Hyers-Ulam-Rassias Stability for Functional Equation in Neutrosophic Normed Spaces

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Abstract

In Neutrosophic Normed spaces, we investigate a unique quadratic function and a unique additive quadratic function of the Hyers-Ulam-Rassias stability for the functional equation $\sum_{i=1}^{n} f(x_i - (1/n)\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf((1/n)\sum_{i=1}^{n} x_i)$ which is said to be a functional equation associated with inner products space.
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1 Introduction

The aim of this article is to prove an Neutrosophic version of the Hyers-Ulam-Rassias stability for the functional equation:

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \tag{A}$$

which is said to be a functional equation associated with inner product spaces. It was shown by Rassias [1] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$ it follows

$$\sum_{i=1}^{n} \left\| x_i - \frac{1}{n}\sum_{j=1}^{n} x_j \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\left\| \frac{1}{n}\sum_{i=1}^{n} x_i \right\|^2 \tag{B}$$

for all $x_i, \ldots, x_n \in X$. Interesting new results concerning functional equations associated with inner product spaces have recently been obtained by Park et al. [2, 3] and Najati and Rassias [4] as well as for the fuzzy stability of a functional equation associated with inner product spaces [5].

Stability problem of a functional equation was first posed by Ulam [6] which was answered by Hyers [7] on approximately additive mappings and then generalized by Aoki [8] and Rassias [9] for additive mappings and linear mappings, respectively. Later there have been proved several new results on stability of various classes of functional equations in the Hyers-Ulam sense (cf. the following books and papers [10-18] and the references cited therein), as well as various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations. Furthermore some stability results concerning Jensen, cubic, mixed-type additive and cubic functional equations were investigated in the spirit of intuitionistic fuzzy normed spaces, while the idea of intuitionistic fuzzy normed space was introduced and further studied.

In future studies on this subject, it is also possible to work with the idea of "Probabilistic metric space" using neutrosophic probability. In 1940, Ulam raised the following question. Under what conditions does there exists an additive mapping near an approximately addition mapping? The case of approximately additive functions was solved by Hyers under certain assumption. In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors. Neutrosophic set(NS) is a new version of the idea of the classical set which is defined by Smarandache [26]. The first world publication related to the concept of neutrosophy was published in 1998 and included in the literature [24].

2 Preliminaries

Definition 2.1. [23] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm [CTN] if it satisfies the following conditions :

1. $*$ is commutative and associative,
2. $*$ is continuous,
3. $\alpha * 1 = \alpha$ for all $\alpha \in [0,1]$,
4. $\alpha * \beta \leq \gamma * \delta$ whenever $\alpha \leq \gamma$ and $\beta \leq \delta$, for each $\alpha, \beta, \gamma, \delta \in [0,1]$.

Definition 2.2. [23] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm [CTCN] if it satisfies the following conditions :

1. $\diamond$ is commutative and associative,
2. $\diamond$ is continuous,
3. $\alpha \diamond 0 = \alpha$ for all $\alpha \in [0,1]$,
4. $\alpha \diamond \beta \leq \gamma \diamond \delta$ whenever $\alpha \leq \gamma$ and $\beta \leq \delta$, for each $\alpha, \beta, \gamma, \delta \in [0,1]$.

Definition 2.3. The six-tuple $(X, \mu, \nu, \omega, *, \diamond, \otimes)$ is said to be an Neutrosophic Normed Spaces(NNS) if X is a vector space, Let $*$ and $\diamond, \otimes$ be the CTN and CTCN, respectively. μ, ν, ω are Normed spaces on $X \times (0, \infty)$ fulfilling the conditions below: For each $x, \beta \in X$ and for each $s, t > 0,\ \Phi \neq 0$,

1. $0 \leq \mu(x,t) \leq 1, 0 \leq \nu(x,t) \leq 1, 0 \leq \omega(x,t) \leq 1,$ for all $t \in (0, \infty)$;
2. $\mu(x,t) + \nu(x,t) + \omega(x,t) \leq 3$;
3. $\mu(x,t) > 0$;
4. $\mu(x,t) = 1$ iff $x = 0$;
5. $\mu(\Phi x, t) = \mu\left(x, \frac{t}{|\Phi|}\right)$ for each $\Phi \neq 0$;
6. $\mu(x,t) * \mu(\beta, s) \leq \mu(x+\beta, t+s)$;
7. $\mu(x, .) : (0, \infty) \to [0,1]$ is continuous increasing function;
8. $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t \to 0} \mu(x,t) = 0$;
9. $\nu(x,t) < 1$;
10. $\nu(x,t) = 0$ iff $x = 0$;
11. $\nu(\Phi x, t) = \nu\left(x, \frac{t}{|\Phi|}\right)$ for each $\Phi \neq 0$;
12. $\nu(x,t) \diamond \nu(\beta, s) \geq \nu(x+\beta, t+s)$;
13. $\nu(x, .) : (0, \infty) \to [0,1]$ is continuous increasing function;

14. $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1.$;
15. $\omega(x,t) < 1$;
16. $\omega(x,t) = 0$ iff $x = 0$;
17. $\omega(\Phi x, t) = \omega\left(x, \frac{t}{|\Phi|}\right)$ for each $\Phi \neq 0$;
18. $\omega(x,t) \otimes\ \omega(\beta,s) \geq \omega(x+\beta, t+s)$;
19. $\omega(x,.) : (0,\infty) \to [0,1]$ is continuous increasing function;
20. $\lim_{t\to\infty} \omega(x,t) = 0$ and $\lim_{t\to 0} \omega(x,t) = 1.$;

Then (μ,ν,ω) is called Neutrosophic Norm(NN).

Example 2.4. Let $(X, \|\cdot\|)$ be a NS. Define CTN and CTCN as follows $x * \beta = x\beta$ and $x \diamond \beta = x + \beta - x\beta$. For $t > \|x\|$,

$$\mu(x,t) = \frac{t}{t+\|x\|}, \nu(x,t) = \frac{\|x\|}{t+\|x\|}, \omega(x,t) = \frac{\|x\|}{t},$$

for all $x, \beta \in X$ and $t > 0$. If $t \leq \|x\|$, then $\mu(x,t) = 0, \nu(x,t) = 1, \omega(x,t) = 1$. Then $(X, \mu, \nu, \omega, *, \diamond, \otimes)$ is NNS

Definition 2.5. Let $(X, \mu, \nu, \omega, *, \diamond, \otimes)$ be a NNS.

1. A sequence (x_n) in X is Neutrosophic convergent to $x \in X$ if $\lim_{n\to\infty} \mu(x_n - x, t) = 1$, $\lim_{n\to\infty} \nu(x_n - x, t) = 0$ and $\lim_{n\to\infty} \omega(x_n - x, t) = 0$ as $t > 0$.
2. A sequence (x_n) is said to be Neutrosophic Cauchy sequence if $\lim_{n\to\infty} \mu(x_{n+p} - x_n, t) = 1$, $\lim_{n\to\infty} \nu(x_{n+p} - x_n, t) = 0$ and $\lim_{n\to\infty} \omega(x_{n+p} - x_n, t) = 0$ to each $t > 0$ and $p = 1, 2, \ldots$.
3. A $(X, \mu, \nu, \omega, *, \diamond, \otimes)$ is said to be Complete if every Neutrosophic Cauchy sequence in $(X, \mu, \nu, \omega, *, \diamond, \otimes)$ is Neutrosophic convergent in $(X, \mu, \nu, \omega, *, \diamond, \otimes)$.

3 Neutrosophic Stability

Throughout this section, assume that $X, (Z, \mu', \nu', \omega')$, and (Y, μ, ν, ω) are linear space, NNS, and Neutrosophic Banach space, respectively. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$\Delta f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right). \tag{C}$$

We begin with the Hyers-Ulam-Rassias type theorem in NNS for the functional (A) which is said to be a functional equation associated with inner product spaces.

