SVN-Ostrowski Type Inequalities for $h-$convex functions

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Abstract

We would like to state well-known Ostrowski inequality via $h-$convex function by using the SVN-Reimann integrals. In addition, we establish some SVN-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $h-$convex functions by using different techniques including Hölder's inequality and power mean inequality. We are introducing very first time that the class of $h-$ convex function, which is the generalization of many important classes including class of Godunova-Levin $s-$convex, $s-$convex in the 2^{nd} kind and hence contains class of convex functions. It also contains class of $P-$convex functions and class of Godunova-Levin functions. In this way we also capture the results with respect to convexity of functions.
Keywords: Ostrowski inequality, $h-$convex functions, Single valued Neutrosophic sets.

1 Introduction

In this section, from literature, we recall and introduce some definitions for various convex functions.

Definition 1.1. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$\eta\left(tx + (1-t)y\right) \le t\eta(x) + (1-t)\eta(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.2. [3] A function $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $MT-$convex, if η is a non-negative and

$$\eta\left(tx + (1-t)y\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.3. [18] We say that $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a $P-$convex function, if η is a non-negative and $\forall x, y \in I$ and $t \in [0,1]$ we have

$$\eta\left(tx + (1-t)y\right) \le \eta(x) + \eta(y).$$

Definition 1.4. [21] We say that $\eta : I \subset \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin convex function, if η is non-negative and $\forall x, y \in I$ and $t \in (0,1)$ we have

$$\eta\left(tx + (1-t)y\right) \le \frac{1}{t}\eta(x) + \frac{1}{1-t}\eta(y).$$

Definition 1.5. [5] Let $s \in [0,1]$. A function $\eta : I \subset [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the 2^{nd} kind, if

$$\eta\left(tx + (1-t)y\right) \le t^s\eta(x) + (1-t)^s\eta(y),$$

$\forall x, y \in I, t \in [0,1]$.

Definition 1.6. [10] We say that the function $\eta : I \subset \mathbb{R} \to [0, \infty)$ is of Godunova-Levin $s-$convex function, with $s \in [0, 1]$, if

$$\eta (tx + (1-t)y) \leq \frac{1}{t^s}\eta(x) + \frac{1}{(1-t)^s}\eta(y),$$

$\forall t \in (0, 1)$ and $x, y \in I$.

Definition 1.7. [34] Let $h : J \subseteq \mathbb{R} \to [0, \infty)$ with h not identical to 0. We say that η is an $h-$convex function if $\forall x, y \in I$, we have

$$\eta (tx + (1-t)y) \leq h(t)\eta(x) + h(1-t)\eta(y),$$

$\forall t \in [0, 1]$.

Remark 1.8. In Definition 1.7, one can see the following.

1. If we take $h(t) = \frac{1}{t^s}$ with $s \in [0, 1]$ in (1), then we get the class of Godunova-Levin $s-$convex (concave) functions.
2. if we put $h(t) = t^s$ with $s \in [0, 1]$ in (1), then we get the concept of $s-$convex (concave) in 2^{nd} kind.
3. If we put $h(t) = \frac{1}{t}$ in (1), then we get the concept of Godunova-Levin convex (concave) function.
4. If we put $h(t) = 1$ in (1), then we get the concept of $P-$convex (concave) function.
5. If we put $h(t) = t$ in (1), then we get the concept of ordinary convex (concave) function.
6. If we put $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ in (1), then we get the concept of $MT-$ordinary convex function.

Theorem 1.9. [33] *Let $\varphi : [\rho_a, \rho_b] \to \mathbb{R}$ be differentiable function on (ρ_a, ρ_b) with the property that $|\varphi'(t)| \leq M$ for all $t \in (\rho_a, \rho_b)$. Then*

$$\left|\varphi(x) - \frac{1}{\rho_b - \rho_a}\int_{\rho_a}^{\rho_b} \varphi(t)dt\right| \leq M(\rho_b - \rho_a)\left[\frac{1}{4} + \left(\frac{x - \frac{\rho_a+\rho_b}{2}}{\rho_b - \rho_a}\right)^2\right], \quad (1)$$

for all $x \in (\rho_a, \rho_b)$. The constant $\frac{1}{4}$ is the best possible in the kind that it cannot be replaced by a smaller quantity.

