	1	0.997549
	6	16	0	1	3	0.999343	7	0	1	2	0.999503	3	0	1	1	0.999686
	8	16	0	1	3	0.999847	7	0	1	2	0.999885	3	0	1	1	0.999927
	10	16	0	1	3	0.999951	7	0	1	2	0.999963	3	0	1	1	0.999977
0.1	2	44	1	3	3	0.950336	18	1	2	2	0.952908	7	1	2	1	0.971505
	4	26	0	1	2	0.990838	11	0	1	2	0.990877	4	0	1	1	0.995586
	6	26	0	1	2	0.998769	11	0	1	2	0.998777	4	0	1	1	0.999427
	8	26	0	1	2	0.999712	11	0	1	2	0.999714	4	0	1	1	0.999867
	10	26	0	1	2	0.999907	11	0	1	2	0.999908	4	0	1	1	0.999958
0.05	2	57	1	3	1	0.951853	23	1	3	1	0.959062	8	1	2	1	0.956869
	4	34	0	1	2	0.984819	14	0	1	1	0.991086	5	0	1	1	0.993101
	6	34	0	1	2	0.997916	14	0	1	1	0.998814	5	0	1	1	0.999095
	8	34	0	1	2	0.999510	14	0	1	1	0.999723	5	0	1	1	0.999790
	10	34	0	1	2	0.999842	14	0	1	1	0.999911	5	0	1	1	0.999933
0.01	2	95	2	5	2	0.950401	38	2	4	2	0.955609	14	2	3	1	0.963268
	4	52	0	1	1	0.978948	21	0	1	1	0.980749	7	0	1	1	0.986679
	6	52	0	1	1	0.997088	21	0	1	1	0.997356	7	0	1	1	0.998212
	8	52	0	1	1	0.999313	21	0	1	1	0.999378	7	0	1	1	0.999582
	10	52	0	1	1	0.999778	21	0	1	1	0.999799	7	0	1	1	0.999866

Discussion. Table 1 presents the optimal parameters of the proposed MDSSP for the EOEGE–E distribution with $\theta = 0.8$, $\lambda = 0.9$, and $\gamma = 3.2$. The optimal sampling plans were obtained for different combinations of the consumer's risk β_c, termination ratio k, and quality ratio r.

Several important observations can be made from the table. First, for any fixed values of β_c, and k, the required sample size decreases considerably as the quality ratio r increases. This behavior is expected because larger values of $r=\frac{t_q}{t_q^0}$ correspond to products with lifetimes substantially exceeding the specified lifetime, resulting in lower probabilities of failure before the termination time. Consequently, fewer inspected units are required to satisfy both the producer's and consumer's risk constraints.

Second, increasing the termination ratio from k = 0.5 to k = 1.0 consistently reduces the required sample size for all considered values of β_c. A longer termination time allows additional lifetime information to be collected during the experiment, thereby improving the effectiveness of the inspection procedure and reducing the sampling effort required to achieve the desired protection levels.

DOI: https://doi.org/10.54216/IJNS.270244

It is also evident that decreasing the consumer's risk from β_c= 0.25 to β_c = 0.01 substantially increases the required sample size. This result reflects the stricter protection imposed on the consumer, since reducing the probability of accepting poor-quality lots inevitably requires more extensive inspection.

Furthermore, the probability of acceptance at the producer's quality level, $P_a(p_1)$, always exceeds the required value of $1 - \alpha_p$, confirming that all reported sampling plans satisfy the producer's risk requirement while maintaining the minimum feasible sample size.

Table 2: Optimal parameters of the proposed MDSSP for EOEGE-*E with* θ = 0.5, λ = 0.5 *and* γ = 3.2

β_c	r	$k = 0.5$					$k = 0.7$					$k = 1.00$				
		n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$
0.25	2	25	2	4	2	0.958358	14	2	4	2	0.967471	7	2	3	2	0.974115
	4	9	0	1	2	0.954439	5	0	1	2	0.958392	3	0	1	1	0.971510
	6	9	0	1	2	0.986050	5	0	1	2	0.987415	3	0	1	1	0.991737
	8	9	0	1	2	0.994179	5	0	1	2	0.994775	3	0	1	1	0.996628
	10	9	0	1	2	0.997080	5	0	1	2	0.997386	3	0	1	1	0.998328
0.1	2	41	3	7	2	0.950906	23	3	5	2	0.952185	9	2	5	2	0.952934
	4	24	1	2	2	0.985496	13	1	2	2	0.989127	4	0	1	1	0.950952
	6	14	0	1	2	0.968439	8	0	1	1	0.981028	4	0	1	1	0.985342
	8	14	0	1	2	0.986466	8	0	1	1	0.992074	4	0	1	1	0.993946
	10	14	0	1	2	0.993114	8	0	1	1	0.996023	4	0	1	1	0.996980
0.05	2	56	4	10	2	0.950239	31	4	6	2	0.952151	13	3	5	2	0.957695
	4	29	1	2	1	0.980315	16	1	2	1	0.984050	8	1	2	1	0.990216
	6	18	0	1	2	0.950678	10	0	1	1	0.971179	5	0	1	1	0.977434
	8	18	0	1	2	0.978394	10	0	1	1	0.987797	5	0	1	1	0.990567
	10	18	0	1	2	0.988884	10	0	1	1	0.993833	5	0	1	1	0.995265
0.01	2	89	6	9	1	0.953700	44	5	8	1	0.950905	20	4	7	1	0.953694
	4	40	1	2	1	0.951337	22	1	2	1	0.959603	11	1	2	1	0.973191
	6	40	1	2	1	0.992265	15	0	2	1	0.956113	7	0	1	1	0.957715
	8	27	0	1	2	0.955167	15	0	1	1	0.973794	7	0	1	1	0.981899
	10	27	0	1	2	0.976356	15	0	1	1	0.986527	7	0	1	1	0.990801