Theorem 3.1. *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 4$. Suppose that an even function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned} &\mu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \geq \mu'(\varphi(x_1), t_1) * \cdots * \mu'(\varphi(x_n), t_n), \\ &\nu(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \leq \nu'(\varphi(x_1), t_1) \diamond \cdots \diamond \nu'(\varphi(x_n), t_n) \quad \textit{and} \\ &\omega(\Delta f(x_1, \ldots, x_n), t_1 + \cdots + t_n) \leq \omega'(\varphi(x_1), t_1) \otimes \cdots \otimes \omega'(\varphi(x_n), t_n) \end{aligned} \tag{3.1.1}$$

for all $x_1, \ldots, x_n \in X$ and all $t_1, \ldots, t_n > 0$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\mu(Q(x) - f(x), t) \geq \mu_1^{''}\left(x, \frac{(4-|\alpha|)t}{8}\right),$$

$$\nu(Q(x) - f(x), t) \leq \nu_1^{''}\left(x, \frac{(4-|\alpha|)t}{8}\right) \quad \textit{and}$$

$$\omega(Q(x)-f(x),t) \leq \omega_1^{''}\left(x, \frac{(4-|\alpha|)t}{8}\right) \tag{3.1.2}$$

for all $x \in X$ and $t > 0$, where

$$\begin{aligned}
\mu_1^{''}(x,t) &:= \mu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad * \mu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right), \\
\nu_1^{''}(x,t) &:= \nu'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad \diamond \nu'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right) \quad \textit{and} \\
\omega_1^{''}(x,t) &:= \omega'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \\
&\quad \otimes \omega'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right).
\end{aligned} \tag{3.1.3}$$

Proof. Put $x_1 = nx_1, x_i = nx_2 (i = 2, \ldots, n), t_i = t (i = 1, \ldots, n)$ in (3.1.1), and, using the evenness of f, we obtain

$$\begin{aligned}
&\mu\begin{pmatrix} nf(x_1+(n-1)x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-f(nx_1)-(n-1)f(nx_2), nt \end{pmatrix} \geq \mu'(\varphi(nx_1),t) * \mu'(\varphi(nx_2),t), \\
&\nu\begin{pmatrix} nf(x_1+(n-1)x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-f(nx_1)-(n-1)f(nx_2), nt \end{pmatrix} \leq \nu'(\varphi(nx_1),t) \diamond \nu'(\varphi(nx_2),t) \quad \text{and} \\
&\omega\begin{pmatrix} nf(x_1+(n-1)x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-f(nx_1)-(n-1)f(nx_2), nt \end{pmatrix} \leq \omega'(\varphi(nx_1),t) \otimes \omega'(\varphi(nx_2),t)
\end{aligned} \tag{3.1.4}$$

for all $x_1, x_2 \in X$ and $t > 0$. Interchanging x_1 with x_2 in (3.1.4) and using the evenness of f, we get

$$\begin{aligned}
&\mu\begin{pmatrix} nf((n-1)x_1+x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-(n-1)f(nx_1)-f(nx_2), nt \end{pmatrix} \geq \mu'(\varphi(nx_1),t) * \mu'(\varphi(nx_2),t), \\
&\nu\begin{pmatrix} nf((n-1)x_1+x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-(n-1)f(nx_1)-f(nx_2), nt \end{pmatrix} \leq \nu'(\varphi(nx_1),t) \diamond \nu'(\varphi(nx_2),t) \quad \text{and} \\
&\omega\begin{pmatrix} nf((n-1)x_1+x_2)+f((n-1)(x_1-x_2)) \\ +(n-1)f(x_1-x_2)-(n-1)f(nx_1)-f(nx_2), nt \end{pmatrix} \leq \omega'(\varphi(nx_1),t) \otimes \omega'(\varphi(nx_2),t)
\end{aligned} \tag{3.1.5}$$

for all $x_1, x_2 \in X$ and $t > 0$. It follows from (3.1.4) and (3.1.5) that

$$\begin{aligned}
&\mu\begin{pmatrix} nf((n-1)x_1+x_2)+nf(x_1+(n-1)x_2) \\ +2f((n-1)(x_1-x_2))+2(n-1)f(x_1-x_2) \\ -nf(nx_1)-nf(nx_2), 2nt \end{pmatrix} \geq \mu'(\varphi(nx_1),t) * \mu'(\varphi(nx_2),t), \\
&\nu\begin{pmatrix} nf((n-1)x_1+x_2)+nf(x_1+(n-1)x_2) \\ +2f((n-1)(x_1-x_2))+2(n-1)f(x_1-x_2) \\ -nf(nx_1)-nf(nx_2), 2nt \end{pmatrix} \leq \nu'(\varphi(nx_1),t) \diamond \nu'(\varphi(nx_2),t) \text{ and} \\
&\omega\begin{pmatrix} nf((n-1)x_1+x_2)+nf(x_1+(n-1)x_2) \\ +2f((n-1)(x_1-x_2))+2(n-1)f(x_1-x_2) \\ -nf(nx_1)-nf(nx_2), 2nt \end{pmatrix} \leq \omega'(\varphi(nx_1),t) \otimes \omega'(\varphi(nx_2),t)
\end{aligned} \tag{3.1.6}$$

for all $x_1, x_2 \in X$ and $t > 0$. Putting $x_1 = nx_2, x_2 = -nx_2,\ x_i = 0\ (i = 3, \ldots, n),\ t_i = t\ (i = 1, \ldots, n)$ in (3.1.1) and using the evenness of f, we obtain

$$\begin{aligned}
&\mu\begin{pmatrix} nf((n-1)x_1+x_2)+f(x_1+(n-1)x_2) \\ +2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2), nt \end{pmatrix} \geq \mu'(\varphi(nx_1),t) * \mu'(\varphi(nx_2),t) * \mu'(\varphi(0),t), \\
&\nu\begin{pmatrix} nf((n-1)x_1+x_2)+f(x_1+(n-1)x_2) \\ +2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2), nt \end{pmatrix} \leq \nu'(\varphi(nx_1),t) \diamond \nu'(\varphi(nx_2),t) \diamond \nu'(\varphi(0),t)
\end{aligned}$$

$$\omega\left(\begin{array}{c} nf((n-1)x_1+x_2)+f(x_1+(n-1)x_2) \\ +2(n-1)f(x_1-x_2)-f(nx_1)-f(nx_2),nt \end{array}\right) \leq \omega'(\varphi(nx_1),t)\otimes\omega'(\varphi(nx_2),t)\otimes\omega'(\varphi(0),t) \tag{3.1.7}$$

for all $x_1, x_2 \in X$ and t¿0. Hence, we obtain from (3.1.6) and (3.1.7) that

$$\begin{aligned} &\mu\left(f((n-1)(x_1-x_2))-(n-1)^2 f(x_1-x_2), \frac{n^2+2n}{2}t\right) \\ &\qquad \geq \mu'(\varphi(nx_1),t)*\mu'(\varphi(nx_2),t)*\mu'(\varphi(-nx_2),t)*\mu'(\varphi(0),t), \\ &\nu\left(f((n-1)(x_1-x_2))-(n-1)^2 f(x_1-x_2), \frac{n^2+2n}{2}t\right) \\ &\qquad \leq \nu'(\varphi(nx_1),t)\diamond\nu'(\varphi(nx_2),t)\diamond\nu'(\varphi(-nx_2),t)\diamond\nu'(\varphi(0),t) \quad \text{and} \\ &\omega\left(f((n-1)(x_1-x_2))-(n-1)^2 f(x_1-x_2), \frac{n^2+2n}{2}t\right) \\ &\qquad \leq \omega'(\varphi(nx_1),t)\otimes\omega'(\varphi(nx_2),t)\otimes\omega'(\varphi(-nx_2),t)\otimes\omega'(\varphi(0),t) \end{aligned} \tag{3.1.8}$$

for all $x_1, x_2 \in X$ and $t>0$. So

$$\begin{aligned} &\mu\left(f((n-1)x)-(n-1)^2 f(x), \frac{n^2+2n}{2}t\right) \geq \mu'(\varphi(nx),t)*\mu'(\varphi(0),t), \\ &\nu\left(f((n-1)x)-(n-1)^2 f(x), \frac{n^2+2n}{2}t\right) \leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and} \\ &\omega\left(f((n-1)x)-(n-1)^2 f(x), \frac{n^2+2n}{2}t\right) \leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \end{aligned} \tag{3.1.9}$$

for all $x \in X$ and $t>0$. Putting $x_1 = nx, x_i = 0 (i=2,\ldots,n), t_i = t(i=1,\ldots,n)$ in (3.1.1), we obtain

$$\begin{aligned} &\mu\left(f(nx)-f((n-1)x)-(2n-1)f(x),nt\right) \geq \mu'(\varphi(nx),t)*\mu'(\varphi(0),t), \\ &\nu\left(f(nx)-f((n-1)x)-(2n-1)f(x),nt\right) \leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and} \\ &\omega\left(f(nx)-f((n-1)x)-(2n-1)f(x),nt\right) \leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \end{aligned} \tag{3.1.10}$$

for all $x \in X$ and $t>0$. It follows from (3.1.9) and (3.1.10) that

$$\begin{aligned} &\mu\left(f(nx)-n^2 f(x), \frac{n^2+4n}{2}t\right) \geq \mu'(\varphi(nx),t)*\mu'(\varphi(0),t), \\ &\nu\left(f(nx)-n^2 f(x), \frac{n^2+4n}{2}t\right) \leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and} \\ &\omega\left(f(nx)-n^2 f(x), \frac{n^2+4n}{2}t\right) \leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \end{aligned} \tag{3.1.11}$$