Now we present the extension of definitions of fuzzy numbers and their results as from the,[7],[8][28] and.[20]

Definition 1.10. [4] A SVN-Number is $\phi : \mathbb{R} \to [0, 1]$ can be defined as

1. $[\phi]^0 =$ Closure$(\{r \in \mathbb{R} : T\phi(r) > 0, I\phi(r) > 0, F\phi(r) > 0\})$ is compact.
2. ϕ is Normal.(i.e, $\exists\ r_0 \in \mathbb{R}$ such that $T\phi(r_0) = 1$, $I\phi(r_0) = 0$ and $F\phi(r_0) = 0$).
3. ϕ is SVN-convex, i.e, $\forall r_1, r_2 \in \mathbb{R}, \eta \in [0, 1]$

$$\begin{aligned} T\phi(\eta r_1 + (1-\eta)r_2) &\geq \min\{T\phi(r_1), T\phi(r_2)\}, \\ I\phi(\eta r_1 + (1-\eta)r_2) &\leq \max\{I\phi(r_1), I\phi(r_2)\}, \\ F\phi(\eta r_1 + (1-\eta)r_2) &\leq \max\{F\phi(r_1), F\phi(r_2)\}. \end{aligned}$$

4. $\forall r_0 \in R$ and $\epsilon > 0, \exists$ Neighborhood $V(r_0)$, such that $\forall r \in \mathbb{R}, T\phi(r) \leq T\phi(r_0) + \epsilon, I\phi(r) \geq I\phi(r_0) - \epsilon$, and $F\phi(r) \geq \phi(r_0) - \epsilon$,

Definition 1.11. [4,22] For any $(\zeta_1, \zeta_2, \zeta_3) \in [0, 1]^3$, and ϕ be any SVN-number, then $\zeta-$level set $[\phi]^{(\zeta_1,\zeta_2,\zeta_3)} = \{r \in \mathbb{R} : T\phi(r) \geq \zeta_1, I\phi(r) \leq \zeta_2, F\phi(r) \leq \zeta_3\}$. Moreover $[\phi]^\zeta = \left[\phi_-^{(\zeta_1,\zeta_2,\zeta_3)}, \phi_+^{(\zeta_1,\zeta_2,\zeta_3)}\right], \forall(\zeta_1, \zeta_2, \zeta_3) \in [0, 1]^3$.

Proposition 1.12. [22,30] *Let $\phi, \varphi \in SVN_{\mathbb{R}}$(Set of all SVN-Numbers) and $\eta \in \mathbb{R}$, then the following properties holds:*

1. $[\phi + \varphi]^{(\zeta_1,\zeta_2,\zeta_3)} = [\phi]^{(\zeta_1,\zeta_2,\zeta_3)} + [\varphi]^{(\zeta_1,\zeta_2,\zeta_3)}$.

2. $[\eta \odot \phi]^{(\zeta_1,\zeta_2,\zeta_3)} = \eta\,[\phi]^{(\zeta_1,\zeta_2,\zeta_3)}$.

3. $\phi \oplus \varphi = \varphi \oplus \phi$.

4. $\eta \odot \phi = \phi \odot \eta$.

5. $1 \odot \phi = \phi$.

$\forall \zeta \in [0,1]$.

Definition 1.13. [31] Let $D : SVN_{\mathbb{R}} \times SVN_{\mathbb{R}} \to \mathbb{R}_+ \cup \{0\}$, defined as

$$D(\phi,\varphi) = \sup_{\zeta\in[0,1]} \max\left\{\left|T\phi_-^{(\zeta)}, T\phi_+^{(\zeta)}\right|, \left|T\varphi_-^{(\zeta)}, T\varphi_+^{(\zeta)}\right|\right\} + \inf_{\zeta\in[0,1]} \min\left\{\left|I\phi_-^{(\zeta)}, I\phi_+^{(\zeta)}\right|, \left|I\varphi_-^{(\zeta)}, I\varphi_+^{(\zeta)}\right|\right\} + \inf_{\zeta\in[0,1]} \min\left\{\left|F\phi_-^{(\zeta)}, F\phi_+^{(\zeta)}\right|, \left|F\varphi_-^{(\zeta)}, F\varphi_+^{(\zeta)}\right|\right\}.$$

$\forall \phi,\varphi \in SVN_{\mathbb{R}}$. Then D is metric on $SVN_{\mathbb{R}}$.

