Neutrosophic Soft Bitopological Spaces

Ahmed B. AL-Nafee[1*], Said Broumi[2], Florentin Smarandache[3]

[1]Ministry of Education Open Educational College, Department of Mathematics, Babylon, Iraq.
E-mail: Ahm_math_88@yahoo.com
[2]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Morocco.
E-mail: broumisaid78@gmail.com
[3] Deparment of Mathematics ,University of New Mexico 705 AVE Gallup, NM 87301, USA.
E-mail: smarand@unm.edu
*Correspondence: Ahm_math_88@yahoo.com

Abstract: In this paper, we built bitopological space on the concept of neutrosophic soft set, we defined the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior and we introduced their properties. In addition, we investigated the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces and we introduced many examples.

Keywords : Neutrosophic soft bitopological spaces, Star bineutrosophic soft open set, star bi neutrosophic soft closed set, fuzzy set.

1. **Introduction**

The concept of soft set is defined by Molodtsov [1] as follows: Let M be an initial universe set and E be a set of parameters. Let P(M) denotes the set of all the subsets of M. Consider $B\neq \emptyset$, $B \subseteq E$. The collecti-on (β,B) is termed to be the soft set, where β is a mapping by $\beta:B\rightarrow P(M)$, and later this concept has been redefined by Naim Cagman [20]. Smarandache [2] introduced neutrosophic set as a generalization of fuzzy set [3] and intuitionistic fuzzy set [4]. P. K. Maji [5] defended the concept of neutrosophic soft set by combining the concept of neutrosophic set and soft set. This the concept is defined as follows: let M be an initial universe set and E be a set of parameters. Let P(M) denote the set of all the neutrosohpic sets of M. Consider $B\neq \emptyset$, $B \subseteq E$. The collection (β,B) is termed to be the soft neutrosophic set, where β is a mapping by $\beta:B\rightarrow P(M)$. This concept has been modified by [6,7].The concept of neutrosohpic soft topological space was introduced by Bera [8]. Taha et al.[9] redefined the neutrosop-hic soft topological spaces differently from the study [8]. Other theoretical studies on these concepts were presented by a number of researchers, for example, Narmada, Georgiou, Cageman, Al-Nafee, Evanzalin and Salama, (see [10, 11, 22, 13, 14, 15, 16, 17, 18, 19, 20]).

Kelly, [21] introduced the concept of bitopological space. This concept is introduced as an extension of topological space. This concept has been introduced with interest in fuzzy set, soft set and neutrosophic set (see [22, 23, 24, 25]). Therefore, we find it important and necessary to build a bitopological spaces on the concept of neutrosophic soft set. In this paper, bitopological space on the concept of neutrosophic soft set is built, the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutros-ophic soft exterior are defined, the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces are investigated and many examples on this concepts are given.

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2. Preliminary

In this section, we will refer to the basic definitions required in our work.

2.1. Definition [26]

The neutrosohpic set N over M is defined as follows:

$$N=\{< m, H_N(m), G_N(m), J_N(m) >) : m \in M\}.$$

where, the functions H,G,J : M→] − 0,+1[and $-0 \leq H_N(m) + G_N(m) + J_N(m) \leq +3$.

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of] − 0,+1[. But in real life application in scientific and engineering problems it is difficult to use a neutrosophic set with value from real standard or non-standard subset of] − 0,+1[. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1] .

Firstly, neutrosophic set defined by Maji [5] and later this concept and its operations have been redefined by [7]. Our work in this research is based on the definition below:

2.2. Definintion [7]

Let M be an initial universe set and B be a set of parameters. Let P(M) denote the set of all the neutron-sohpic sets of M. Then, a neutrosophic soft set β_B over M is a set defined by a set valued function β representing a mapping from B to P(M), where e β is called approximate function of the neutrosophic soft set β_B.

In other words, β_B is a parameterized family of some elements of the set P(M) and therefore it can be written as a set of ordered pairs,

$$\beta_B = \{(r, \{< m^{(H_{\beta(r)}(m), G_{\beta(r)}(m), J_{\beta(r)}(m))} > : m \in M\}), r \in B\}.$$

Where,

$H_{\beta(r)}(m)$, $G_{\beta(r)}(m)$, $J_{\beta(r)}(m) \in [0,1]$, respectively called the truth-membership, indeterminacymembership,falsity-membership function of β(r). Since supremum of each H, G, J is 1 so the inequality,

$0 \leq H_{\beta(r)}(m) + G_{\beta(r)}(m) + J_{\beta(r)}(m) \leq 3$ is obvious.

