Neutrosophic Quotient Algebra

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Abstract

The algebraic properties of neutrsphic ideals over algebra, isomorphism properties of neutrosophic ideal and neutrosophic modules over algebra are discussed in this paper. Some of the charactrisations of Neutrosophic quotient algebra are derived and the role of algebraic structures is studied in the context of neutrosophic set. This paper expands the definition of quotient algebra within the context of neutrosophical set.

Keywords: Neutrosophic algebra over a neutrosophic subfield, Neutrosophic ideal, Neutrosophic quotient algebra.

1.Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms 0 and 1; according to a bivalent condition-an element either belongs or does not belong to the set. As an extension, fuzzy set theory permits the gradual assessment of the membership of elements in a set. A fuzzy set A in X is characterised by a membership function which is associated with each element in X, a real number in the interval [0,1**]**. Lotfi A Zadeh [1] introduced a theory whose objects fuzzy sets-are sets with imprecise boundaries which allow us to represent vague concepts and contexts in natural language. Fuzzy set theory is limited to modelling a situation involving uncertainty. As an extension of fuzzy set concept, the theory of intuitionistic fuzzy sets introduced whose elements have degree of membership and non membership. Intuitionistic fuzzy sets have been introduced by Krassimir Atanassov [2] as an extension of Lotfi Zadeh's notion of fuzzy set. Let us have a fixed universe X and A is a subset of X .The intuitionistic fuzzy set can be defined as $A = \{(x, \mu_A(x), \nu_A(x) / x \in X\}$ where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. μ for membership and ν for non membership, which belongs to the real unit interval [0,1] and sum belongs to the same interval.

Neutrosophy is a new branch of philosophy and logic introduced by Florentin Smarandache [3,4] in 1995 which studies the origin and features of neutralities in nature. Each proposition in Neutrosophic logic is approximated to have the percentage of truth (T), the percentage of indeterminacy (I) and the percentage of falsity (F). So this Neutrsophic logic is called generalization of classical logic, conventional fuzzy logic, intuitionistic fuzzy logic and interval valued fuzzy logic. This mathematical tool is used to handle problems like imprecise, indeterminate and inconsistent data. The use of neutrosophic theory becomes inevitable when a situation involving indeterminacy is to be modelled. The introduction of Neutrosophic theory has led to the establishment of the concept of

DOI: 10.5281/zenodo.3930148
Received: May 07, 2019 Revised: July 25, 2019 Accepted: octobre 02, 2019

neutrosophic algebraic structures in this article [5,6]. For more information on real applications of neutrosophic theory, the readers can see [13-15]

The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Among the different branches of applied and pure mathematics, abstract algebra was one of the first few area where research was conducted using the concept of neutrosophic set. Initially, B. Vasantha Kandasamy and Florentin Smarandache [7] introduced and applied fundamental algebraic neutrosophic structures. This paper focuses on algebra over a field , quotient algebra over a field and algebraic structures ideal in neutrosophic domain and derive some algebraic properties. This paper focuses on algebra over a field , quotient algebra over a field and algebraic structure ideal in neutrosophic domain and derives some algebraic properties.

2. Preliminaries

Abraham Robinson [8] introduced the non-standard analysis in the 1960s, a formalization of the analysis and a branch of mathematical logic that describes the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < 1/n$. Let $\varepsilon > 0$ be a infinitesimal number. Let us consider the non-standard finite numbers $1^+ = 1+\varepsilon$, where 1 is its standard part and ε its non-standard part, and $^-0 = 0-\in$, where 0 is its standard part and $\in$ its non-standard part. Then, we call] - 0, 1+ [a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. Generally the left and right borders of non standard interval $]^-a, b^+[$ are vague, imprecise and themselves being a non standard subsets.