Proposition 1.14. [31] *Let $\phi_1,\phi_2,\phi_3,\phi_4 \in SVN_{\mathbb{R}}$ and $\eta \in SVN_{\mathbb{R}}$, we have*

1. $(SVN_{\mathbb{R}}, D)$ *is complete.*

2. $D(\phi_1 \oplus \phi_3, \phi_2 \oplus \phi_3) = D(\phi_1,\phi_2)$.

3. $D(\eta \odot \phi_1, \eta \odot \phi_2) = |\eta| D(\phi_1,\phi_2)$.

4. $D(\phi_1 \oplus \phi_2, \phi_3 \oplus \phi_4) = D(\phi_1,\phi_3) + D(\phi_2,\phi_4)$.

5. $D(\phi_1 \oplus \phi_2, \widetilde{0}) = D(\phi_1,\widetilde{0}) + D(\phi_2,\widetilde{0})$.

6. $D(\phi_1 \oplus \phi_2, \phi_3) = D(\phi_1,\phi_3) + D(\phi_2,\widetilde{0})$,

where $\widetilde{0} \in SVN_{\mathbb{R}}$, defined by $\forall r \in \mathbb{R}, \widetilde{0}(r) = (0,0,1)$.

Definition 1.15. [30] Let $\phi,\varphi \in SVN_{\mathbb{R}}$, if $\exists\, \theta \in SVN_{\mathbb{R}}$, such that $\phi = \varphi \oplus \theta$, then θ is $H-$difference of ϕ and φ, denoted by $\theta = \phi \ominus \varphi$.

Definition 1.16. [31] A function $\phi : [r_0, r_0+\epsilon] \to SVN_{\mathbb{R}}$ is $H-$differentiable at r, if $\exists\, \phi'(r) \in SVN_{\mathbb{R}}$, i.e both limits

$$\lim_{h\to 0^+} \frac{\phi(r+h)\ominus\phi(r)}{h}, \quad \lim_{h\to 0^+} \frac{\phi(r)\ominus\phi(r-h)}{h}$$

exists and are equal to $\phi'(r)$.

Definition 1.17. [30,31] Let $\phi : [\rho_a,\rho_b] \to SVN_{\mathbb{R}}$, if $\forall \zeta > 0, \exists \eta > 0$, for any partition $P = \{[u,v] : \delta\}$ of $[\rho_a,\rho_b]$ with norm $\Delta(P) < \eta$, we have

$$D\left(\sum_P^{*} (v-u)\phi(\delta), \varphi\right) < \zeta,$$

then we say that ϕ is SVN-Riemann integrable to $\varphi \in SVN_{\mathbb{R}}$, we write it as

$$\varphi = (SVNR)\int_{\rho_a}^{\rho_b} \phi(x)dx.$$

2 SVN-Ostrowski type inequalities via $\phi-$convex functions

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $\varphi : [\rho_a, \rho_b] \to SVN_{\mathbb{R}}$ be an absolutely continuous mapping on (ρ_a, ρ_b) with $\rho_a < \rho_b$. If $\varphi' \in C_F[\rho_a, \rho_b] \cap L_F[\rho_a, \rho_b]$, then for $x \in (\rho_a, \rho_b)$ the following identity holds:*

$$\frac{1}{\rho_b-\rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \oplus \frac{(x-\rho_a)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_a)dt$$
$$= \varphi(x) \oplus \frac{(\rho_b-x)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_b)dt. \tag{2}$$

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

Theorem 2.2. *Suppose all the assumptions of Lemma 2.1 hold. Additionally, $h(t) \neq \frac{1}{t}$, $D(\varphi', \widetilde{0})$ be a $h-$convex function on $[\rho_a, \rho_b]$ and $D(\varphi'(x), \widetilde{0}) \leq M$. Then for each $x \in (\rho_a, \rho_b)$ the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq M \left(\int_0^1 (t\,h(t) + t\,h(1-t))\,dt\right) I(x), \tag{3}$$

where $I(x) = \frac{(x-\rho_a)^2+(\rho_b-x)^2}{\rho_b-\rho_a}$.