From now on, the set of all neutrosohpic sets over M is denoted by $N_3(M)$.

2.3. Definition [9,5]

Let β_B and $\mu_B \in N_3(M)$ such that;

$\beta_B = \{(r, \{< m^{(H_{\beta(r)}(m), G_{\beta(r)}(m), J_{\beta(r)}(m))} > : m \in M\}), r \in B\}$.

$\mu_B = \{(r, \{< m^{(H_{\mu(r)}(m), G_{\mu(r)}(m), J_{\mu(r)}(m))} > : m \in M\}), r \in B\}$.Then:

- $\widetilde{M}_B = \{(r, \{< m^{(1,1,0)} > : m \in M\}), r \in B\}$ [Absolute neutrosohpic soft set].
- $\widetilde{\emptyset}_B = \{(r, \{< m^{(0,0,1)} > : m \in M\}), r \in B\}$ [Null neutrosohpic soft set].
- $\beta_B \sqsubseteq \mu_B \leftrightarrow \{(r, \{< m^{(H_{\beta(r)}(m) \leq H_{\mu(r)}(m), G_{\beta(r)}(m) \leq G_{\mu(r)}(m), J_{\beta(r)}(m) \geq J_{\mu(r)}(m))} > : m \in M\}), r \in B\}$.
- $\beta_B \sqcup \mu_B = \{(r, \{< m^{(H_{\beta(r)}(m) \vee H_{\mu(r)}(m), G_{\beta(r)}(m) \vee G_{\mu(r)}(m), J_{\beta(r)}(m) \wedge J_{\mu(r)}(m))} >) : m \in M\}, r \in B\}$.
- $\beta_B \sqcap \mu_B = \{(r, \{< m^{(H_{\beta(r)}(m) \wedge H_{\mu(r)}(m), G_{\beta(r)}(m) \wedge G_{\mu(r)}(m), J_{\beta(r)}(m) \vee J_{\mu(r)}(m))} > : m \in M\}), r \in B\}$.

2.4. Definition

Let $\beta_B \in N_3(M)$, The complement of β_B is denoted by $(\beta_B)^C$ and is definded as:

$$(\beta_B)^C = (r, \{< m^{(1-H_{\beta(r)}(m), 1-G_{\beta(r)}(m), 1-J_{\beta(r)}(m))} > : m \in M\}), r \in B\}.$$

2.5. Definition [9]

Let T ⊆ (N_3(M)). The collection T is called a neutrosophic soft topology on M, if the following conditions are true:

1) $\widetilde{M}_B$, $\widetilde{\emptyset}_B$ belong to T.
2) If $\beta_{j_B} \in T$; j ∈ J, then $\sqcup_{j\in J} \beta_{j_B} \in T$ ∀ j ∈ J.
3) If $\beta_B, \mu_B \in T$, then $\beta_B \sqcap \mu_B \in T$.

Then the triplet (M,B,T) is a neutrosophic soft topological space or (N_3-Top for short).
Members of T are called a neutrosohpic soft open sets (N_3-T-open for short) and their complements are a neutrosohpic soft open sets (N_3-T-closed for short).

The neutrosophic soft interior of $\beta_B \in N_3(M)$ ($(\beta_B)^0$ for short) is defended as:

$$(\beta_B)^0 = \sqcup\{(\omega_B): \omega_B \text{ is a } N_3\text{-T-open set, } \omega_B \sqsubseteq \beta_B\}.$$

The neutrosophic soft closure of $\beta_B \in N_3(M)$ ($\overline{(\beta_B)}$ for short) is defended as:

$$\overline{(\beta_B)} = \sqcap\{(\omega_B): \omega_B \text{ is a } N_3\text{-T-closed set, } \beta_B \sqsubseteq \omega_B\}.$$

2.6. Example

Let M = {m_1, m_2, m_3}, B = {r} and $\beta_B, \mu_B, \gamma_B \in N_3(M)$.

Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\},$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\},$$

$$\gamma_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

Then, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is a neutrosohpic soft topology on M.