for all $x \in X$ and $t>0$. Letting $x_2 = -(n-1)x_1$ in (3.1.7) and replacing x_1 by $\frac{x}{n}$ in the obtained inequality, we get

$$\begin{aligned} &\mu\left(\begin{array}{c} f((n-1)x)-f((n-2)x) \\ -(2n-3)f(x),nt \end{array}\right) \geq \mu'(\varphi(nx),t)*\mu'(\varphi((n-1)x),t)*\mu'(\varphi(0),t), \\ &\nu\left(\begin{array}{c} f((n-1)x)-f((n-2)x) \\ -(2n-3)f(x),nt \end{array}\right) \leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi((n-1)x),t)\diamond\nu'(\varphi(0),t) \quad \text{and} \\ &\omega\left(\begin{array}{c} f((n-1)x)-f((n-2)x) \\ -(2n-3)f(x),nt \end{array}\right) \leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi((n-1)x),t)\otimes\omega'(\varphi(0),t) \end{aligned} \tag{3.1.12}$$

for all $x \in X$ and $t>0$. It follows from (3.1.9), (3.1.10), (3.1.11) and (3.1.12) that

$$\begin{aligned} &\mu\left(f((n-2)x)-(n-1)^2 f(x), \frac{n^2+4n}{2}t\right) \\ &\qquad \geq \mu'(\varphi(nx),t)*\mu'(\varphi((n-1)x),t)*\mu'(\varphi(nx),t)*\mu'(\varphi(0),t), \end{aligned}$$

$$\nu\left(f((n-2)x)-(n-1)^2f(x),\frac{n^2+4n}{2}t\right)$$
$$\leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi((n-1)x),t)\diamond\nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and}$$
$$\omega\left(f((n-2)x)-(n-1)^2f(x),\frac{n^2+4n}{2}t\right)$$
$$\leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi((n-1)x),t)\otimes\omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \tag{3.1.13}$$

for all $x\in X$ and $t>0$. Applying (3.1.11) and (3.1.13), we obtain

$$\mu\left(f(nx)-f((n-2)x)-4(n-1)f(x),(n^2+4n)t\right)$$
$$\geq \mu'(\varphi(nx),t)*\mu'(\varphi((n-1)x),t)*\mu'(\varphi(nx),t)*\mu'(\varphi(0),t),$$
$$\nu\left(f(nx)-f((n-2)x)-4(n-1)f(x),(n^2+4n)t\right)$$
$$\leq \nu'(\varphi(nx),t)\diamond\nu'(\varphi((n-1)x),t)\diamond\nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and}$$
$$\omega\left(f(nx)-f((n-2)x)-4(n-1)f(x),(n^2+4n)t\right)$$
$$\leq \omega'(\varphi(nx),t)\otimes\omega'(\varphi((n-1)x),t)\otimes\omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \tag{3.1.14}$$

for all $x\in X$ and $t>0$. Setting $x_1=x_2=nx, x_i=0(i=3,\ldots,n), t_i=t(i=1,\ldots,n)$ in (3.1.1), we obtain

$$\mu\left(f((n-2)x)+(n-1)f(2x)-f(nx),\frac{n}{2}t\right)\geq\mu'(\varphi(nx),t)*\mu'(\varphi(0),t),$$
$$\nu\left(f((n-2)x)+(n-1)f(2x)-f(nx),\frac{n}{2}t\right)\leq\nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and}$$
$$\omega\left(f((n-2)x)+(n-1)f(2x)-f(nx),\frac{n}{2}t\right)\leq\omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \tag{3.1.15}$$

for all $x\in X$ and $t>0$. It follows from (3.1.14) and (3.1.15) that

$$\mu\begin{pmatrix} f(2x)-4f(x), \\ \dfrac{2n^2+9n}{2n-2} \end{pmatrix}\geq\mu'(\varphi(nx),t)*\mu'(\varphi((n-1)x,t),t)*\mu'(\varphi(nx),t)*\mu'(\varphi(0),t),$$
$$\nu\begin{pmatrix} f(2x)-4f(x), \\ \dfrac{2n^2+9n}{2n-2} \end{pmatrix}\leq\nu'(\varphi(nx),t)\diamond\nu'(\varphi((n-1)x,t),t)\diamond\nu'(\varphi(nx),t)\diamond\nu'(\varphi(0),t) \quad \text{and}$$
$$\omega\begin{pmatrix} f(2x)-4f(x), \\ \dfrac{2n^2+9n}{2n-2} \end{pmatrix}\leq\omega'(\varphi(nx),t)\otimes\omega'(\varphi((n-1)x,t),t)\otimes\omega'(\varphi(nx),t)\otimes\omega'(\varphi(0),t) \tag{3.1.16}$$

It follows that

$$\mu\left(f(x)-4^{-1}f(2x),t\right)\geq\begin{pmatrix} \mu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)*\mu'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \\ *\mu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)*\mu'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix},$$
$$\nu\left(f(x)-4^{-1}f(2x),t\right)\leq\begin{pmatrix} \nu'(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t)\diamond\nu'(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t) \\ \diamond\nu'(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t)\diamond\nu'(\varphi(0),\frac{8(n-1)}{2n^2+9n}t) \end{pmatrix} \quad \text{and}$$
$$\omega\left(f(x)-4^{-1}f(2x),t\right)\leq\begin{pmatrix} \omega'(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t)\otimes\omega'(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t) \\ \otimes\omega'(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t)\otimes\omega'(\varphi(0),\frac{8(n-1)}{2n^2+9n}t) \end{pmatrix}. \tag{3.1.17}$$

Define

$$\mu_1''(x,t):=\begin{pmatrix} \mu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)*\mu'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \\ *\mu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)*\mu'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix},$$
$$\nu_1''(x,t):=\begin{pmatrix} \nu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)\diamond\nu'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \\ \diamond\nu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)\diamond\nu'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix} \quad \text{and}$$
$$\omega_1''(x,t):=\begin{pmatrix} \omega'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)\otimes\omega'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \\ \otimes\omega'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right)\otimes\omega'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix}. \tag{3.1.18}$$

Then, by our assumption,

$$\mu_1^{''}(2x,t)=\mu_1^{''}\left(x,\frac{t}{\alpha}\right),\nu_1^{''}(2x,t)=\nu_1^{''}\left(x,\frac{t}{\alpha}\right) \quad \text{and } \omega_1^{''}(2x,t)=\omega_1^{''}\left(x,\frac{t}{\alpha}\right) \tag{3.1.19}$$

Replacing x by $2^n x$ in (3.1.17) and applying (3.1.19), we get

$$\begin{aligned}
\mu\left(\frac{2^n x}{4^n}-\frac{f(2^{n+1}x)}{4^{n+1}},\frac{\alpha^n t}{4^n}\right)&=\mu\left(f(2^n x)-\frac{f(2^{n+1}x)}{4},\alpha^n t\right)\geq \mu_1^{''}(2^n x,\alpha^n t)=\mu_1^{''}(x,t),\\
\nu\left(\frac{2^n x}{4^n}-\frac{f(2^{n+1}x)}{4^{n+1}},\frac{\alpha^n t}{4^n}\right)&=\nu\left(f(2^n x)-\frac{f(2^{n+1}x)}{4},\alpha^n t\right)\leq \nu_1^{''}(2^n x,\alpha^n t)=\nu_1^{''}(x,t) \quad \text{and}\\
\omega\left(\frac{2^n x}{4^n}-\frac{f(2^{n+1}x)}{4^{n+1}},\frac{\alpha^n t}{4^n}\right)&=\omega\left(f(2^n x)-\frac{f(2^{n+1}x)}{4},\alpha^n t\right)\leq \omega_1^{''}(2^n x,\alpha^n t)=\omega_1^{''}(x,t).
\end{aligned} \tag{3.1.20}$$