Discussion. Table 2 reports the optimal MDSSP parameters for the EOEGE–E distribution with θ = 0.5, λ = 0.5, and γ = 3.2. Although the distribution parameters differ from those used in Table 1, the proposed sampling plan exhibits similar monotonic behavior with respect to the design parameters.

For all values of the consumer's risk, increasing the quality ratio leads to a noticeable reduction in the optimal sample size. This trend confirms that products possessing higher lifetime performance require less inspection effort because the probability of failure before the truncation time decreases as the lifetime quality improves.

The influence of the termination ratio is also pronounced. Larger values of k consistently produce smaller sample sizes, indicating that extending the duration of the life test enhances the capability of the proposed MDSSP to discriminate between acceptable and unacceptable lots. Consequently, reliable inspection decisions can be achieved with fewer tested units.

Compared with Table 1, it can be observed that the optimal sample sizes are generally larger under the current parameter configuration. This indicates that the underlying lifetime distribution plays an important role in determining the efficiency of the proposed sampling plan. Different parameter combinations produce different failure probabilities, which directly influence the optimal design through the probability of lot acceptance.

Finally, the reported values of $P_a(p_1)$ remain above the target producer's acceptance probability, demonstrating that the proposed optimization procedure successfully satisfies both risk requirements for all reported sampling plans.

DOI: https://doi.org/10.54216/IJNS.270244

Table 3: Optimal parameters of the proposed MDSSP for EOEGE - E with θ = 0.5, λ = 0.9 *and* γ = 3.5

β_c	r	$k=0.5$					$k=0.7$					$k=1.00$				
		n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$
0.25	2	26	2	3	2	0.958049	14	2	3	2	0.965716	5	1	2	1	0.951935
	4	9	0	1	3	0.962259	5	0	1	2	0.971778	3	0	1	1	0.978981
	6	9	0	1	3	0.989687	5	0	1	2	0.992545	3	0	1	1	0.994629
	8	9	0	1	3	0.996049	5	0	1	2	0.997182	3	0	1	1	0.997996
	10	9	0	1	3	0.998148	5	0	1	2	0.998688	3	0	1	1	0.999073
0.1	2	44	3	5	2	0.959679	20	2	5	1	0.953032	9	2	4	2	0.960562
	4	17	0	2	1	0.958438	9	0	2	1	0.961619	4	0	1	1	0.963476
	6	15	0	1	2	0.980233	8	0	1	2	0.981735	4	0	1	1	0.990410
	8	15	0	1	2	0.992317	8	0	1	2	0.992934	4	0	1	1	0.996385
	10	15	0	1	2	0.996371	8	0	1	2	0.996671	4	0	1	1	0.998319
0.05	2	53	3	6	1	0.950892	27	3	7	2	0.950215	11	2	5	1	0.950919
	4	30	1	2	3	0.984334	16	1	2	2	0.989074	5	0	2	1	0.958490
	6	19	0	1	2	0.969543	10	0	1	2	0.972417	5	0	1	1	0.985139
	8	19	0	1	2	0.987961	10	0	1	2	0.989166	5	0	1	1	0.994339
	10	19	0	1	2	0.994266	10	0	1	2	0.994857	5	0	1	1	0.997354
0.01	2	85	5	8	1	0.959326	45	5	7	1	0.958181	19	4	6	2	0.951890
	4	42	1	2	1	0.973043	22	1	2	1	0.977852	11	1	2	1	0.983239
	6	29	0	1	1	0.958162	15	0	1	1	0.963288	7	0	1	1	0.971795
	8	29	0	1	1	0.983354	15	0	1	1	0.985511	7	0	1	1	0.989033
	10	29	0	1	1	0.992046	15	0	1	1	0.993105	7	0	1	1	0.994821

Discussion. Table 3 summarizes the optimal MDSSP parameters for the EOEGE–E distribution with θ = 0.5, λ = 0.9, and γ = 3.5. Comparing the results with those reported in Tables 1 and 2 reveals the influence of the distribution parameters on the optimal sampling design.