2.7. Example

Let M = {m_1, m_2, m_3}, B = {r} and $\beta_B, \mu_B, \gamma_B, \delta_B \in N_3(M)$. Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\}.$$

$$\delta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}.$$

Then, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is a neutrosohpic soft topology on M.

3. Neutrosophic soft bitopological space

In this section, we defined the neutrosophic soft bitopological space or (N_3-Bi-Top for short) on the concept of neutrosophic soft set and the basic topological concepts of this spaces which are N_3-biopen and N_3-biclosed.

3.1. Definition

Let (M,B,T_1) and (M,B,T_2) be two N_3-Top spaces defined on M. Then (M,B,T_1,T_2) is called a neutrosophic soft bitopological space or (N_3-Bi-Top for short).

3.2. Example

Let M = {m_1,m_2}, B = {r} and $\beta_B, \mu_B \in N_3(M)$ such that

$\beta_B = \{(r, \{< m_1^{(0.6, 0.2, 0.5)} >, < m_2^{(0.5, 0.4, 0.9)}>\})\}$, $\mu_B = \{(r, \{< m_1^{(0.6, 0.2, 0.4)} >, < m_2^{(0.6, 0.4, 0.7)} >\})\}$.

Then, $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \mu_B\}$ is an N_3-Top on M.

Therefore, (M,B,T_1,T_2) is an N_3-Bi-Top space.

3.3. Definition

Let (M,B,T_1,T_2) be an N_3-Bi-Top space. The members of (M,B,T_1,T_2) are called bineutrosohpic soft open sets (N_3-biopen for short) and their complements are bineutrosohpic soft closed sets (N_3-biclosed for short).

3.4. Remark

a) Every neutrosophic soft open (closed) set in (M,B,T_1) or (M,B,T_2) is an N_3-biopen (N_3-biclosed) set.

b) Every N_3-Bi-Top space (M,B,T_1,T_2) induces two N_3-Top spaces as (M,B,T_1) and (M,B,T_2).

c) If (M,B,T_1) is an N_3-Top space then (M,B,T_1,T_1) is an N_3-Bi-Top space.

3.5. Theorem

If (M,B,T_1,T_2) is an N_3-Bi-Top space, then $(M,B,T_1 \cap T_2)$ is an N_3-Top space.

Proof

Let (M,B,T_1,T_2) be an N_3-Bi-Top space.

(1) Clearly that $\widetilde{\emptyset}_B, \widetilde{M}_B \in (T_1 \cap T_2)$.

(2) Let $\beta_B, \mu_B \in (T_1 \cap T_2)$, then $\beta_B, \mu_B \in T_1$ and $\beta_B, \mu_B \in T_2$. This implies that, $\beta_B \sqcap \mu_B \in T_1$ and $\beta_B \sqcap \mu_B \in T_1$. Therefore, $\beta_B \sqcap \mu_B \in (T_1 \cap T_2)$.

(3) Let $\beta_{j_B} \in (T_1 \cap T_2)$; $j \in J$. Then $\beta_{j_B} \in T_1$ and $\beta_{j_B} \in T_2$; $j \in J$. Therefore $\sqcup_{j\in J} \beta_{j_B} \in T_1$ and $\sqcup_{j\in J} \beta_{j_B} \in T_2$ $\forall j \in J$. Thus, we have $\sqcup_{j\in J} \beta_{j_B} \in (T_1 \cap T_2)$.

Hence, $(M,B,T_1 \cap T_2)$ is an N_3-Top space.

3.6. Remark

If we take the operation of union instead of the operation of intersection, then the above theorem is not generally correct.

3.7. Example

Let $M = \{m_1,m_2\}$, $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$\beta_B = \{(r, \{< m_1^{(0.3, 0.5, 0.7)} >, < m_2^{(0.2, 0.4, 0.6)} >\})\}$, $\mu_B = \{(r, \{< m_1^{(0.5, 0.7, 0.8)} >, < m_2^{(0.3, 0.6, 0.8)} >\})\}$.

Then, $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \mu_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space. But, $(M,B,T_1 \cup T_2)$ is not an N_3-Top space. Because, $\beta_B \sqcup \mu_B$ does not belong to $(T_1 \cup T_2)$.

4. N_3-(bi)*-open set in neutrosophic soft bitopological space

In this section, N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined based on the idea of δ-open set which was defined in [27].