Thus for each $n>m$, we have

$$\begin{aligned}
\mu\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)&=\mu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)\\
&\geq \prod_{k=m}^{n-1}\left(\frac{2^k x}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\frac{\alpha^k t}{4^k}\right)\geq \mu_1^{''}(x,t),\\
\nu\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)&=\nu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)\\
&\leq \coprod_{k=m}^{n-1}\left(\frac{2^k x}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\frac{\alpha^k t}{4^k}\right)\leq \nu_1^{''}(x,t) \quad \text{and}\\
\omega\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)&=\omega\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{4^k}\right)\\
&\leq \coprod_{k=m}^{n-1}\left(\frac{2^k x}{4^k}-\frac{f(2^{k+1}x)}{4^{k+1}},\frac{\alpha^k t}{4^k}\right)\leq \omega_1^{''}(x,t)
\end{aligned} \tag{3.1.21}$$

where $\prod_{j=1}^n a_j=a_1*a_2*\cdots,*a_n$, $\coprod_{j=1}^n a_j=a_1\diamond a_2\diamond\cdots a_n$ and $\coprod_{j=1}^n a_j=a_1\otimes a_2\otimes\cdots a_n$. Let $\epsilon>0$ and $\delta>0$ be given. Since $\lim_{t\to\infty}\mu_1^{''}(x,t)=1$, $\lim_{t\to\infty}\nu_1^{''}(x,t)=0$ and $\lim_{t\to\infty}\omega_1^{''}(x,t)=0$ there exists some $t_0>0$ such that $\mu_1^{''}(x,t_0)>1-\epsilon$ $\nu_1^{''}(x,t_0)<\epsilon$ and $\omega_1^{''}(x,t_0)<\epsilon$. Since $\sum_{k=0}^{\infty}\left(\frac{a^k t_0}{4^k}\right)<\infty$, there is some $n_0\in\mathbb{N}$ such that $\sum_{k=m}^{n-1}\left(\frac{a^k t_0}{4^k}\right)<\delta$ for each $n>m\geq n_0$. It follows that

$$\begin{aligned}
\mu\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\delta\right)&\geq \mu\left(\frac{f(2^m x)}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{4^k}\right)\geq \mu_1^{''}(x,t_0)>1-\epsilon\\
\nu\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\delta\right)&\leq \nu\left(\frac{f(2^m x)}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{4^k}\right)\leq \nu_1^{''}(x,t_0)<\epsilon \quad \text{and}\\
\omega\left(\frac{2^m x}{4^m}-\frac{f(2^n x)}{4^n},\delta\right)&\leq \omega\left(\frac{f(2^m x)}{4^m}-\frac{f(2^n x)}{4^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{4^k}\right)\leq \omega_1^{''}(x,t_0)<\epsilon
\end{aligned} \tag{3.1.22}$$

for all $t>t_0$. This shows that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is Cauchy in (Y,μ,ν,ω). Since (Y,μ,ν,ω) is Neutrosophic Banach space, $\{\frac{f(2^n x)}{4^n}\}$ converges to some point $Q(x)\in Y$. Thus, we can define a mapping $Q(x):X\to Y$ such that $Q(x):=(\mu,\nu,\omega)-\lim_{n\to\infty}\left(\frac{f(2^n x)}{4^n}\right)$. Moreover, if we put $m=0$ in (3.1.21), we get

$$\mu\left(\frac{2^n x}{4^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{4^k}\right)\geq \mu_1^{''}(x,t),$$

$$\nu\left(\frac{2^n x}{4^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^k}\right) \le \nu_1^{''}(x,t), \quad \text{and}$$
$$\omega\left(\frac{2^n x}{4^n} - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^k}\right) \le \omega_1^{''}(x,t) \tag{3.1.23}$$

Thus,

$$\mu\left(\frac{2^n x}{4^n} - f(x), t\right) \ge \mu_1^{''}\left(x, \frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{4}\right)^k}\right),$$
$$\nu\left(\frac{2^n x}{4^n} - f(x), t\right) \le \nu_1^{''}\left(x, \frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{4}\right)^k}\right) \quad \text{and}$$
$$\omega\left(\frac{2^n x}{4^n} - f(x), t\right) \le \omega_1^{''}\left(x, \frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{4}\right)^k}\right). \tag{3.1.24}$$

Now, we will show that Q is quadratic. Setting $x_i = 2^m x_i (i = 1, \ldots, n)$ and $t_i = \left(\frac{t}{n}\right)$ $(i = 1, \ldots, n)$ in (3.1.1), we obtain

$$\begin{aligned}
\mu\left(\frac{\Delta f(2^m x_1, \ldots, 2^m x_n)}{4^m}, t\right) &\ge \mu'\left(\varphi(2^m x_1), 4^m \frac{t}{n}\right) * \cdots * \mu'\left(\varphi(2^m x_n), 4^m \frac{t}{n}\right) \\
&= \mu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) * \cdots * \mu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right), \\
\nu\left(\frac{\Delta f(2^m x_1, \ldots, 2^m x_n)}{4^m}, t\right) &\le \nu'\left(\varphi(2^m x_1), 4^m \frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(2^m x_n), 4^m \frac{t}{n}\right) \\
&= \nu'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) \diamond \cdots \diamond \nu'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) \quad \text{and} \\
\omega\left(\frac{\Delta f(2^m x_1, \ldots, 2^m x_n)}{4^m}, t\right) &\le \omega'\left(\varphi(2^m x_1), 4^m \frac{t}{n}\right) \otimes \cdots \otimes \omega'\left(\varphi(2^m x_n), 4^m \frac{t}{n}\right) \\
&= \omega'\left(\varphi(x_1), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right) \otimes \cdots \otimes \omega'\left(\varphi(x_n), \left(\frac{4}{\alpha}\right)^m \frac{t}{n}\right)
\end{aligned} \tag{3.1.25}$$

for all $x_1, \ldots, x_n \in X$ and $t > 0$. Letting $n \to \infty$ in (3.1.25), we obtain

$$\mu(\Delta Q(x_1, \ldots, x_n), t) = 1, \nu(\Delta Q(x_1, \ldots, x_n), t) = 0 \quad \text{and } \omega(\Delta Q(x_1, \ldots, x_n), t) = 0 \tag{3.1.26}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. This means that Q satisfies the functional (A) and so it is quadratic (see Lemma 2.2 of [4]).

Next, we approximate the difference between f and Q in Neutrosophic sense. By (3.1.24), we have

$$\begin{aligned}
\mu\left(Q(x) - f(x), t\right) &\ge \mu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) * \mu'\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\
&\ge \mu_1^{''}\left(x, \frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^k}\right) = \mu_1^{''}\left(x, \frac{(4-\alpha)t}{8}\right), \\
\nu\left(Q(x) - f(x), t\right) &\le \nu\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) \diamond \nu'\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\
&\le \nu_1^{''}\left(x, \frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^k}\right) = \nu_1^{''}\left(x, \frac{(4-\alpha)t}{8}\right) \quad \text{and} \\
\omega\left(Q(x) - f(x), t\right) &\le \omega\left(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}\right) \otimes \omega'\left(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2}\right) \\
&\le \omega_1^{''}\left(x, \frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{4}\right)^k}\right) = \omega_1^{''}\left(x, \frac{(4-\alpha)t}{8}\right)
\end{aligned} \tag{3.1.27}$$

for every $x \in X, t > 0$ and large enough n. To prove the uniqueness of Q, assume that Q' is another quadratic mapping from X to Y, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\mu(Q(x)-Q'(x),t) \geq \mu\left(Q(x)-f(x),\frac{t}{2}\right) * \mu\left(Q'(x)-f(x),\frac{t}{2}\right) \geq \mu_1^{''}\left(x,\frac{(4-\alpha)t}{16}\right),$$
$$\nu(Q(x)-Q'(x),t) \leq \nu\left(Q(x)-f(x),\frac{t}{2}\right) \diamond \nu\left(Q'(x)-f(x),\frac{t}{2}\right) \leq \nu_1^{''}\left(x,\frac{(4-\alpha)t}{16}\right) \quad \text{and}$$
$$\omega(Q(x)-Q'(x),t) \leq \omega\left(Q(x)-f(x),\frac{t}{2}\right) \otimes \omega\left(Q'(x)-f(x),\frac{t}{2}\right) \leq \omega_1^{''}\left(x,\frac{(4-\alpha)t}{16}\right). \quad (3.1.28)$$

Since Q and Q' are quadratic, we have

$$\mu(Q(x)-Q'(x),t) = \mu\left(Q(2^n x)-Q'(2^n x),4^n t\right) \geq \mu_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right),$$
$$\nu(Q(x)-Q'(x),t) = \nu\left(Q(2^n x)-Q'(2^n x),4^n t\right) \leq \nu_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right) \quad \text{and}$$
$$\omega(Q(x)-Q'(x),t) = \omega\left(Q(2^n x)-Q'(2^n x),4^n t\right) \leq \omega_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right) \quad (3.1.29)$$

for all $x \in X, t > 0$ and $n \in \mathbb{N}$. Since $0 \leq \alpha < 4$ and $\lim_{n\to\infty}\left(\frac{4}{\alpha}\right)^n = \infty$, we get

$$\lim_{n\to\infty}\mu_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right) = 1,$$
$$\lim_{n\to\infty}\nu_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right) = 0 \text{ and}$$
$$\lim_{n\to\infty}\omega_1^{''}\left(x,\frac{\left(\frac{4}{\alpha}\right)^n(4-\alpha)t}{16}\right) = 0 \quad (3.1.30)$$