As in the previous tables, increasing the quality ratio substantially decreases the required sample size, whereas increasing the termination ratio leads to more economical inspection plans. These consistent trends demonstrate the robustness of the proposed MDSSP under different parameter settings of the EOEGE–E distribution.

A comparison with Table 2 shows that modifying the distribution parameters changes the optimal combinations of (n, c_1, c_2, m). This behavior is expected because the parameters θ, λ, and γ directly affect the failure probability before the truncation time, which in turn determines the probability of lot acceptance and the resulting optimal sampling plan.

Another noteworthy observation is that the proposed optimization procedure continues to satisfy the producer's risk requirement across all parameter settings while maintaining relatively small sample sizes, confirming the flexibility of the proposed MDSSP for different lifetime distributions generated by the EOEGE–E model.

The numerical results demonstrate that, although the optimal sampling parameters depend on the values of the EOEGE–E distribution parameters, the overall behavior of the proposed MDSSP remains stable. Specifically, increasing the quality ratio or the termination ratio consistently reduces the required sample size, whereas imposing stricter consumer protection increases the inspection effort. This stability indicates that the proposed MDSSP possesses robust operating characteristics over a wide range of distributional parameter settings.

4.2. *Comparison with the Single Sampling Plan.*

The performance of the proposed Multiple Dependent State Sampling Plan (MDSSP) is further evaluated by comparing it with the conventional Single Sampling Plan (SSP) under identical design specifications. Since both sampling plans are designed to satisfy the same producer's and consumer's risk requirements, the comparison focuses primarily on the required sample size, which directly reflects the inspection cost and testing effort.

Table 4 presents the optimal sampling parameters obtained using both sampling schemes. It is evident that the proposed MDSSP consistently requires a smaller sample size than the corresponding SSP for almost all combinations of the design parameters considered in this study.

DOI: https://doi.org/10.54216/IJNS.270244

Table 4: Comparative overview of MDSSP and SSP under the EOEGE - E distribution using fixed values of θ = 0.5, λ = 0.9, γ = 2.6

β_c	r	$a=0.5$								$a=1$							
		MDSSP					SSP			MDSSP					SSP		
		n	c_1	c_2	m	$P_a(P_1)$	n	c	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$	n	c	$P_a(P_1)$
0.25	2	25	3	5	2	0.950098	41	6	0.9515	8	2	5	1	0.954850	14	5	0.9531
	4	13	1	2	2	0.983728	19	2	0.9757	5	1	2	1	0.992057	7	2	0.9848
	6	7	0	1	2	0.962423	13	1	0.9737	3	0	1	1	0.974774	5	1	0.9786
	8	7	0	1	2	0.981011	13	1	0.9870	3	0	1	1	0.987608	5	1	0.9895
	10	7	0	1	2	0.988973	13	1	0.9925	3	0	1	1	0.992919	2	0	0.9505
0.10	2	44	5	8	2	0.951460	74	10	0.9604	14	4	6	2	0.955793	21	7	0.9533
	4	18	1	2	2	0.956510	25	2	0.9502	7	1	2	1	0.974768	9	2	0.9678
	6	12	0	2	1	0.953487	18	1	0.9516	4	0	1	1	0.956401	7	1	0.9578
	8	11	0	1	2	0.956735	18	1	0.9755	4	0	1	1	0.978187	7	1	0.9790
	10	11	0	1	2	0.974309	18	1	0.9858	4	0	1	1	0.987408	7	1	0.9879
0.05	2	57	6	9	1	0.950909	86	11	0.9504	18	5	7	2	0.950047	28	9	0.9570
	4	23	1	3	1	0.962504	36	3	0.9685	8	1	2	1	0.961658	13	3	0.9820
	6	22	1	2	1	0.986831	29	2	0.9808	5	0	2	1	0.950457	11	2	0.9858
	8	14	0	1	1	0.957145	22	1	0.9644	5	0	1	1	0.966672	8	1	0.9726
	10	14	0	1	1	0.974701	22	1	0.9791	5	0	1	1	0.980565	8	1	0.9841
0.01	2	89	9	14	1	0.956374	132	16	0.9558	28	7	11	1	0.958052	40	12	0.9517
	4	39	2	4	1	0.971980	54	4	0.9658	14	2	3	1	0.968290	17	3	0.9540
	6	30	1	2	1	0.967490	39	2	0.9584	11	1	2	1	0.977769	14	2	0.9720
	8	30	1	2	1	0.988823	39	2	0.9841	7	0	2	1	0.954110	14	2	0.9897
	10	21	0	2	1	0.961697	30	1	0.9627	7	0	1	1	0.963403	11	1	0.9703

Discussion. The results presented in Table 4 clearly demonstrate the effciency of the proposed MDSSP over the conventional SSP. For every combination of the consumer's risk, termination ratio, and quality ratio, the proposed plan achieves the required producer's and consumer's protection levels with fewer inspected units.