4.1. Definition

A subset $\beta_B \in N_3(M)$ of an N_3-Bi-Top space (M,B,T_1,T_2) is called star bineutrosophic soft open (N_3-(bi)*-open, for short) in (M,B,T_1,T_2) if and only if $\beta_B \sqsubseteq \overline{(\beta_B)^{oT}}^{(T1)^{oT2}}$ and their complement is an N_3-(bi)*-closed set. The set of all N_3-(bi)*-open [N_3-(bi)*-closed] sets in (M,B,T_1,T_2) is denoted by $M^{(Bi)*-N}$ [$M^{(Bi)*-NSC}$] respectively.

4.2. Example

Let $M = \{m_1,m_2,m_3\}$, $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\},$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\}.$$

$T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N_3-Top on M. Thus, (M, B,T_1,T_2) is an N_3-Bi-Top space.

Note that:

$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\} \sqsubseteq \overline{(\beta_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore \beta_B \sqsubseteq \overline{(\beta_B)^{oT}}^{(T1)^{oT2}}$.

$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\} \sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore \mu_B \sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}}$.

$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\} \sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore \gamma_B \sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}}$.

$\delta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\} \not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore \delta_B \not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}}$.

$\varepsilon_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,0,0)} >\})\} \not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore \varepsilon_B \not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}}$.

$\vartheta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\} \not\sqsubseteq \overline{(\vartheta_B)^{oT}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore \vartheta_B \not\sqsubseteq \overline{(\vartheta_B)^{oT}}^{(T1)^{oT2}}$.

In general in any N_3-Bi-Top space, $\widetilde{\emptyset}_B, \widetilde{M}_B$ are clearly N_3-(bi)*-open sets.

Hence:

$$M^{(Bi)*-NSO} = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}.$$

$$\begin{aligned} M^{(Bi)*-NSC} = \{ & \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}, \\ & \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}, \\ & \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\}, \\ & \widetilde{\emptyset}_B, \\ & \widetilde{M}_B \}. \end{aligned}$$

4.3. Remark

Let β_B and μ_B be an N3-(bi)*-open sets, then $\beta_B \sqcap \mu_B$ is not necessary an N3-(bi)*-open set.

4.4. Example

Let M = $\{m_1, m_2, m_3, m_4, m_5\}$, B = {r} and $\beta_B, \mu_B, \gamma_B, \varepsilon_B, \vartheta_B, \alpha_B \in$ N3(M).

Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(1,1,0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(1,1,0)} >\})\}.$$

$$\varepsilon_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\vartheta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\alpha_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(1,1,0)} >\})\}.$$

T1 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is an N3-Top on M and T2 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B, \varepsilon_B, \vartheta_B, \alpha_B\}$ is an N3-Top on M. Thus, (M,B,T1,T2) is an N3-Bi-Top space. Then

ε_B and $\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ are an N3-(bi)*-open sets, but the intersection of them $\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ is not an N3-(bi)*-open set.

4.5. Theorem

Let (M,B,T1,T2) be an N3-Bi-Top space, then every neutrosophic soft open set in (M,B,T2) is an N3-(bi)*-open set in (M,B,T1,T2).

Proof

Let β_B be a neutrosophic soft open set in (M,B,T2). Then $(\beta_B)^{oT2} = \beta_B$. Since $\beta_B \sqsubseteq \overline{(\beta_B)}^{T1}$,
$\beta_B \sqsubseteq \overline{(\beta_B)^{oT2}}^{T1}$, $(\beta_B)^{oT} \sqsubseteq \overline{(\beta_B)^{oT}}^{T1\,oT2}$. Thererfor $\beta_B \sqsubseteq \overline{(\beta_B)^{oT}}^{T1\,oT2}$ and thus β_B is an N3-(bi)*-open set in (M,B,T1,T2).

4.6. Remark

The converse of above remark is not true in general. In Example 3.4 note that, $\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ is an N3-(bi)*-open set in (M,B,T1,T2), but not a neutrosophic soft open set in (M,B,T2).