Proof. From the Lemma 2.1,

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt\right)$$
$$\leq D\left(\frac{(x-\rho_a)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_a)dt, \frac{(\rho_b-x)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_b)dt\right),$$

$$\leq D\left(\frac{(x-\rho_a)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_a)dt, \widetilde{0}\right)$$
$$+D\left(\frac{(\rho_b-x)^2}{\rho_b-\rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_b)dt, \widetilde{0}\right),$$

$$= \frac{(x-\rho_a)^2}{\rho_b-\rho_a} D\left((SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_a)dt, \widetilde{0}\right)$$
$$+\frac{(\rho_b-x)^2}{\rho_b-\rho_a} D\left((SVNR) \int_0^1 t \odot \varphi'(tx+(1-t)\rho_b)dt, \widetilde{0}\right),$$

$$\leq \frac{(x-\rho_a)^2}{\rho_b-\rho_a} \int_0^1 tD\left(\varphi'(tx+(1-t)\rho_a), \widetilde{0}\right)dt$$
$$+\frac{(\rho_b-x)^2}{\rho_b-\rho_a} \int_0^1 tD\left(\varphi'(tx+(1-t)\rho_b), \widetilde{0}\right)dt, \tag{4}$$

Since $D(\varphi', \widetilde{0})$ be $h-$convex function and $D(\varphi'(x), \widetilde{0}) \leq M$, we have

$$\begin{aligned} D\left(\varphi'(tx+(1-t)\rho_a), \widetilde{0}\right) &\leq h(t)D\left(\varphi'(x), \widetilde{0}\right) + h(1-t)D\left(\varphi'(\rho_a), \widetilde{0}\right) \\ &\leq M\left[h(t)+h(1-t)\right] \end{aligned} \tag{5}$$

$$\begin{aligned} D\left(\varphi'(tx+(1-t)\rho_b), \widetilde{0}\right) &\leq h(t)D\left(\varphi'(x), \widetilde{0}\right) + h(1-t)D\left(\varphi'(\rho_b), \widetilde{0}\right) \\ &\leq M\left[h(t)+h(1-t)\right]. \end{aligned} \tag{6}$$

Now using (5) and (6) in (4) we get (3). □

Corollary 2.3. *In Theorem* 2.2*, one can see the following.*

1. *If one takes* $h(t) = t^{-s}$ *in* (3)*, then one has the* $SVN-$*Ostrowski inequality for Godunova-Levin* $s-$*convex functions:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le M\left(\frac{1}{1-s}\right)I(x).$$

2. *If one takes* $h(t) = t^{s}$ *where* $s \in (0,1]$ *in* (3)*, then one has the* $SVN-$*Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le M\left(\frac{1}{1+s}\right)I(x).$$

3. *If one takes* $h(t) = 1$ *in* (3)*, then one has the* $SVN-$*Ostrowski inequality for* $P-$*convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le MI(x).$$

4. *If one takes* $h(t) = t$ *in* (3)*, then one has the* $SVN-$*Ostrowski inequality for convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le \frac{M}{2}I(x).$$

5. *If one takes* $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ *in in* (3)*, then one has the* $SVN-$*Ostrowski inequality for* $MT-$*convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le \frac{M\pi}{4}I(x).$$

Theorem 2.4. *Suppose all the assumptions of Lemma* 2.1 *hold. Additionally,* $h(t) \neq \frac{1}{t}$, $[D(\varphi', \widetilde{0})]^q$ *for* $q \ge 1$ *be* $h-$*convex function on* $[\rho_a, \rho_b]$ *and* $D(\varphi'(x), \widetilde{0}) \le M$*. Then* $\forall x \in (\rho_a, \rho_b)$ *the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right) \le \frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1 (t\,h(t)+t\,h(1-t))\,dt\right)^{\frac{1}{q}} I(x), \tag{7}$$

where $I(x) = \frac{(x-\rho_a)^2+(\rho_b-x)^2}{\rho_b-\rho_a}$.