Therefore $\mu(Q(x)-Q'(x),t) = 1$, $\nu(Q(x)-Q'(x),t) = 0$ and $\omega(Q(x)-Q'(x),t) = 0$ for all $x \in X$ and $t > 0$. Hence $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Theorem 3.2. *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 2$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$\mu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \geq \mu'(\varphi(x_1),t_1) * \cdots * \mu'(\varphi(x_n),t_n),$$
$$\nu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \leq \nu'(\varphi(x_1),t_1) \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \quad \textit{and}$$
$$\omega(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \leq \omega'(\varphi(x_1),t_1) \otimes \cdots \otimes \omega'(\varphi(x_n),t_n) \quad (3.2.1)$$

for all $x_1,\ldots,x_n \in X$ and all $t_1,\ldots,t_n > 0$. Then there exists a unique additive quadratic function $A : X \to Y$ such that

$$\mu(A(x)-f(x),t) \geq \mu_2^{''}\left(x,\frac{(2-|\alpha|)t}{4}\right), \nu(A(x)-f(x),t) \leq \nu_2^{''}\left(x,\frac{(2-|\alpha|)t}{4}\right)$$
$$\textit{and } \omega(A(x)-f(x),t) \leq \omega_2^{''}\left(x,\frac{(2-|\alpha|)t}{4}\right) \quad (3.2.2)$$

for all $x \in X$ and $t > 0$, where

$$\mu_2^{''}(x,t) := \begin{pmatrix} \mu'\left(\varphi(2x),\frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x),\frac{4}{n^2+4n}t\right) \\ *\mu'\left(\varphi(-x),\frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0),\frac{4}{n^2+4n}t\right) \end{pmatrix},$$
$$\nu_2^{''}(x,t) := \begin{pmatrix} \nu'\left(\varphi(2x),\frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi((n-1)x),\frac{4}{n^2+4n}t\right) \\ \diamond\nu'\left(\varphi(x),\frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0),\frac{4}{n^2+4n}t\right) \end{pmatrix} \quad \textit{and}$$
$$\omega_2^{''}(x,t) := \begin{pmatrix} \omega'\left(\varphi(2x),\frac{4}{n^2+4n}t\right) \otimes \omega'\left(\varphi((n-1)x),\frac{4}{n^2+4n}t\right) \\ \otimes\omega'\left(\varphi(x),\frac{4}{n^2+4n}t\right) \otimes \omega'\left(\varphi(0),\frac{4}{n^2+4n}t\right) \end{pmatrix} \quad (3.2.3)$$

Proof. Put $x_1 = nx_1, x_i = nx_1^{'}(i = 2, \dots, n), t_i = t(i = 1, \dots, n)$ in (3.2.1) and using the oddness of f, we obtain

$$\mu\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) + f((n-1)(x_1 - x_1^{'})) \\ -(n-1)f(x_1 - x_1^{'}) - f(nx_1) - (n-1)f(nx_1^{'}), nt \end{array}\right) \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1^{'}), t),$$
$$\nu\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) + f((n-1)(x_1 - x_1^{'})) \\ +(n-1)f(x_1 - x_1^{'}) - f(nx_1) - (n-1)f(nx_1^{'}), nt \end{array}\right) \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1^{'}), t) \quad \text{and}$$
$$\omega\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) + f((n-1)(x_1 - x_1^{'})) \\ +(n-1)f(x_1 - x_1^{'}) - f(nx_1) - (n-1)f(nx_1^{'}), nt \end{array}\right) \leq \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(nx_1^{'}), t) \quad (3.2.4)$$

for all $x_1, x_1^{'} \in X$ and $t > 0$. Interchanging x_1 with $x_1^{'}$ in (3.2.4) and using the oddness of f, we get

$$\mu\left(\begin{array}{c} nf((n-1)x_1 + x_1^{'}) - f((n-1)(x_1 - x_1^{'})) \\ +(n-1)f(x_1 - x_1^{'}) - (n-1)f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1^{'}), t),$$
$$\nu\left(\begin{array}{c} nf((n-1)x_1 + x_1^{'}) - f((n-1)(x_1 - x_1^{'})) \\ +(n-1)f(x_1 - x_1^{'}) - (n-1)f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1^{'}), t) \quad \text{and}$$
$$\omega\left(\begin{array}{c} nf((n-1)x_1 + x_1^{'}) - f((n-1)(x_1 - x_1^{'})) \\ +(n-1)f(x_1 - x_1^{'}) - (n-1)f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \leq \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(nx_1^{'}), t) \quad (3.2.5)$$

for all $x_1, x_1^{'} \in X$ and $t > 0$. It follows from (3.2.4) and (3.2.5) that

$$\mu\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) - nf((n-1)x_1 + x_1^{'}) \\ +2f((n-1)(x_1 - x_1^{'})) - 2(n-1)f(x_1 - x_1^{'}) \\ +(n-2)f(nx_1) - (n-2)f(nx_1^{'}), 2nt \end{array}\right) \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1^{'}), t),$$
$$\nu\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) - nf((n-1)x_1 + x_1^{'}) \\ +2f((n-1)(x_1 - x_1^{'})) - 2(n-1)f(x_1 - x_1^{'}) \\ +(n-2)f(nx_1) - (n-2)f(nx_1^{'}), 2nt \end{array}\right) \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1^{'}), t) \quad \text{and}$$
$$\omega\left(\begin{array}{c} nf(x_1 + (n-1)x_1^{'}) - nf((n-1)x_1 + x_1^{'}) \\ +2f((n-1)(x_1 - x_1^{'})) - 2(n-1)f(x_1 - x_1^{'}) \\ +(n-2)f(nx_1) - (n-2)f(nx_1^{'}), 2nt \end{array}\right) \leq \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(nx_1^{'}), t) \quad (3.2.6)$$

for all $x_1, x_1^{'} \in X$ and $t > 0$. Setting $x_1 = nx_1, x_2 = -nx_1^{'}, x_i = 0(i = 3, \dots, n), t_i = t\ (i = 1, \dots, n)$ in (3.2.1) and using the oddness of f, we get

$$\mu\left(\begin{array}{c} f((n-1)x_1 + x_1^{'}) - f(x_1 + (n-1)x_1^{'}) \\ +2f(x_1 - x_1^{'}) - f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(-nx_1^{'}), t) * \mu'(\varphi(0), t),$$
$$\nu\left(\begin{array}{c} f((n-1)x_1 + x_1^{'}) - f(x_1 + (n-1)x_1^{'}) \\ +2f(x_1 - x_1^{'}) - f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(-nx_1^{'}), t) \diamond \nu'(\varphi(0), t) \quad \text{and}$$
$$\omega\left(\begin{array}{c} f((n-1)x_1 + x_1^{'}) - f(x_1 + (n-1)x_1^{'}) \\ +2f(x_1 - x_1^{'}) - f(nx_1) - f(nx_1^{'}), nt \end{array}\right) \leq \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(-nx_1^{'}), t) \otimes \omega'(\varphi(0), t) \quad (3.2.7)$$

for all $x_1, x_1^{'} \in X$ and $t > 0$. It follows from (3.2.6) and (3.2.7) that

$$\mu\left(f((n-1)(x_1 - x_1^{'})) + f(x_1 - x_1^{'}) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 2n}{2}t\right)$$
$$\geq \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1^{'}), t) * \mu'(\varphi(-nx_1^{'}), t) * \mu'(\varphi(0), t),$$
$$\nu\left(f((n-1)(x_1 - x_1^{'})) + f(x_1 - x_1^{'}) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 2n}{2}t\right)$$
$$\leq \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1^{'}), t) \diamond \nu'(\varphi(-nx_1^{'}), t) \diamond \nu'(\varphi(0), t) \quad \text{and}$$
$$\omega\left(f((n-1)(x_1 - x_1^{'})) + f(x_1 - x_1^{'}) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 2n}{2}t\right)$$
$$\leq \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(nx_1^{'}), t) \otimes \omega'(\varphi(-nx_1^{'}), t) \otimes \omega'(\varphi(0), t) \quad (3.2.8)$$

for all $x \in X$ and $t > 0$. Putting $x_1 = n(x_1 - x_1^{'}), x_i = 0 (i = 2, \ldots, n), t_i = t (i = 1, \ldots, n)$ in (3.2.1), we obtain