4.7. Definition

If (M,B,T1,T2) is an N3-(Bi)*-Top space and $\beta_B \in$ N3(M), then the largest N3-(bi)*-open set contained in β_B is called (bi)*-neutrosophic soft interior of β_B, ($(\beta_B)^{0(bi)*}$ for short). i.e.

$$(\beta_B)^{0(bi)*} = \sqcup\{(\omega_B): \omega_B \text{ is a N3-(bi)*-open set}, \omega_B \sqsubseteq \beta_B\}.$$

4.8. Theorem

Let (M,B,T1,T2) be an N3-(Bi)*-Top space and $\beta_B \in$ N3(M). Then β_B is an N3-(bi)*-open set if and only if $\beta_B = (\beta_B)^{0(bi)*}$.

Proof

Let β_B be an N3-(bi)*-open set. Then β_B is itself an N3-(bi)*-open set which contains β_B. Therefore, β_B is the largest N3-(bi)*-open set contained in β_B and $\beta_B = (\beta_B)^{0(bi)*}$. Conversely, suppose that $\beta_B = (\beta_B)^{0(bi)*}$,then β_B is the largest N3-(bi)*-open set contained in β_B. Thus, β_B is an N3-(bi)*-open set.

4.9. Theorem

Let $\beta_B, \mu_B \in N_3(M)$.

a) $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$.
b) $((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.
c) $(\beta_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.
e) $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.
f) $(\widetilde{M}_B)^{0(bi)*} = \widetilde{M}_B$.
g) $(\widetilde{\emptyset}_B)^{0(bi)*} = \widetilde{\emptyset}_B$.

Proof

(a), (f), (g), (c) (Straightforward).

(b) Let $\mu_B = (\beta_B)^{0(bi)*}$. Then $\mu_B = (\mu_B)^{0(bi)*}$(from Theorem 4.8). Thus$((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.

(d) Since, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*}$ and $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$. Then, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$...(1.

Since, $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$ and $(\mu_B)^{0(bi)*} \sqsubseteq \mu_B$, then $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*} \sqsubseteq \beta_B \sqcap \mu_B$. But $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$ is a N_3-(bi)*-open subset of $\beta_B \sqcap \mu_B$. Therefore, from the detention, we have that $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$…(2.

Hence, $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.

(e) Since, $\beta_B \sqsubseteq (\beta_B \sqcup \mu_B)$ and $\mu_B \sqsubseteq (\beta_B \sqcup \mu_B)$, therefore $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*}$and $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\mu_B)^{0(bi)*}$. So, $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.

4.10. Example

Let us consider $\beta_B, \mu_B, \gamma_B \in N_3(M)$ in Example 2.6. Such that, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N_3-Top on M and $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space.

Note that: 1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.11. Definition

If (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the intersection of all N_3-(bi)*-closed sets containing β_B is called a (bi)*-neutrosophic soft closure of β_B, ($\overline{(\beta_B)}^{(bi)*}$ for short). i.e.

$$\overline{(\beta_B)}^{(bi)*} = \sqcap\{(\omega_B): \omega_B \text{ is an } N_3\text{-(bi)*-closed set}, \beta_B \sqsubseteq \omega_B\}.$$

4.12. Theorem

Let $\beta_B, \mu_B \in N_3(M)$.

a) $\beta_B \sqsubseteq \overline{(\beta_B)}^{(bi)*}$.
b) $\overline{\left(\overline{(\beta_B)}^{(bi)*}\right)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*}$.
c) $\overline{(\beta_B)}^{(bi)*} \sqsubseteq \overline{(\mu_B)}^{(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $\overline{(\beta_B \sqcap \mu_B)}^{(bi)*} \sqsubseteq \overline{(\beta_B)}^{(bi)*} \sqcap \overline{(\mu_B)}^{(bi)*}$.
e) $\overline{(\beta_B \sqcup \mu_B)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*} \sqcup \overline{(\mu_B)}^{(bi)*}$.

f) $\overline{(\widetilde{M}_B)}^{(bi)*} = \widetilde{M}_B$.

g) $\overline{(\widetilde{\emptyset}_B)}^{(bi)*} = \widetilde{\emptyset}_B$.

Proof Straightforward.

4.13. Remark

In above theorem, it is not necessary the converse of (a) and (d) be true.

4.14. Example

Let us take, $\beta_B, \mu_B, \gamma_B, \delta_B \in N_3(M)$ in Example 2.7.

$T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is an N3-Top on M and $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N3-Top on M. Thus, (M,B,T_1,T_2) is an N3-Bi-Top space.