The reduction in sample size is primarily attributed to the dependent-state mechanism incorporated into the proposed sampling plan. Unlike the SSP, which bases the acceptance decision solely on the current lot, the MDSSP utilizes information from previously accepted lots. Incorporating historical quality information improves the efficiency of the inspection process and consequently reduces the number of units that must be tested before reaching a reliable decision.

Another important observation is that the advantage of the proposed MDSSP becomes more pronounced when the consumer's risk requirement becomes stricter. Under these con- ditions, the SSP requires a substantial increase in sample size to satisfy the specified risk constraints, whereas the proposed MDSSP maintains considerably smaller sample sizes by effectively exploiting the information from preceding lots.

Overall, the comparison confirms that the proposed MDSSP provides a more economical inspection strategy than the traditional SSP while preserving the desired protection levels for both producers and consumers. Therefore, the proposed procedure offers an attractive alter- native for lifetime acceptance sampling, particularly in industrial applications where reducing inspection cost and testing time is of primary importance.

4.3. *Operating Characteristic Curve.*

Figure 1 compares the operating characteristic (OC) curves of the proposed Multiple Dependent State Sampling Plan (MDSSP) and the conventional Single Sampling Plan (SSP) under the EOEGE–E distribution. The comparison is based on the optimal sampling plans obtained under identical design specifications.

As expected, the probability of lot acceptance decreases monotonically as the failure prob- ability increases for both sampling schemes. When the failure probability is small, corresponding to high-quality lots, both plans exhibit acceptance probabilities close to one, indicating a very low likelihood of rejecting conforming lots. Conversely, as the failure probabil- ity increases, the probability of lot acceptance declines rapidly, providing effective protection against accepting poor-quality lots.

DOI: https://doi.org/10.54216/IJNS.270244

A closer examination of Figure 1 shows that the OC curve of the proposed MDSSP lies below that of the conventional SSP over most of the intermediate range of failure probabilities. This indicates that the proposed plan is more discriminative, as it reduces the probability of accepting marginal-quality lots while preserving a high probability of accepting good-quality lots.

It is also noteworthy that the proposed MDSSP achieves this improved discriminatory performance with a considerably smaller sample size (n = 18) than the corresponding SSP (n = 28). Therefore, the proposed plan provides a more economical inspection procedure by reducing the number of tested units without compromising the required producer's and consumer's protection levels.

Overall, the OC curves confirm that incorporating the dependent-state mechanism en- hances the efficiency of lifetime acceptance sampling. By utilizing information from preceding lots, the proposed MDSSP simultaneously reduces inspection effort and maintains reliable decision-making performance.

For the design considered in Figure 1, the proposed MDSSP reduces the required sample size from 28 to 18 units, corresponding to a reduction of approximately 35.7% in inspection effort compared with the conventional SSP. This substantial reduction highlights the practical advantage of the proposed sampling plan in applications where inspection cost and testing time are critical considerations.

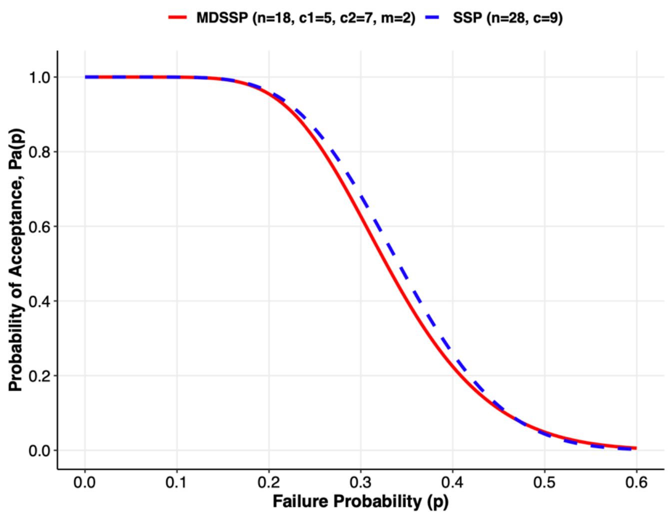

Figure 1. OC Curves for MDSSP and SSP under EOEGE-E.

Overall, the numerical results confirm that the proposed MDSSP provides a practical and efficient alternative to conventional acceptance sampling procedures. The advantages become increasingly apparent for higher-quality products and larger dependent-state param- eters, where the reduction in sample size is more pronounced.

5 Real Data Application.

5.1. *Description of the Dataset*.