$$\begin{aligned} &\mu(f(n(x_1 - x_1^{'})) - f((n-1)(x_1 - x_1^{'})) - f(x_1 - x_1^{'}), nt) \geq \mu'(\varphi(n(x_1 - x_1^{'})), t) * \mu'(\varphi(0), t), \\ &\nu(f(n(x_1 - x_1^{'})) - f((n-1)(x_1 - x_1^{'})) - f(x_1 - x_1^{'}), nt) \leq \nu'(\varphi(n(x_1 - x_1^{'})), t) \diamond \nu'(\varphi(0), t) \quad \text{and} \\ &\omega(f(n(x_1 - x_1^{'})) - f((n-1)(x_1 - x_1^{'})) - f(x_1 - x_1^{'}), nt) \leq \omega'(\varphi(n(x_1 - x_1^{'})), t) \otimes \omega'(\varphi(0), t) \quad (3.2.9) \end{aligned}$$

for all $x_1, x_1^{'} \in X$ and $t > 0$. It follows from (3.2.8) and (3.2.9) that

$$\begin{aligned} &\mu\left(f(n(x_1 - x_1^{'})) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 4n}{2} t\right) \\ &\quad \geq \mu'(\varphi(n(x_1 - x_1^{'}), t) * \mu'(\varphi(nx_1), t) * \mu'(\varphi(nx_1^{'}), t) * \mu'(\varphi(-nx_1^{'}), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(n(x_1 - x_1^{'})) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 4n}{2} t\right) \\ &\quad \leq \nu'(\varphi(n(x_1 - x_1^{'}), t) \diamond \nu'(\varphi(nx_1), t) \diamond \nu'(\varphi(nx_1^{'}), t) \diamond \nu'(\varphi(-nx_1^{'}), t) \diamond \nu'(\varphi(0), t) \quad \text{and} \\ &\omega\left(f(n(x_1 - x_1^{'})) - f(nx_1) + f(nx_1^{'}), \frac{n^2 + 4n}{2} t\right) \\ &\quad \leq \omega'(\varphi(n(x_1 - x_1^{'}), t) \otimes \omega'(\varphi(nx_1), t) \otimes \omega'(\varphi(nx_1^{'}), t) \otimes \omega'(\varphi(-nx_1^{'}), t) \otimes \omega'(\varphi(0), t) \quad (3.2.10) \end{aligned}$$

for all $x \in X$ and $t > 0$. Replacing x_1 and $x_1^{'}$ by $\frac{x}{n}$ and $\frac{-x}{n}$ in (3.2.10)respectively, we have

$$\begin{aligned} &\mu\left(f(2x) - 2f(x), \frac{n^2 + 4n}{2} t\right) \geq \mu'(\varphi(2x), t) * \mu'(\varphi(x), t) * \mu'(\varphi(-x), t) * \mu'(\varphi(0), t), \\ &\nu\left(f(2x) - 2f(x), \frac{n^2 + 4n}{2} t\right) \leq \nu'(\varphi(2x), t) \diamond \nu'(\varphi(x), t) \diamond \nu'(\varphi(-x), t) \diamond \nu'(\varphi(0), t) \quad \text{and} \\ &\omega\left(f(2x) - 2f(x), \frac{n^2 + 4n}{2} t\right) \leq \omega'(\varphi(2x), t) \otimes \omega'(\varphi(x), t) \otimes \omega'(\varphi(-x), t) \otimes \omega'(\varphi(0), t). \quad (3.2.11) \end{aligned}$$

It follows that

$$\begin{aligned} \mu\left(f(x) - 2^{-1} f(2x), t\right) &\geq \begin{pmatrix} \mu'\left(\varphi(2x), \frac{4}{n^2+4n} t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ * \mu'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix}, \\ \nu\left(f(x) - 2^{-1} f(2x), t\right) &\leq \begin{pmatrix} \nu'(\varphi(2x), \frac{4}{n^2+4n} t) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix} \quad \text{and} \\ \omega\left(f(x) - 2^{-1} f(2x), t\right) &\leq \begin{pmatrix} \omega'(\varphi(2x), \frac{4}{n^2+4n} t) \otimes \omega'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ \otimes \omega'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) \otimes \omega'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix}. \quad (3.2.12) \end{aligned}$$

Define

$$\begin{aligned} \mu_2^{''}(x, t) &:= \begin{pmatrix} \mu'\left(\varphi(2x), \frac{4}{n^2+4n} t\right) * \mu'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ * \mu'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) * \mu'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix}, \\ \nu_2^{''}(x, t) &:= \begin{pmatrix} \nu'\left(\varphi(2x), \frac{4}{n^2+4n} t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix} \quad \text{and} \\ \omega_2^{''}(x, t) &:= \begin{pmatrix} \omega'\left(\varphi(2x), \frac{4}{n^2+4n} t\right) \otimes \omega'\left(\varphi(x), \frac{4}{n^2+4n} t\right) \\ \otimes \omega'\left(\varphi(-x), \frac{4}{n^2+4n} t\right) \otimes \omega'\left(\varphi(0), \frac{4}{n^2+4n} t\right) \end{pmatrix}. \quad (3.2.13) \end{aligned}$$

Then, by the assumption,

$$\mu_2^{''}(2x, t) = \mu_2^{''}\left(x, \frac{t}{\alpha}\right), \nu_2^{''}(2x, t) = \nu_2^{''}\left(x, \frac{t}{\alpha}\right) \quad \text{and } \omega_2^{''}(2x, t) = \omega_2^{''}\left(x, \frac{t}{\alpha}\right) \quad (3.2.14)$$

Replacing x by $2^n x$ in (3.2.12) and applying (3.2.14), we get

$$\mu\left(\frac{2^n x}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}},\frac{\alpha^n t}{2^n}\right)=\mu\left(f(2^n x)-\frac{f(2^{n+1}x)}{2},\alpha^n t\right)\geq \mu_2^{''}(2^n x,\alpha^n t)=\mu_2^{''}(x,t),$$
$$\nu\left(\frac{2^n x}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}},\frac{\alpha^n t}{2^n}\right)=\nu\left(f(2^n x)-\frac{f(2^{n+1}x)}{2},\alpha^n t\right)\leq \nu_2^{''}(2^n x,\alpha^n t)=\nu_2^{''}(x,t)\quad\text{and}$$
$$\omega\left(\frac{2^n x}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}},\frac{\alpha^n t}{2^n}\right)=\omega\left(f(2^n x)-\frac{f(2^{n+1}x)}{2},\alpha^n t\right)\leq \omega_2^{''}(2^n x,\alpha^n t)=\omega_2^{''}(x,t). \tag{3.2.15}$$

Thus for each $n>m$, we have

$$\begin{aligned}\mu\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)&=\mu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)\\&\geq\prod_{k=m}^{n-1}\left(\frac{2^k x}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\frac{\alpha^k t}{2^k}\right)\geq\mu_2^{''}(x,t),\\ \nu\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)&=\nu\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)\\&\leq\coprod_{k=m}^{n-1}\left(\frac{2^k x}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\frac{\alpha^k t}{2^k}\right)\leq\nu_2^{''}(x,t)\quad\text{and}\\ \omega\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)&=\omega\left(\sum_{k=m}^{n-1}\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\sum_{k=m}^{n-1}\frac{\alpha^k t}{2^k}\right)\\&\leq\coprod_{k=m}^{n-1}\left(\frac{2^k x}{2^k}-\frac{f(2^{k+1}x)}{2^{k+1}},\frac{\alpha^k t}{2^k}\right)\leq\omega_2^{''}(x,t)\end{aligned}\tag{3.2.16}$$

where $\prod_{j=1}^{n}a_j=a_1*a_2*\cdots,*a_n$, $\coprod_{j=1}^{n}a_j=a_1\diamond a_2\diamond\cdots a_n$ and $\coprod_{j=1}^{n}a_j=a_1\otimes a_2\otimes\cdots a_n$. Let $\epsilon>0$ and $\delta>0$ be given. Since $\lim_{t\to\infty}\mu_2^{''}(x,t)=1$, $\lim_{t\to\infty}\nu_2^{''}(x,t)=0$ and $\lim_{t\to\infty}\omega_2^{''}(x,t)=0$ there exists some $t_0>0$ such that $\mu_2^{''}(x,t_0)>1-\epsilon$, $\nu_2^{''}(x,t_0)<\epsilon$ and $\omega_2^{''}(x,t_0)<\epsilon$. Since $\sum_{k=0}^{\infty}\left(\frac{a^k t_0}{2^k}\right)<\infty$, there is some $n_0\in\mathbb{N}$ such that $\sum_{k=m}^{n-1}\left(\frac{a^k t_0}{2^k}\right)<\delta$ for each $n>m\geq n_0$. It follows that