Proof. From the inequality (4) and power mean inequality[35]

$$\begin{aligned}
&D\left(\varphi(x), \frac{1}{\rho_b-\rho_a}\odot(SVNR)\int_{\rho_a}^{\rho_b}\varphi(t)dt\right)\\
&\le \frac{(x-\rho_a)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)\rho_a), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}\\
&+\frac{(\rho_b-x)^2}{\rho_b-\rho_a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)\rho_b), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{8}$$

Since $[D(\varphi', \widetilde{0})]^q$ be $h-$convex function and $D(\varphi'(x), \widetilde{0}) \le M$, we have

$$\begin{aligned}
\left[D\left(\varphi'(tx+(1-t)\rho_a), \widetilde{0}\right)\right]^q &\le h(t)\left[D\left(\varphi'(x), \widetilde{0}\right)\right]^q + h(1-t)\left[D\left(\varphi'(\rho_a), \widetilde{0}\right)\right]^q\\
&\le M^q\left[h(t)+h(1-t)\right],
\end{aligned} \tag{9}$$

$$\begin{aligned}
\left[D\left(\varphi'(tx+(1-t)\rho_b), \widetilde{0}\right)\right]^q &\le h(t)\left[D\left(\varphi'(x), \widetilde{0}\right)\right]^q + h(1-t)\left[D\left(\varphi'(\rho_b), \widetilde{0}\right)\right]^q\\
&\le M^q\left[h(t)+h(1-t)\right],
\end{aligned} \tag{10}$$

Now using (9) and (10) in (8) we get (7). □

Corollary 2.5. *In Theorem* 2.4, *one can see the following.*

1. *If one takes* $q = 1$, *one has the Theorem* 2.2.
2. *If one takes* $h(t) = t^{-s}$ *in* (7), *then one has* $SVN-$*Ostrowski inequality for Godunova-Levin* $s-$*convex functions:*
$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}} I(x).$$
3. *If one takes* $h(t) = t^s$ *where* $s \in [0,1]$ *in* (7), *then one has* $SVN-$*Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*
$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}} I(x).$$
4. *If one takes* $h(t) = 1$, *in* (7), *then one has the* $SVN-$*Ostrowski inequality for* $P-$*convex function:*
$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} I(x).$$
5. *If one takes* $h(t) = t$, *in* (7), *then one has the* $SVN-$*Ostrowski inequality for convex function:*
$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2} I(x).$$
6. *If one takes* $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ *in* (7), *then one has the* $SVN-$*Ostrowski inequality for* $MT-$*convex function:*
$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} I(x).$$

Theorem 2.6. *Suppose all the assumptions of Lemma* 2.1 *hold. Additionally,* $h(t) \neq \frac{1}{t^2}$, $[D(\varphi', \widetilde{0})]^q$ *be a* $h-$*convex function on* $[\rho_a, \rho_b], q > 1$ *and* $D(\varphi'(x), \widetilde{0}) \leq M$. *Then for each* $x \in (\rho_a, \rho_b)$, *the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1 (h(t) + h(1-t))\, dt\right)^{\frac{1}{q}} I(x), \tag{11}$$

where $p^{-1} + q^{-1} = 1$.

Proof. From the inequality (4) and Hölder's inequality[36]

$$\begin{aligned} &D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \\ &\leq \frac{(x-\rho_a)^2}{\rho_b - \rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(\varphi'(tx + (1-t)\rho_a), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}} \\ &+ \frac{(\rho_b - x)^2}{\rho_b - \rho_a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(\varphi'(tx + (1-t)\rho_b), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{12}$$

Since $[D(\varphi', \widetilde{0})]^q$ be $h-$convex function and $D(\varphi'(x), \widetilde{0}) \leq M$, we have

$$\begin{aligned} \left[D\left(\varphi'(tx + (1-t)\rho_a), \widetilde{0}\right)\right]^q &\leq h(t)\left[D\left(\varphi'(x), \widetilde{0}\right)\right]^q + h(1-t)\left[D\left(\varphi'(\rho_a), \widetilde{0}\right)\right]^q \\ &\leq M^q\left[h(t) + h(1-t)\right], \end{aligned} \tag{13}$$

$$\begin{aligned} \left[D\left(\varphi'(tx + (1-t)\rho_b), \widetilde{0}\right)\right]^q &\leq h(t)\left[D\left(\varphi'(x), \widetilde{0}\right)\right]^q + h(1-t)\left[D\left(\varphi'(\rho_b), \widetilde{0}\right)\right]^q \\ &\leq M^q\left[h(t) + h(1-t)\right], \end{aligned} \tag{14}$$