To demonstrate the practical applicability of the proposed Multiple Dependent State Sampling Plan (MDSSP), a real lifetime dataset was analyzed. The dataset consists of 76 daily COVID-19 mortality observations recorded in the United Kingdom over the period from April 15 to June 30, 2020. These data were previously analyzed by El-Metwally and Mubarak (2021) and have been widely employed as a benchmark for evaluating flexible lifetime distributions.

The observations are positive continuous measurements exhibiting considerable variability, making them suitable for reliability and lifetime modeling. Consequently, this dataset provides an appropriate benchmark for evaluating the capability of the proposed Extended Odd Exponential Generalized Exponential (EOEGE–E) distribution to represent real lifetime data and for illustrating the practical implementation of the proposed MDSSP under truncated life tests. The complete dataset is reported in Table 5.

DOI: https://doi.org/10.54216/IJNS.270244

Table 5: The UK's COVID-19 data covering 76 consecutive days between April 15 and June 30, 2020.

0.0587	0.0863	0.1165	0.1247	0.1277	0.1303
0.1652	0.2079	0.2395	0.2751	0.2845	0.2992
0.3188	0.3317	0.3446	0.3553	0.3622	0.3926
0.3926	0.4110	0.4633	0.4690	0.4954	0.5139
0.5696	0.5837	0.6197	0.6365	0.7096	0.7193
0.7444	0.8590	1.0438	1.0602	1.1305	1.1468
1.1533	1.2260	1.2707	1.3423	1.4149	1.5709
1.6017	1.6083	1.6324	1.6998	1.8164	1.8392
1.8721	1.9844	2.1360	2.3987	2.4153	2.5225
2.7087	2.7946	3.3609	3.3715	3.7840	3.9042
4.1969	4.3451	4.4627	4.6477	5.3664	5.4500
5.7522	6.4241	7.0657	7.4456	8.2307	9.6315
10.1870	11.1429	11.2019	11.4584		

5.2. *Model Fitting and Goodness-of-Fit Assessment*.

Before implementing the proposed Multiple Dependent State Sampling Plan, it is necessary to verify that theunderlying lifetime distribution provides an adequate representation of the observed data. To this end, the parameters of the EOEGE-E distribution were estimated by the maximum likelihood method, followed by graphical and numerical goodness-of-fit analyses.

Figure 2 presents the graphical diagnostics of the fitted EOEGE-E distribution. Overall, the diagnostic plots indicate an excellent agreement between the fitted model and the observed data. The P–P plot shows that the empirical probabilities closely follow the theoretical proba- bilities, whereas the Q–Q plot demonstrates a nearly linear relationship between the empirical and theoretical quantiles, with only minor deviations appearing in the upper tail.

The histogram superimposed with the fitted probability density function illustrates that the proposed distribution accurately captures the overall shape of the data, including its pronounced asymmetry. Likewise, the empirical cumulative distribution function closely matches the fitted cumulative distribution function throughout the entire range of observa- tions. The violin plot further confirms the asymmetric nature and variability of the data, while the TTT plot suggests a non-constant hazard rate, supporting the use of a flexible life- time distribution such as the proposed EOEGE–E model.

Collectively, these graphical diagnostics provide strong visual evidence that the EOEGE– E distribution adequately represents the observed COVID–19 mortality data.

The maximum likelihood estimates of the model parameters together with their standard errors and goodness-of-fit statistics are summarized in Table 6.

The estimated parameters indicate that the EOEGE–E distribution provides sufficient flexi- bility for modeling the observed lifetime data. More importantly, the goodness-of-fit statistics strongly support the adequacy of the fitted model. The Kolmogorov–Smirnov statistic is rela- tively small (KS = 0.0578), while the corresponding p-value (0.9614) is considerably larger than the conventional significance level of 0.05. Therefore, there is insufficient evidence to reject the null hypothesis that the data are consistent with the fitted EOEGE–E distribution.

Furthermore, the proposed model yields smaller information criteria values than the competing Weibull distribution while simultaneously producing a larger maximized log- likelihood. These results indicate that the proposed EOEGE–E distribution achieves a better balance between model flexibility and model complexity, leading to an improved representation of the observed data.

Consequently, the fitted EOEGE–E distribution provides a reliable probabilistic basis for constructing the proposed Multiple Dependent State Sampling Plan in the subsequent analysis.