$$\begin{aligned}\mu\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\delta\right)&\geq\mu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{2^k}\right)\geq\mu_2^{''}(x,t_0)>1-\epsilon\\ \nu\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\delta\right)&\leq\nu\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{2^k}\right)\leq\nu_2^{''}(x,t_0)<\epsilon\quad\text{and}\\ \omega\left(\frac{2^m x}{2^m}-\frac{f(2^n x)}{2^n},\delta\right)&\leq\omega\left(\frac{f(2^m x)}{2^m}-\frac{f(2^n x)}{2^n},\sum_{k=m}^{n-1}\frac{\alpha^k t_0}{2^k}\right)\leq\omega_2^{''}(x,t_0)<\epsilon\end{aligned}\tag{3.2.17}$$

for all $t>t_0$. This shows that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy in (Y,μ,ν,ω). Since (Y,μ,ν,ω) is Neutrosophic Banach space, $\{\frac{f(2^n x)}{2^n}\}$ converges to some point $A(x)\in Y$. Thus, we can define a mapping $A(x):X\to Y$ such that $A(x):=(\mu,\nu,\omega)-\lim_{n\to\infty}\left(\frac{f(2^n x)}{2^n}\right)$. Moreover, if we put $m=0$ in (3.2.16), we get

$$\mu\left(\frac{2^n x}{2^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}\right)\geq\mu_2^{''}(x,t),$$
$$\nu\left(\frac{2^n x}{2^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}\right)\leq\nu_2^{''}(x,t)\quad\text{and}$$

$$\omega\left(\frac{2^n x}{2^n}-f(x),\sum_{k=0}^{n-1}\frac{\alpha^k t}{2^k}\right)\leq \omega_2^{''}(x,t). \tag{3.2.18}$$

Thus,

$$\mu\left(\frac{2^n x}{2^n}-f(x),t\right)\geq \mu_2^{''}\left(x,\frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{2}\right)^k}\right),$$
$$\nu\left(\frac{2^n x}{2^n}-f(x),t\right)\leq \nu_2^{''}\left(x,\frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{2}\right)^k}\right) \quad \text{and}$$
$$\omega\left(\frac{2^n x}{2^n}-f(x),t\right)\leq \omega_2^{''}\left(x,\frac{t}{\sum_{k=0}^{n-1}\left(\frac{\alpha}{2}\right)^k}\right). \tag{3.2.19}$$

Next, we will show that A is Additive. Putting $x_i = 2^m x_i (i = 1,\ldots,n)$ and $t_i = \left(\frac{t}{n}\right)$ $(i = 1,\ldots,n)$ in (3.2.1), we obtain

$$\begin{aligned}
\mu\left(\frac{\Delta f(2^m x_1,\ldots,2^m x_n)}{2^m},t\right) &\geq \mu'\left(\varphi(2^m x_1),2^m\frac{t}{n}\right)*\cdots*\mu'\left(\varphi(2^m x_n),2^m\frac{t}{n}\right)\\
&=\mu'\left(\varphi(x_1),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right)*\cdots*\mu'\left(\varphi(x_n),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right),\\
\nu\left(\frac{\Delta f(2^m x_1,\ldots,2^m x_n)}{2^m},t\right) &\leq \nu'\left(\varphi(2^m x_1),2^m\frac{t}{n}\right)\diamond\cdots\diamond\nu'\left(\varphi(2^m x_n),2^m\frac{t}{n}\right)\\
&=\nu'\left(\varphi(x_1),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right)\diamond\cdots\diamond\nu'\left(\varphi(x_n),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right) \quad \text{and}\\
\omega\left(\frac{\Delta f(2^m x_1,\ldots,2^m x_n)}{2^m},t\right) &\leq \omega'\left(\varphi(2^m x_1),2^m\frac{t}{n}\right)\otimes\cdots\otimes\omega'\left(\varphi(2^m x_n),2^m\frac{t}{n}\right)\\
&=\omega'\left(\varphi(x_1),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right)\otimes\cdots\otimes\omega'\left(\varphi(x_n),\left(\frac{2}{\alpha}\right)^m\frac{t}{n}\right)
\end{aligned} \tag{3.2.20}$$

for all $x_1,\ldots,x_n \in X$ and $t > 0$. Letting $n \to \infty$ in (3.2.20), we obtain

$$\mu(\Delta A(x_1,\ldots,x_n),t)=1, \nu(\Delta A(x_1,\ldots,x_n),t)=0 \quad \text{and } \omega(\Delta A(x_1,\ldots,x_n),t)=0 \tag{3.2.21}$$

for all $x_1,\ldots,x_n \in X$ and all $t > 0$. This means that A satisfies the functional (A) and so it is additive (see Lemma 2.1 of [4]).

Next, we approximate the difference between f and A in Neutrosophic sense. For every $x \in X, t > 0$ and sufficiently large n, by (3.2.19), we have

$$\begin{aligned}
\mu\left(A(x)-f(x),t\right) &\geq \mu\left(A(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)*\mu'\left(\frac{f(2^n x)}{2^n}-f(x),\frac{t}{2}\right)\\
&\geq \mu_2^{''}\left(x,\frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{2}\right)^k}\right)=\mu_2^{''}\left(x,\frac{(2-\alpha)t}{4}\right),
\end{aligned}$$

$$\begin{aligned}
\nu\left(A(x)-f(x),t\right) &\leq \nu\left(A(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)\diamond\nu'\left(\frac{f(2^n x)}{2^n}-f(x),\frac{t}{2}\right)\\
&\leq \nu_2^{''}\left(x,\frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{2}\right)^k}\right)=\nu_2^{''}\left(x,\frac{(2-\alpha)t}{4}\right) \quad \text{and}\\
\omega\left(A(x)-f(x),t\right) &\leq \omega\left(A(x)-\frac{f(2^n x)}{2^n},\frac{t}{2}\right)\otimes\omega'\left(\frac{f(2^n x)}{2^n}-f(x),\frac{t}{2}\right)\\
&\leq \omega_2^{''}\left(x,\frac{t}{2\sum_{k=0}^{\infty}\left(\frac{\alpha}{2}\right)^k}\right)=\omega_2^{''}\left(x,\frac{(2-\alpha)t}{4}\right),
\end{aligned} \tag{3.2.22}$$

To prove the uniqueness of A, assume that A' is another additive mapping from X to Y, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$,

$$\mu(A(x)-A'(x),t)\geq \mu\left(A(x)-f(x),\frac{t}{2}\right)*\mu\left(A'(x)-f(x),\frac{t}{2}\right)\geq \mu_2^{''}\left(x,\frac{(2-\alpha)t}{8}\right),$$

$$\nu(A(x)-A'(x),t) \leq \nu\left(A(x)-f(x),\frac{t}{2}\right) \diamond \nu\left(A'(x)-f(x),\frac{t}{2}\right) \leq \nu_2^{''}\left(x,\frac{(2-\alpha)t}{8}\right) \quad \text{and}$$
$$\omega(A(x)-A'(x),t) \leq \omega\left(A(x)-f(x),\frac{t}{2}\right) \otimes \omega\left(A'(x)-f(x),\frac{t}{2}\right) \leq \omega_2^{''}\left(x,\frac{(2-\alpha)t}{8}\right) \tag{3.2.23}$$

Therefore, by the additivity of A and A', we have

$$\mu(A(x)-A'(x),t) = \mu\left(A(2^n x)-A'(2^n x),2^n t\right) \geq \mu_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right),$$
$$\nu(A(x)-A'(x),t) = \nu\left(A(2^n x)-A'(2^n x),2^n t\right) \leq \nu_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right) \quad \text{and}$$
$$\omega(A(x)-A'(x),t) = \omega\left(A(2^n x)-A'(2^n x),2^n t\right) \leq \omega_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right) \tag{3.2.24}$$

for all $x \in X, t > 0$ and $n \in \mathbb{N}$. Since $0 \leq \alpha < 2$ and $\lim_{n\to\infty}\left(\frac{2}{\alpha}\right)^n = \infty$, we get

$$\lim_{n\to\infty}\mu_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right) = 1,$$
$$\lim_{n\to\infty}\nu_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right) = 0,$$
$$\lim_{n\to\infty}\omega_2^{''}\left(x,\frac{\left(\frac{2}{\alpha}\right)^n(2-\alpha)t}{8}\right) = 0 \tag{3.2.25}$$