Now using (13) and (14) in (12) we get (11). □

Corollary 2.7. *In Theorem* 2.6*, one can see the following.*

1. *If one takes* $h(t) = t^{-s}$ *where* $s \in [0, 1)$ *in* (11)*, then one has the* $SVN-$*Ostrowski inequality for Godunova-Levin* $s-$*convex functions:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}} I(x).$$

2. *If one takes* $h(t) = t^s$*, where* $s \in (0, 1]$ *in* (11)*, then one has the* $SVN-$*Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1+s}\right)^{\frac{1}{q}} I(x).$$

3. *If one takes* $h(t) = 1$*, in* (11)*, then one has the* $SVN-$*Ostrowski inequality for* $P-$*convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}} I(x).$$

4. *If one takes* $h(t) = t$*, in* (11)*, then one has the* $SVN-$*Ostrowski inequality for convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}} I(x).$$

5. *If one takes* $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ *in* (11)*, then one has the* $SVN-$*Ostrowski inequality for* $MT-$*convex function:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} I(x).$$

2.1 SVN-Ostrowski type midpoint inequalities via $h-$convex functions

Remark 2.8. In Theorem 2.4, one can see the following.

1. If one takes $x = \frac{\rho_a+\rho_b}{2}$ in (7), then one has the $SVN-$Ostrowski Midpoint inequality for $h-$convex function:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M(\rho_b-\rho_a)}{2^{2-\frac{1}{q}}}\left(\int_0^1 (th(t) + th(1-t))\,dt\right)^{\frac{1}{q}}.$$

2. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t^{-s}$ in (7), then one has $SVN-$Ostrowski Midpoint inequality for Godunova-Levin $s-$convex functions:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

3. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t^s$ where $s \in [0, 1]$ in (7), then one has $SVN-$Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}(\rho_b-\rho_a).$$

4. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = 1$ in (7), then one has the $SVN-$Ostrowski Midpoint inequality for $P-$convex function:

$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}}(\rho_b-\rho_a).$$

5. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t$ in (7), then one has the $SVN-$Ostrowski Midpoint inequality for convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{4}\left(\rho_b-\rho_a\right).$$

6. If one takes $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ in (7), then one has the $SVN-$Ostrowski inequality for $MT-$convex function:
$$D\left(\varphi(x), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} I(x).$$

Remark 2.9. In Theorem 2.6, one can see the following.

1. If one takes $x = \frac{\rho_a+\rho_b}{2}$ in (11), one has the $SVN-$Ostrowski Midpoint inequality for $h-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\left(\rho_b-\rho_a\right)}{2\left(p+1\right)^{\frac{1}{p}}}\left(\int_0^1 \left(h(t)+h(1-t)\right)dt\right)^{\frac{1}{q}}.$$

2. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t^{-s}$ where $s \in [0,1)$ in (11), then one has the $SVN-$Ostrowski Midpoint inequality for Godunova-Levin $s-$convex functions:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}\left(\rho_b-\rho_a\right).$$

3. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t^s$, where $s \in (0,1]$ in (11), then one has the $SVN-$Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}\left(\rho_b-\rho_a\right).$$

4. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = 1$ in (11), then one has the $SVN-$Ostrowski Midpoint inequality for $P-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\rho_b-\rho_a\right).$$

5. If one takes $x = \frac{\rho_a+\rho_b}{2}$ and $h(t) = t$ in (11), then one has the $SVN-$Ostrowski Midpoint inequality for convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M}{2\left(p+1\right)^{\frac{1}{p}}}\left(\rho_b-\rho_a\right).$$

6. If one takes $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ in (11), then one has the $SVN-$Ostrowski inequality for $MT-$convex function:
$$D\left(\varphi\left(\frac{\rho_a+\rho_b}{2}\right), \frac{1}{\rho_b-\rho_a} \odot (SVNR)\int_{\rho_a}^{\rho_b} \varphi(t)dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{\frac{1}{q}+1}(1+p)^{\frac{1}{p}}}\left(\rho_b-\rho_a\right).$$