DOI: https://doi.org/10.54216/IJNS.270244

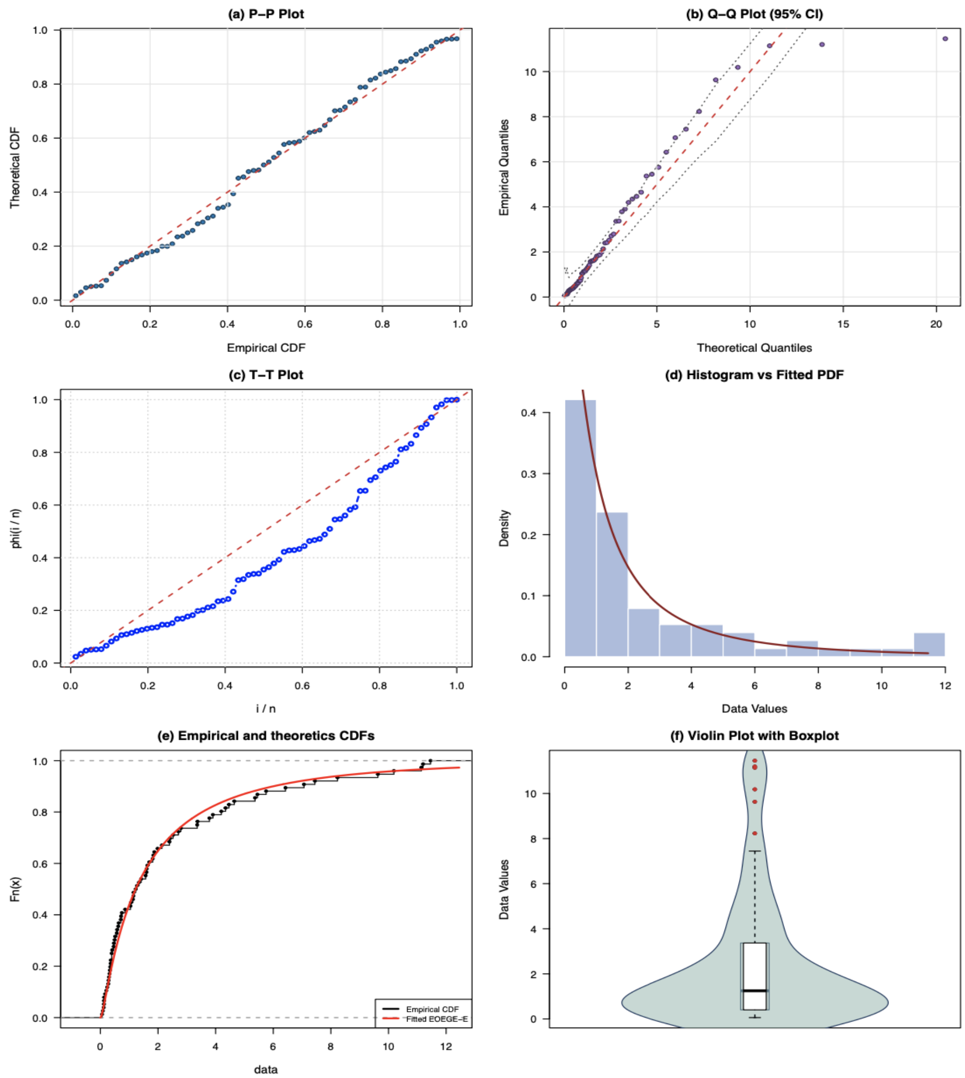

Figure 2. Graphical goodness-of-fit diagnostics for the fitted EOEGE–E distribution.

Table 6 :Maximum likelihood estimates, standard errors, and goodness-of-fit statistics for the fitted models.

Distribution	Estimates	Std error	LL	AIC	BIC	CAIC	KS	P-value
Weibull	$\hat{\theta} = 0.8465908$ $\hat{\lambda} = 2.2199924$	0.0743471 0.3184661	-141.7519	287.5037	292.1652	287.6681	0.0805580	0.7074000
Proposed Model	$\hat{\alpha} = 0.1251997$ $\hat{\theta} = 12.4042703$ $\hat{\lambda} = 3.6492896$ $\hat{\gamma} = 0.2232199$	0.1282037 23.3189604 7.2354399 0.2203047	-139.5988	287.1977	296.5206	287.7610	0.05779388	0.9614390

5.3. *Implementation of the Proposed MDSSP*.

Having established the adequacy of the EOEGE–E distribution for modeling the COVID–19 mortality data, the estimated parameter values were substituted into the design equations developed in Section 3 to construct the proposed Multiple Dependent State Sampling Plan (MDSSP). The optimal sampling plans were obtained by minimizing the sample size while satisfying the prescribed producer's and consumer's risk requirements.

DOI: https://doi.org/10.54216/IJNS.270244

The resulting optimal plan parameters are summarized in Table 7. For every combination of the design parameters, feasible sampling plans were successfully obtained, demonstrating the applicability of the proposed optimization algorithm to real lifetime data.

Several important observations can be drawn from Table 7. First, the required sample size increases as the consumer's risk β becomes more restrictive. For example, under a fixed termination ratio and quality ratio, reducing β_c from 0.25 to 0.01 leads to a substantial increase in the minimum sample size. This behavior is expected because stronger consumer protection requires additional inspection evidence before accepting a submitted lot.