Note that:

1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.15. Theorem

Let (M,B,T_1,T_2) be an N3-(Bi)*-Top space and $\beta_B \in N_3(M)$.

a) $((\beta_B)^C)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

b) $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

Proof

(a) We know that, $\overline{(\beta_B)}^{(bi)*} = \sqcap\{\omega_B : (\omega_B)^C$ is a N3-(bi)*-open set, $\beta_B \sqsubseteq \omega_B\}$. So, we have that, $\left(\overline{(\beta_B)}^{(bi)*}\right)^C = \sqcup\{(\omega_B)^C : (\omega_B)^C$ is an N3-(bi)*- open set, $(\omega_B)^C \sqsubseteq (\beta_B)^C\} = ((\beta_B)^C)^{0(bi)*}$. Thus, $((\beta_B)^C)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

(b) If we take, $(\beta_B)^C$ instead of β_B in (a), we get that,

$\left(\overline{((\beta_B)^c)}^{(bi)*}\right)^C = (((\beta_B)^C)^C)^{0(bi)*} = ((\beta_B))^{0(bi)*}$. So, $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

4.16. Theorem

If (M,T_1,T_2) is an N3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then β_B is an N3-(bi)*-closed set if and only if $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Proof

Let β_B be an N3-(bi)*-closed set, then β_B is itself an N3-(bi)*-closed set which contains β_B. Therefore, β_B is the intersection of all N3-(bi)*-closed sets containing β_B and $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Conversely, suppose that $\beta_B = \overline{(\beta_B)}^{(bi)*}$, then β_B is the intersection of all N3-(bi)*-closed sets containing β_B. Thus, β_B is an N3-(bi)*-closed set.

4.17. Definition

If (M,T_1,T_2) is an N3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the (bi)*-neutrosophic soft exterior of β_B, (bi)*-ext(β_B) for short) is defined as, (bi)*-ext(β_B) = $((\beta_B)^C)^{0(bi)*}$.

4.18. Definition

If (M,B,T_1,T_2) is an N3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the (bi)*-neutrosophic soft boundary of β_B, ((bi)*-br(β_B) for short) is defined as, (bi)*-br(β_B) = $\overline{(\beta_B)^C}^{(bi)*} \sqcap \overline{(\beta_B)}^{(bi)*}$.

4.19. Theorem

Assume that (M,B,T_1,T_2) is an N_3-$(Bi)^*$-Top space and $\beta_B \in N_3(M)$.

- $(bi)^*$-br $((\beta_B)^C) = (bi)^*$-ext$(\beta_B) \sqcup (\beta_B)^{0(bi)*}$.
- $\overline{(\beta_B)}^{(bi)*} = (bi)^*$-br $(\beta_B) \sqcup (\beta_B)^{0(bi)*}$.
- $(bi)^*$-br $(\beta_B) \sqcap (\beta_B)^{0(bi)*} = \widetilde{\emptyset}_B$.
- $(bi)^*$-br $(\beta_B)^{0(bi)*} \sqsubseteq (bi)^*$-br (β_B).

Proof Straightforward.

4.20. Theorem

Assume that (M,B,T_1,T_2) is an N_3-$(Bi)^*$-Top space and $\beta_B \in N_3(M)$.

- $\beta_B \in M^{(Bi)*-NSO}$ if and only if $(bi)^*$-br $(\beta_B) \sqcap \beta_B = \widetilde{\emptyset}_B$.
- $\beta_B \in M^{(Bi)*-NSC}$ if and only if $(bi)^*$-br $(\beta_B) \sqsubseteq \beta_B$.

Proof Straightforward.

Conclusion

In this research, bitopological space on the concept of neutrosophic soft set is built , the basic topological concepts of this spaces which are N_3-$(bi)^*$-open set, N_3-$(bi)^*$-closed set, $(bi)^*$-neutrosophic soft interior, $(bi)^*$-neutrosophic soft closure, $(bi)^*$-neutrosophic soft boundary, $(bi)^*$-neutrosophic soft exterior are defined and many examples on this concepts are given.

This paper is just a beginning of a new structure and we have studied a few ideas only , it will be necessary to carry out more theoretical research to establish a general framework for the practical application .

we hope that the findings in this paper will help researchers enhance and promote the further study on neutrosophic soft bitopological space .

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