Definition 2.1 [9,10] A Neutrosophic set A on the universal set X is defined as $A = \{\langle x, t_A(x), i_A(x), f_A(x)\rangle\}$ where $x \in X$ and $t_A, i_A, f_A : X \to]0^-, 1^+[$ where t, i & f are known as Neutrosophic components which are subsets of $]0^-, 1^+[$ and $^-0 \le t_A(x) + i_A(x) + f_A(x) \le 3^+$. A Neutrosophic set A can be written as $A = \sum_i^n (t_A(x_i), i(x_i), f(x_i))/x_i,\ x_i \in X$. Thus a Neutrosophic set has 3 components.

i) t represents membership value (Percentage of truth)

ii) i represents indeterminacy (Percentage of indeterminacy)

iii) f represents non membership value (Percentage of falsity)

Since the membership function t_A, i_A, f_A defined in real life and scientific applications are from X in to the unit interval $[0,1]$ as $t_A, i_A, f_A : X \to [0,1]$, a Neutrosophic set A will be denoted by a mapping defined by $A : X \to [0,1] \times [0,1] \times [0,1]$.

Example 2.1 Assume that $X = \{x, y, z\}$, x is hard work, y is capability and z is knowledge in particular area. They are obtained from the questionnaire of some domain experts about the question good researcher. A is single valued Neutrosophic set of X defined by

DOI: 10.5281/zenodo.3930148

$$A = {(0.4,0.5,0.2)}/{x} + {(0.5,02.0.3)}/{y} + {(0.6,0.2,0.3)}/{z}$$

Definition 2.2 [3,9] Let A and B be two Neutrosophic sets on X. Then

i) A is contained in B, denoted as $A \subseteq B$ if and only if $A(x) \leq B(x)$, $\forall x \in X$ this means that

$$t_A(x) \leq t_B(x), i_A(x) \leq i_B(x) \,\&\, f_A(x) \geq f_B(x)$$

ii) The union of A and B is denoted by $C = A \bigcup B$ and defined as $C(x) = A(x) \vee B(x)$ where

for each $x \in X$. This means that $A(x) \vee B(x) = \{t_A(x) \vee t_B(x), i_A(x) \vee i_B(x), f_A(x) \vee f_B(x)\}$. i.e.

$$t_C(x) = \max\{t_A(x), t_B(x)\}, i_C(x) = \max\{i_A(x), i_B(x)\} \,\&\, f_C(x) = \min\{f_A(x), f_B(x)\}$$

iii) The intersection of A and B is denoted by $C = A \bigcap B$ and defined as $C(x) = A(x) \wedge B(x)$

where $A(x) \wedge B(x) = \{t_A(x) \wedge t_B(x), i_A(x) \wedge i_B(x), f_A(x) \wedge f_B(x)\}$ for each $x \in X$.

i.e $t_C(x) = \min\{t_A(x), t_B(x)\}, i_C(x) = \min\{i_A(x), i_B(x)\} \,\&\, f_C(x) = \max\{f_A(x), f_B(x)\}$

iv) The compliment of A is denoted by A^C and defined as $A^C(x) = (f_A(x), 1 - i_A(x), t_A(x))$,

for each $x \in X$. Here $(A^C)^C = A$

Definition 2.3 [11] An algebra is an algebraic structure which consist of a set, together with multiplication, addition and scalar multiplication by elements of underline field and satisfies the axioms implied by vector field and bilinear. An algebra over a field is a vector space equipped with bilinear product.

Definition 2.4 [12] Let V be a vector space over a field F equipped with binary operation from $V \times V \rightarrow V$. Then V is an algebra over a field F if the following conditions hold $\forall x, y, z \in V$ and $a, b \in F$

- $(x + y).z = x.z + y.z$
- $z.(x + y) = z.x + z.y$
- $(ax).(by) = (ab)(x.y)$

3 Neutrosophic quotient algebra

This section defines neutrosophic quotient algebra over a field and derive some elementary properties by extending the concept of algbra over a field in neutrosophic set.

Definition 3.1 Let A be algebra over field F, then the neutrosophic subset N_A of A is called neutrosophic algebra over F if for all $x, y \in A, \alpha \in F$, we have