3 Conclusion and Remarks

3.1 Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $h-$convex function which is the generalization of many important classes including class Godunova-Levin $s-$convex,[10] $s-$convex in the 2^{nd} kind[5] and hence contains class of convex functions.[3] It also contains class of $P-$convex functions[18] and class of Godunova-Levin functions.[21] We would like to state the SVN-Ostrowski inequality via $h-$ convex function. In addition, we establish some SVN-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $h-$convex functions by using different techniques including Hölder's inequality[36] and power mean inequality.[35]

3.2 Remarks and Future Ideas

1. One may do similar work to generalize all results stated in this article by applying weights.
2. One may also do similar work by using various different classes of functions.
3. One may also generalize this work in fractional integral form.
4. One may try to state all results stated in this article for fractional integral with respect to another function.
5. One may also state all results stated in this article for higher dimensions.

References

[1] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, "Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense", *Appl. Math. Lett.*, **23**(1), (2010), pp. 1071–1076.

[2] A. Arshad and A. R. Khan, "Hermite−Hadamard−Fejer Type Integral Inequality for $s-p-$Convex Functions of Several Kinds", *TJMM.*, **11** (2), (2019), pp. 25–40.

[3] E. F. Beckenbach, "Convex functions", *Bull. Amer. Math. Soc.*, **54**(1), (1948), pp. 439–460.

[4] P. Biswas, S. Pramanik, and B. C. Giri, "TOPSIS Strategy for Multi-Attribute Decision Making with Trapezoidal Neutrosophic Numbers", *Neutrosophic Sets and Systems*, **19** (1), (2018), pp. 29-39.

[5] W. W. Breckner, "Stetigkeitsaussagen Fur Eine Klasse Verallgemeinerter Konvexer Funktionen in Topologischen Linearen Raumen. (German)", *Publ. Inst. Math.*, **23** (37), (1978), pp. 13–20.

[6] P. Cerone and S.S. Dragomir, "Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions", *Demonstratio Math*, **37** (2), (2004), pp. 299–308.

[7] W. Congxin and M. Ming, "On embedding problem of fuzzy number space: Part 1", *Fuzzy Sets and Systems*, **44** (1), (1991), pp. 33–38.

[8] W.Congxin and G. Zengtai, "On Henstock integral of fuzzy-number-valued functions (I)", *Fuzzy Sets and Systems*, **120** (1), (2001) pp. 523–532.

[9] S. S. Dragomir, "A Functional Generalization of Ostrowski Inequality via Montgomery identity", *Acta Math. Univ. Comenianae, LXXXIV.*, **1**(2), (2015), pp. 63–78.

[10] S. S. Dragomir, "Integral inequalities of Jensen type for λ-convex functions", *In Proceedings of RGMIA, Res. Rep. Coll.*, **1** (17), (2014).

[11] S. S. Dragomir, "Inequalities of Jensen Type for $\phi-$Convex Functions", *Fasciculi Mathematici*, **5**(1), (2015), pp. 35–52.

[12] S. S. Dragomir, "A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications", *Int. J. Nonlinear Anal. Appl.*, **5** (2), (2014), pp. 89–97.

[13] S. S. Dragomir, "On the Ostrowski's Integral Inequality for Mappings with Bounded Variation and Applications", *Math. Inequal. Appl.*, **4** (1), (2001), pp. 59–66.

[14] S. S. Dragomir, "Refinements of the Generalised Trapozoid and Ostrowski Inequalities for Functions of Bounded Variation", *Arch. Math.*, **91** (5), (2008), pp. 450–460.

[15] S. S. Dragomir and N. S. Barnett, "An Ostrowski Type Inequality for Mappings whose Second Derivatives are Bounded and Applications", *J. Indian Math. Soc. (N.S.)*, **66** (4), (1999), pp. 237–245.

[16] S. S. Dragomir, P. Cerone, N. S. Barnett and J. Roumeliotis, "An Inequality of the Ostrowski Type for Double Integrals and Applications for Cubature Formulae", *Tamsui Oxf. J. Math. Sci.*, **16**(1), (2000), pp. 1–16.