Therefore $\mu(A(x)-A'(x),t) = 1$, $\nu(A(x)-A'(x),t) = 0$ and $\omega(A(x)-A'(x),t) = 0$ for all $x \in X$ and $t > 0$. Hence $A(x) = A'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Theorem 3.3. *Let $\varphi : X \to Z$ be a function such that $\varphi(2x) = \alpha\varphi(x)$ for some real number α with $0 < |\alpha| < 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\mu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \geq \mu'(\varphi(x_1),t_1) * \cdots * \mu'(\varphi(x_n),t_n),$$
$$\nu(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \leq \nu'(\varphi(x_1),t_1) \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \quad \text{and}$$
$$\omega(\Delta f(x_1,\ldots,x_n),t_1+\cdots+t_n) \leq \omega'(\varphi(x_1),t_1) \otimes \cdots \otimes \omega'(\varphi(x_n),t_n) \tag{3.3.1}$$

for all $x_1,\ldots,x_n \in X$ and all $t_1,\ldots,t_n > 0$. Then there exists a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ such that

$$\mu(Q(x)-A(x)-f(x),t) \geq M_1^{''}\left(x,\frac{(4-|\alpha|)t}{16}\right) * \widetilde{M_1}^{''}\left(x,\frac{(2-|\alpha|)t}{8}\right),$$
$$\nu(Q(x)-A(x)-f(x),t) \leq M_2^{''}\left(x,\frac{(4-|\alpha|)t}{16}\right) \diamond \widetilde{M_2}^{''}\left(x,\frac{(2-|\alpha|)t}{8}\right) \quad \text{and}$$
$$\omega(Q(x)-A(x)-f(x),t) \leq M_3^{''}\left(x,\frac{(4-|\alpha|)t}{16}\right) \otimes \widetilde{M_3}^{''}\left(x,\frac{(2-|\alpha|)t}{8}\right) \tag{3.3.2}$$

for all $x \in X$ and $t > 0$, where

$$M_1^{''}(x,t) := \begin{pmatrix} \mu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(x),\frac{8(n-1)}{2n^2+9n}t\right) \\ *\mu'\left(\varphi(-nx),\frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(-(n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \\ *\mu'\left(\varphi(-x),\frac{8(n-1)}{2n^2+9n}t\right) * \mu'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix},$$
$$\widetilde{M_1}^{''}(x,t) := \begin{pmatrix} \mu'\left(\varphi(2x),\frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(x),\frac{4(n-1)}{n^2+4n}t\right) * \mu'\left(\varphi(-x),\frac{4}{n^2+4n}t\right) \\ *\mu'\left(\varphi(-2x),\frac{4}{n^2+4n}t\right) * \mu'\left(\varphi(0),\frac{4}{n^2+4n}t\right) \end{pmatrix},$$
$$M_2^{''}(x,t) := \begin{pmatrix} \nu'\left(\varphi(nx),\frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi((n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(x),\frac{8(n-1)}{2n^2+9n}t\right) \\ \diamond\nu'\left(\varphi(-nx),\frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(-(n-1)x),\frac{8(n-1)}{2n^2+9n}t\right) \diamond \nu'\left(\varphi(-x),\frac{8(n-1)}{2n^2+9n}t\right) \\ \diamond\nu'\left(\varphi(0),\frac{8(n-1)}{2n^2+9n}t\right) \end{pmatrix},$$

$$\widetilde{M}_2''(x,t) := \left(\begin{array}{c} \nu'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \\ \diamond \nu'\left(\varphi(-2x), \frac{4}{n^2+4n}t\right) \diamond \nu'\left(\varphi(0), \frac{4}{n^2+4n}t\right) \end{array} \right) \quad \textit{and}$$

$$M_3''(x,t) := \left(\begin{array}{c} \omega'\left(\varphi(nx), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi((n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi(x), \frac{8(n-1)}{2n^2+9n}t\right) \\ \otimes\omega'\left(\varphi(-nx), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi(-(n-1)x), \frac{8(n-1)}{2n^2+9n}t\right) \otimes \omega'\left(\varphi(-x), \frac{8(n-1)}{2n^2+9n}t\right) \\ \otimes\omega'\left(\varphi(0), \frac{8(n-1)}{2n^2+9n}t\right) \end{array} \right),$$

$$\widetilde{M}_3''(x,t) := \left(\begin{array}{c} \omega'\left(\varphi(2x), \frac{4}{n^2+4n}t\right) \otimes \omega'\left(\varphi(x), \frac{4}{n^2+4n}t\right) \otimes \omega'\left(\varphi(-x), \frac{4}{n^2+4n}t\right) \\ \otimes\omega'\left(\varphi(-2x), \frac{4}{n^2+4n}t\right) \otimes \omega'\left(\varphi(0), \frac{4}{n^2+4n}t\right) \end{array} \right). \tag{3.3.3}$$

Proof. Passing to the even part f_e and odd part f_0 of f, we deduce from (3.3.1) that

$$\begin{aligned} &\mu(\Delta f_e(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \geq \mu'(\varphi(x_1),t_1) * \mu'(\varphi(-x_1),t_1) * \cdots * \mu'(\varphi(x_n),t_n) * \mu'(\varphi(-x_n),t_n), \\ &\nu(\Delta f_e(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \leq \nu'(\varphi(x_1),t_1) \diamond \nu'(\varphi(-x_1),t_1) \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \diamond \nu'(\varphi(-x_n),t_n) \quad \text{and} \\ &\omega(\Delta f_e(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \leq \omega'(\varphi(x_1),t_1) \otimes \omega'(\varphi(-x_1),t_1) \otimes \cdots \otimes \omega'(\varphi(x_n),t_n) \otimes \omega'(\varphi(-x_n),t_n) \end{aligned} \tag{3.3.4}$$

On the other hand,

$$\begin{aligned} &\mu(\Delta f_0(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \geq \mu'(\varphi(x_1),t_1) * \mu'(\varphi(-x_1),t_1) * \cdots * \mu'(\varphi(x_n),t_n) * \mu'(\varphi(-x_n),t_n), \\ &\nu(\Delta f_0(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \leq \nu'(\varphi(x_1),t_1) \diamond \nu'(\varphi(-x_1),t_1) \diamond \cdots \diamond \nu'(\varphi(x_n),t_n) \diamond \nu'(\varphi(-x_n),t_n) \quad \text{and} \\ &\omega(\Delta f_0(x_1,\dots,x_n), t_1+\cdots+t_n) \\ &\qquad \leq \omega'(\varphi(x_1),t_1) \otimes \omega'(\varphi(-x_1),t_1) \otimes \cdots \otimes \omega'(\varphi(x_n),t_n) \otimes \omega'(\varphi(-x_n),t_n) \end{aligned} \tag{3.3.5}$$

Applying the proofs of the Theorem 3.1 and 3.2, we get a unique quadratic function Q and a unique additive function A satisfying

$$\begin{aligned} \mu(Q(x)-f_e(x),t) &\geq M_1''\left(x, \frac{(4-|\alpha|)t}{8}\right), \\ \nu(Q(x)-f_e(x),t) &\leq M_2''\left(x, \frac{(4-|\alpha|)t}{8}\right) \quad \text{and} \\ \omega(Q(x)-f_e(x),t) &\leq M_3''\left(x, \frac{(4-|\alpha|)t}{8}\right). \end{aligned} \tag{3.3.6}$$

Also,

$$\begin{aligned} \mu(A(x)-f_0(x),t) &\geq \tilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{4}\right), \\ \nu(A(x)-f_0(x),t) &\leq \tilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{4}\right) \quad \text{and} \\ \omega(A(x)-f_0(x),t) &\leq \tilde{M}_3''\left(x, \frac{(2-|\alpha|)t}{4}\right) \end{aligned} \tag{3.3.7}$$

Therefore,

$$\begin{aligned} \mu(Q(x)-A(x)-f(x),t) &\geq M_1''\left(x, \frac{(4-|\alpha|)t}{16}\right) * \tilde{M}_1''\left(x, \frac{(2-|\alpha|)t}{8}\right), \\ \nu(Q(x)-A(x)-f(x),t) &\leq M_2''\left(x, \frac{(4-|\alpha|)t}{16}\right) \diamond \tilde{M}_2''\left(x, \frac{(2-|\alpha|)t}{8}\right) \quad \text{and} \\ \omega(Q(x)-A(x)-f(x),t) &\leq M_3''\left(x, \frac{(4-|\alpha|)t}{16}\right) \diamond \tilde{M}_3''\left(x, \frac{(2-|\alpha|)t}{8}\right) \end{aligned} \tag{3.3.8}$$

This completes the proof of the theorem. □

Conclusion

In this article, we prove existence of unique quadratic function and unique additive quadratic function between linear space and Neutrosophic Banach Space.

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