DOI: 10.5281/zenodo.3930148

i) $N_A(x-y) \geq N_A(x) \wedge N_A(y)$

$\Rightarrow t_A(x-y) \geq t_A(x) \wedge t_A(y), i_A(x-y) \geq i_A(x) \wedge i_A(y), f_A(x-y) \leq f_A(x) \vee f_B(y)$

ii) $N_A(xy) \geq N_A(x) \wedge N_A(y)$

$\Rightarrow t_A(xy) \geq t_A(x) \wedge t_A(y), i_A(xy) \geq i_A(x) \wedge i_A(y), f_A(xy) \leq f_A(x) \vee f_B(y)$

iii) $N_A(\alpha x) \geq N_A(x)$

$\Rightarrow t_A(ax) \geq t_A(x), i_A(\alpha x) \geq i_A(x) \wedge i_A(y), f_A(\alpha x) \leq f_A(x)$

iv) $N_A(0)=1$

$\Rightarrow t_A(0)=1, i_A(0)=1, f_A(0)=0$

Definition 3.2 Let N_F be a neutrosophic subset of a Field F. If $\lambda_1, \lambda_2 \in F$,

i) $N_F(\lambda_1 - \lambda_2) \geq N_F(\lambda_1) \wedge N_F(\lambda_2)$

$\Rightarrow t_F(x-y) \geq t_F(x) \wedge t_F(y), i_F(x-y) \geq i_F(x) \wedge i_F(y), f_F(x-y) \leq f_F(x) \vee f_F(y)$

ii) $N_F(\lambda_1 \lambda_2^{-1}) = N_F(\lambda_1) \wedge N_F(\lambda_2)$

$\Rightarrow t_F(\lambda_1\lambda_2^{-1}) \geq t_F(\lambda_1) \wedge t_F(\lambda_2), i_F(\lambda_1\lambda_2^{-1}) \geq i_F(\lambda_1) \wedge i_F(\lambda_2), f_F(\lambda_1\lambda_2^{-1}) \leq f_F(\lambda_1) \vee f_F(\lambda_2)$

then N_F is called neutrosophic subfield of F

Definition 3.3 Let A be an algebra over a field F and N_F be a neutrosophic subfield of a Field F. A neutrosophic subset N_A of A is called neutrosophic algebra N_F^A of A over the neutrosophic subfield N_F if it satisfies the following condition. If for $a_1, a_2 \in A$ and $\lambda \in F$

i) $N_A(a_1 - a_2) \geq N_A(a_1) \wedge N_A(a_2)$

$\Rightarrow t_A(a_1 - a_2) \geq t_A(a_1) \wedge t_A(a_2), i_A(a_1 - a_2) \geq i_A(a_1) \wedge i_A(a_2), f_A(a_1 - a_2) \leq f_A(a_1) \vee f_A(a_2)$

ii) $N_A(\lambda a_1) \geq N_F(\lambda) \wedge N_A(a_1)$

$\Rightarrow t_A(\lambda a_1) \geq t_F(\lambda) \wedge t_A(a_1), i_A(\lambda a_1) \geq i_F(\lambda) \wedge i_A(a_1), f_A(\lambda a_1) \leq f_F(\lambda) \vee f_A(a_1)$

iii) $N_A(a_1 a_2) \geq N_A(a_1) \wedge N_A(a_2)$

DOI: 10.5281/zenodo.3930148

$$\Rightarrow t_A(a_1a_2) \geq t_A(a_1) \wedge t_A(a_2), i_A(a_1a_2) \geq i_A(a_1) \wedge i_A(a_2), f_A(a_1a_2) \leq f_A(a_1) \vee f_A(a_2)$$

It is denoted as neutrosophic algebra N_F^A

Definition 3.4 Let U be a neutrosophic algebra N_F^A. If for $a_1, a_2 \in A$ and $\lambda \in F$

i) $U(a_1a_2) \geq U(a_1) \wedge U(a_2)$

$$\Rightarrow t_U(a_1a_2) \geq t_U(a_1) \wedge t_U(a_2), i_U(a_1a_2) \geq i_U(a_1) \wedge i_U(a_2), f_U(a_1a_2) \leq f_U(a_1) \vee f_U(a_2)$$

ii) $U(\lambda a_1) \geq U(\lambda) \wedge U(a_1)$

$$\Rightarrow t_U(\lambda a_1) \geq t_F(\lambda) \wedge t_U(a_1), i_U(\lambda a_1) \geq i_F(\lambda) \wedge i_U(a_1), f_U(\lambda a_1) \leq f_F(\lambda) \vee f_U(a_1)$$

then U is called neutrosophic N_F^A ideal

Definition 3.5 Let A be neutrosophic N_X^Y-ideal and Y be an algebra over a field X, then $y \in Y$, define neutrosophic subset $(Y + A)(y_1) = A(y_1 - y), y_1 \in Y$