[17] S. S. Dragomir, P. Cerone and J. Roumeliotis, "A new Generalization of Ostrowski Integral Inequality for Mappings whose Derivatives are Bounded and Applications in Numerical Integration and for Special Means, *Appl. Math. Lett.*, **13**(1), (2000), pp. 19–25.

[18] S. S.Dragomir, J. Pečarić and L. Persson, "Some inequalities of Hadamard type", *Soochow J. Math.*, **21** (3), (1995), pp. 335–341.

[19] A. Ekinci, "Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için Integral Eşitsizlikler", *Ph.D. Thesis*, Thesis ID: 361162 in tez2.yok.gov.tr Atatürk University, 2014.

[20] S. Gal, "Approximation theory in fuzzy setting, Chapter 13 in Handbook of Analytic Computational Methods in Applied Mathematics (edited by G. Anastassiou)", *Chapman and Hall, CRC Press, Boca Raton, New York*, (2000), pp. 617–666.

[21] E. K. Godunova, V. I. Levin, "Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions", *Numerical Mathematics and Mathematical Physics, (Russian)*, **166**(1), (1985), pp. 138–142.

[22] G. Á. Gómez, J. V. Moya, and J. E. Ricardo, "Method to measure the formation of pedagogical skills through neutrosophic numbers of unique value," *Revista Asociación Latinoamericana de Ciencias Neutrosóficas*, ISSN 2574-1101, **11** (1), (2020), pp. 41-48.

[23] N. Irshad, A. R. Khan and A. Nazir, "Extension of Ostrowki Type Inequality Via Moment Generating Function", *Adv. Inequal. Appl.*, **2**(1), (2020), pp. 1–15.

[24] N. Irshad, A. R. Khan and M. A. Shaikh, "Generalization of Weighted Ostrowski Inequality with Applications in Numerical Integration", *Adv. Ineq. Appl.*, **7**(1), (2019), pp. 1–14.

[25] N. Irshad, A. R. Khan, and Muhammad Awais Shaikh, "Generalized Weighted Ostrowski-Gruss Type Inequality with Applications", *Global J. Pure Appl. Math.*, **15** (5), (2019), pp. 675–692.

[26] N. Irshad and A. R. Khan, "On Weighted Ostrowski Gruss Inequality with Applications", *TJMM*, **10** (1), (2018), pp. 15–22.

[27] N. Irshad and A. R. Khan, "Generalization of Ostrowski Inequality for Differentiable functions and its applications to numerical quadrature rules", *J. Math. Anal*, **8**(1), (2017), pp. 79–102.

[28] O. Kaleva, "Fuzzy diferential equations", *Fuzzy Sets and Systems*, **24**(1) (1987) pp. 301–317.

[29] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, "Inequalities Involving Functions and Their Integrals and Derivatives", *Kluwer Academic, Dordrecht*, (1991).

[30] A. S. Molina, W. A. C. Calle, and J. D. B. Remache, "The application of Microsoft Solution Framework Software Testing using Neutrosophic Numbers," *Neutrosophic Sets and Systems*, **37**(1), (2020), pp. 267-276.

[31] M. Mullai and R. Surya, "Neutrosophic Inventory Backorder Problem Using Triangular Neutrosophic Numbers", *Neutrosophic Sets and Systems*, **31**(1), (2020), pp. 148-155.

[32] M. A. Noor and M. U. Awan, "Some integral inequalities for two kinds of convexities via fractional integrals", *TJMM*, **5**(1), (2013), pp. 129 – 136.

[33] A. M. Ostrowski, "Uber die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert", *Comment. Math. Helv.*, **10**(1), (1938), pp. 226–227.

[34] S. Varošanec, "On h-convexity", *J. Math. Anal. Appl.*, **326**(1), (2007), pp. 303–311.

[35] Z. G. Xiao, and A. H. Zhang, "Mixed power mean inequalities", *Research Communication on Inequalities*, **8** (1), (2002), pp. 15—17.

[36] X. Yang, "A note on Hölder inequality", *Appl. Math. Comput.*, **134**(1), (2003), pp. 319–322