Second, increasing the quality ratio $r = \frac{t_q}{t_q^0}$ consistently reduces the required sample size. Larger values of r correspond to products whose true lifetime substantially exceeds the specified lifetime, resulting in lower failure probabilities before the truncation time. Consequently, fewer test units are required to satisfy the prescribed producer's and consumer's risk constraints.

The termination ratio also has a noticeable influence on the optimal sampling plans. As the termination ratio k increases, the required sample size generally decreases. A longer truncation time allows more lifetime information to be collected during the experiment, thereby improving the efficiency of the inspection procedure and reducing the number of units that must be tested.

Another noteworthy observation is that the acceptance constants (c_1, c_2) and the dependent- state parameter m remain relatively stable for moderate and large quality ratios. Most of the adjustments required to satisfy the risk constraints are therefore achieved through changes in the sample size rather than through substantial modifications of the acceptance criteria. This behavior indicates that the proposed optimization procedure naturally converges toward simple and practically implementable inspection rules.

Finally, the producer's acceptance probabilities reported in Table 7 consistently exceed the required protection level of $1 - \alpha_p$, confirming that all obtained sampling plans simultaneously satisfy both the producer's and consumer's risk requirements. Overall, these results demonstrate that the proposed MDSSP can be effectively implemented in practical reliability applications while maintaining economical inspection costs and reliable decision-making performance.

Table 7: Optimal MDSSP parameters obtained from the fitted EOEGE–E distribution for the COVID–19 mortality dataset.

β_c	r	$k=0.5$					$k=0.7$					$k=1.00$				
		n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$	n	c_1	c_2	m	$P_a(P_1)$
0.25	2	26	5	8	1	0.950883	21	6	9	2	0.951959	21	8	11	2	0.951906
	4	9	1	2	1	0.970411	6	1	2	3	0.951874	7	2	3	2	0.977237
	6	5	0	1	1	0.958002	4	0	2	1	0.950375	5	1	2	1	0.987320
	8	4	0	1	3	0.972461	3	0	1	2	0.970328	3	0	1	1	0.954989
	10	4	0	1	3	0.985440	3	0	1	2	0.983806	3	0	1	1	0.974298
0.1	2	43	8	12	1	0.950232	37	10	16	2	0.951209	33	12	19	2	0.950237
	4	13	1	4	1	0.954100	12	2	3	2	0.953858	9	2	4	2	0.954999
	6	12	1	2	1	0.985903	9	1	2	1	0.977321	7	1	2	1	0.960913
	8	7	0	2	2	0.951810	6	0	2	1	0.950280	7	1	2	1	0.986762
	10	7	0	1	2	0.969622	5	0	1	2	0.958120	4	0	1	1	0.955604
0.05	2	55	10	16	1	0.952101	47	12	17	1	0.952207	43	15	20	1	0.955618
	4	19	2	3	1	0.954119	14	2	6	2	0.950490	13	3	5	2	0.964112
	6	14	1	2	2	0.970200	11	1	2	1	0.958395	8	1	3	1	0.966049
	8	10	0	3	1	0.950569	11	1	2	1	0.986798	8	1	2	1	0.979534
	10	9	0	1	1	0.969566	7	0	1	1	0.951028	8	1	2	1	0.991717
0.01	2	84	15	21	1	0.950669	72	18	24	1	0.952412	62	21	28	1	0.951031
	4	30	3	5	1	0.970681	22	3	5	1	0.951816	19	4	7	2	0.952628
	6	20	1	3	1	0.964247	18	2	3	1	0.972657	14	2	4	1	0.972655
	8	20	1	2	1	0.980812	14	1	2	1	0.972438	11	1	3	1	0.969744
	10	14	0	2	1	0.950430	14	1	2	1	0.989295	11	1	2	1	0.977119