Preposition 3.1 Let A be neutrosophic N_X^Y-ideal, then for all $y_1, y_2 \in Y$,
$y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(0)$

Proof

Necessary part

Given $y_1 + A = y_2 + A$

$(y_2 + A)(y_1) = A(y_1 - y_2)...(1)$

$(y_1 + A)(y_1) = A(y_1 - y_1) = A(0)...(2)$

i.e., $y_1 + A = y_2 + A \rightarrow A(y_1 - y_2) = A(0)$

Sufficient part

Consider $(y_1 + A)(y) = A(y - y_1)$

$(y_1 + A)(y) = A((y - y_2) - (y_1 - y_2))$

$(y_1 + A)(y) \geq A((y - y_2) \wedge A(y_1 - y_2))$

$(y_1 + A)(y) \geq A((y - y_2) \wedge A(0) = (y_2 + A)(y)$

$y_1 + A \geq y_2 + A \ldots (3)$

DOI: 10.5281/zenodo.3930148

Similarly we can prove that $y_2 + A \geq y_1 + A$ ... (4)

From (3) and (4) $y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(0)$

Proposition 3.2: Let A be neutrosophic N_X^Y-ideal, then for all $y_1, y_2, x_1, x_2 \in Y, \lambda \in X$ then

i). $x_1 + A = y_1 + A, x_2 + A = y_2 + A \Rightarrow (x_1 + x_2) + A = (y_1 + y_2) + A$

ii). $x_1 x_2 + A = y_1 y_2 + A$

iii). $x_1 + A = y_1 + A \Rightarrow \lambda x_1 + A = \lambda y_1 + A$

Proof. i) and ii)

$$A((x_1 + x_2) - (y_1 - y)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0)$$
$$A(x_1 x_2 - y_1 y_2) = A((x_1 - y_1)x_2 + y(x_2 - y_2)) \geq A(x_1 - y_1) \wedge A(x_2 - y_2) = A(0)$$

So, $A((x_1 + x_2) - (y_1 + y_2)) = A(x_1 x_2 - y_1 y_2) = A(0)$

From preposition 1, $(x_1 + x_2) + A = (y_1 + y_2) + A$ and $x_1 x_2 + A = y_1 y_2 + A$

iii). $x_1 + A = y_1 + A \Rightarrow \lambda x_1 + A = \lambda y_1 + A$

Proof

$$A(\lambda x_1 - \lambda y_1) = A(\lambda(x_1 - y_1)) \geq A(x_1 - y_1) = A(0)$$

i.e. $A(\lambda x_1 - \lambda y_1) = A(0)$, hence $\lambda x_1 + A = \lambda y_1 + A$

Preposition 3.3 Let A be neutrosophic N_X^Y-ideal, then Y/A is algebra over X and $Y/A \cong Y/A_0$ where $Y/A = \{y + A \setminus y \in A\}, A_0 = \{y \in y / A(y) = A(0)\}$ and

$$(y_1 + A) + (y_2 + A) = (y_1 + y_2) + A$$
$$(y_1 + A)(y_2 + A) = y_1 y_2 + A$$
$$\lambda(y_1 + A) = \lambda y_1 + A, \forall y_1, y_2 \in Y, \lambda \in X$$

Proof.

From prepositions stated above ,we can conclude that Y/A is an algebra over X and $f : Y/A \rightarrow Y/A_0$ defined by $f(y + A) = y + A_0$ is an isomorphism. Hence $Y/A \cong Y/A_0$

DOI: 10.5281/zenodo.3930148

Definition 3.3 Let A be a neutrosophic N_X^Y ideal, then Y/A is called neutrosophic quotient algebra of Y concern with A

5. Conclusions

Neutrosophic quotient algebra is one of the generalizations of quotient algebra. This paper has developed a combination of an algebraic structure , quotient algebra with neutrosophic set theory. Neutrosophic quotient algebra becomes a key element in the study of neutrosophic quotient modules of an R- module and their properties. This study leads to algebraic nature of neutrosophic algebraic structure and the evolution of new neutrosophic algebraic structures.

Funding: "This research received no external funding"

Conflicts of Interest: "The authors declare no conflict of interest."

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DOI: 10.5281/zenodo.3930148