DOI: https://doi.org/10.54216/IJNS.270244

5.4. *Comparison with the Weibull-Based MDSSP.*

To further evaluate the practical performance of the proposed methodology, the optimal Multiple Dependent State Sampling Plans obtained under the fitted EOEGE–E distribution were compared with those constructed using the classical Weibull distribution. The comparison was performed using the same COVID–19 mortality dataset and identical design specifications in terms of the producer's risk, consumer's risk, quality ratio, and termination ratio. The corresponding optimal sampling plans are reported in Table 8. For each design configuration, both sampling plans satisfy the prescribed producer's and consumer's risk requirements, allowing the comparison to focus primarily on the inspection effort measured by the required sample size. The results clearly demonstrate that the proposed EOEGE–E-based MDSSP consistently requires fewer inspected units than the Weibull-based MDSSP. The reduction in sample size ranges from approximately 24% to 33%, depending on the selected design parameters. Such reductions represent a substantial improvement in inspection efficiency while maintaining the same protection levels for both producers and consumers. The observed improvement can be attributed to the superior fitting performance of the EOEGE–E distribution for the COVID–19 mortality data. As demonstrated in the previous subsection, the proposed distribution provides a better representation of the underlying lifetime behavior than the Weibull model. Consequently, the estimated failure probabilities more accurately reflect the actual reliability characteristics of the data, leading to more efficient sampling plans with smaller required sample sizes. It is also noteworthy that the producer's acceptance probabilities remain very close for both distributions, indicating that the reduction in sample size is not achieved at the expense of producer protection. Similarly, the consumer's risk remains within the specified limits for all reported sampling plans. Therefore, the proposed methodology achieves greater inspection efficiency without compromising the prescribed operating characteristics. Overall, the comparison confirms that employing the EOEGE–E distribution as the un- derlying lifetime model leads to more economical Multiple Dependent State Sampling Plans than those obtained under the Weibull distribution. This improvement is particularly attractive in truncated life testing applications, where reducing the required sample size directly decreases inspection cost and testing time while preserving the desired reliability protection.

Table 8: Comparison of the optimal MDSSP parameters obtained from the Weibull and EOEGE–E distributions for the COVID–19 mortality dataset.

β_c	r	Distribution	n	c_1	c_2	m	$P_a(p_1)$	$P_a(p_2)$	Reduction in n (%)
0.25	2	Weibull	29	9	14	2	0.9518	0.2406	–
		EOEGE-E	21	6	9	2	0.9520	0.2422	**27.59%**
	4	Weibull	9	2	5	3	0.9529	0.2381	–
		Proposed	6	1	2	3	0.9519	0.2486	**33.33%**
0.10	2	Weibull	49	14	20	1	0.9503	0.0993	–
		Proposed	37	10	16	2	0.9512	0.0891	**24.49%**
	4	Weibull	16	3	6	1	0.9568	0.0936	–
		Proposed	12	2	3	2	0.9539	0.0922	**25.00%**
0.05	2	Weibull	64	18	26	1	0.9501	0.0496	–
		Proposed	47	12	17	1	0.9522	0.0467	**26.56%**
	4	Weibull	21	4	7	1	0.9601	0.0475	–
		Proposed	14	2	6	2	0.9505	0.0456	**33.33%**
0.01	2	Weibull	97	27	34	1	0.9502	0.0095	–
		Proposed	72	18	24	1	0.9524	0.0090	**25.77%**
	4	Weibull	32	6	9	1	0.9569	0.0098	–
		Proposed	22	3	5	1	0.9518	0.0096	**31.25%**

6. Conclusion

This paper developed a Multiple Dependent State Sampling Plan (MDSSP) based on the Extended Odd Exponential Generalized Exponential (EOEGE–E) distribution for truncated life tests. The proposed methodology determines the optimal sampling plan by minimizing the required sample size while satisfying the specified producer's and consumer's risk requirements. An efficient optimization procedure was employed to obtain the optimal design parameters under different combinations of the termination ratio, quality ratio, and consumer's risk. The numerical results demonstrated that the proposed MDSSP exhibits stable and reli- able operating characteristics across a wide range of design settings. In particular, the required sample size decreases with increasing quality ratio and termination

DOI: https://doi.org/10.54216/IJNS.270244

ratio, whereas more stringent consumer protection results in larger sample sizes. Moreover, all obtained sampling plans satisfy the prescribed producer's and consumer's risk requirements, confirm- ing the effectiveness of the proposed design methodology. The practical applicability of the proposed methodology was illustrated using a real COVID–19 mortality dataset. The goodness-of-fit analysis showed that the EOEGE–E dis- tribution provides an adequate representation of the observed lifetime data, supporting its suitability as the underlying lifetime model for constructing the proposed sampling plan. Furthermore, comparison with the Weibull-based MDSSP demonstrated that the proposed methodology achieves the required operating characteristics while requiring fewer inspected units, leading to a more economical inspection procedure. Overall, the proposed MDSSP provides an efficient and practical framework for lifetime acceptance sampling under truncated life tests. By combining a flexible lifetime distribution with a dependent-state sampling strategy, the proposed methodology enhances inspection efficiency while reducing the required sample size, inspection cost, and testing effort, making it suitable for reliability assessment and industrial quality control applications. Future research may consider extending the proposed methodology to group and repetitive group sampling plans, Bayesian and adaptive sampling schemes, accelerated life testing, progressively censored data, and other flexible lifetime distributions. Such extensions may further improve the applicability and efficiency of acceptance sampling procedures in mod- ern reliability and quality engineering.

Acknowledgment: The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4340290GSSR01

Funding Statement: This research work was funded by Umm Al-Qura University, Saudi Arabia, under grant number: 26UQU4340290GSSR01.

Conflicts of Interest: The authors declare no conflict of interest.

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