



A Study of Some Neutrosophic Clean Rings

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Abstract

The objective of this paper is to introduce a necessary and sufficient condition for a neutrosophic ring to be clean. This work proves the equivalence between case of classical clean ring R and the corresponding neutrosophic ring $R(I)$, refined neutrosophic ring $R(I_1, I_2)$, and n -refined neutrosophic ring $R_n(I)$.

Keywords: refined neutrosophic ring; n -refined neutrosophic ring; classical clean ring; neutrosophic clean ring

1. Introduction

Neutrosophy is a new kind of generalized logic founded by Smarandache to handle the indeterminacy and uncertainty in science and life.

The concept of neutrosophic rings was defined by Kandasamy and Samarandache in [1], as an extension of any algebraic ring R .

These rings played an important role in algebraic studies such as as number theory [2,3], functional analysis [4], and algebraic geometry [5].

Recently, neutrosophic rings have been expanded to refined neutrosophic rings [6], and n -refined neutrosophic rings [7].

In [11-13], we find many theorems about the structure of idempotents and special sets of these rings.

In [9], the concept of neutrosophic clean ring was studied with many elementary interesting properties.

This motivates us to extend the previous works about clean rings to refined and n -refined neutrosophic rings, where we show the equivalence between clean conditions in any ring R and its corresponding neutrosophic ring $R(I)$, refined neutrosophic ring $R(I_1, I_2)$, and n -refined neutrosophic ring $R_n(I)$. All rings are considered with unity 1.

2. Preliminaries.

Definition 1.[1]

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$$R(I) = \{a + bI; a, b \in R\}.$$

Definition 2. [6]

(a) The element I can be split into two indeterminacies I_1, I_2 with conditions:

$$I_1^2 = I_1, I_2^2 = I_2, I_1 I_2 = I_2 I_1 = I_1.$$

(b) If X is a set then $X(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in X\}$ is called the refined neutrosophic set generated by X, I_1, I_2 .

(c) Let $(R, +, \times)$ be a ring, $(R(I_1, I_2), +, \times)$ is called a refined neutrosophic ring generated by R, I_1, I_2 .

Definition 3.

Let $(R, +, \times)$ be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define $R_n(I) = \{a_0 + a_1 I + \dots + a_n I_n; a_i \in R\}$ to be n -refined neutrosophic ring.

Addition and multiplication on $R_n(I)$ are defined as:

$$\sum_{i=0}^n x_i I_i + \sum_{i=0}^n y_i I_i = \sum_{i=0}^n (x_i + y_i) I_i, \sum_{i=0}^n x_i I_i \times \sum_{i=0}^n y_i I_i = \sum_{i,j=0}^n (x_i \times y_j) I_i I_j.$$

Where \times is the multiplication defined on the ring R .

Definition 4. [10]

Let R be any ring, x be an arbitrary element, then.

a). x is called a unit if there exists $y \in R$, such that $xy = yx = 1$.

b). x is called an idempotent if and only if $x^2 = x$.

Theorem 5.[10]

Let $R(I)$ a neutrosophic ring, then.

a). $x = a + bI$ is idempotent if and only if $a, a + b$ are idempotents in R .

b). $x = a + bI$ is unit if and only if $a, a + b$ units in R .

Theorem 6.[10]

Let $R(I_1, I_2)$ be a refined neutrosophic ring, then.

a). $x = (a, bI_1, cI_2)$ is an idempotent if and only if $a, a + c, a + b + c$ are idempotents in R .

b). $x = (a, bI_1, cI_2)$ is a unit if and only if $a, a + c, a + b + c$ are units in R .

Definition 7. [9]

Let R be a ring, then it is called clean if and only if for each $x \in R$, we have $x = e + l$, where e is a unit and l is an idempotent.

Example 8.

$Z_3 = \{0,1,2\}$ is clean, that is because.

$0 = 2 + 1$ (2 is a unit and 1 is an idempotent).

$1 = 1 + 0$ (1 is a unit and 0 is an idempotent).

$2 = 1 + 1$ (1 is a unit and an idempotent too).

3. Main discussion.

Theorem 3.1:

Let R be any ring, $R(I)$ be its corresponding neutrosophic ring.

$R(I)$ is clean if and only if R is clean.

Proof.

If $R(I)$ is clean, then R is clean. That is because $R \subset R(I)$.

Conversely, suppose that R is clean, we must prove that $R(I)$ is clean.

$\forall x = a + bI \in R(I)$, then $a, b, a + b \in R$.

By the assumption, we can write.

$a = e_1 + l_1, b = e_2 + l_2, a + b = e_3 + l_3$; where e_i are units and l_i are idempotents.

Now, we have $x = a + bI = a + (a + b - a)I = (e_1 + l_1) + I[(e_3 + l_3) - (e_1 + l_1)] = (e_1 + (e_3 - e_1)I) + (l_1 + (l_3 - l_1)) = x_1 + x_2$.

x_1 is a unit, that is because $e_1, (e_3 - e_1) + e_1 = e_3$ are units in R . Also, x_2 is an idempotent, that is because $l_1, (l_3 - l_1) + l_1 = l_3$ are idempotents in R .

[See theorem 5].

Thus x is a sum of an idempotent with a unit, hence $R(I)$ is a clean.

Example 3.2:

We have shown that Z_3 is a clean ring.

According to the previous theorem $Z_3(I) = \{a + bI; a, b \in Z_3\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$ is clean.

Remark that $0 = 2 + 1, 1 = 1 + 0, 2 = 1 + 1, I = (2 + 2I) + (1 + 2I), 2I = (2 + 2I) + 1, 1 + I = (1) + (I),$

$1 + 2I = (1 + I) + I, 2 + I = (3) + (I), 2 + 2I = (2 + 2I) + 0$. Where $\{2,1,2 + 2I, 1 + I\}$ are units and $\{1,0,1 + 2I, I\}$ are idempotents.

Theorem 3.3:

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Let R be any ring, $R(I_1, I_2)$ be its corresponding refined neutrosophic ring.

$R(I_1, I_2)$ is clean if and only if R is clean.

Proof.

R is a homomorphic image of $R(I_1, I_2)$, hence if $R(I_1, I_2)$ is clean, then R is clean.

Conversely, assume that R is clean, we must prove that $R(I_1, I_2)$ is clean.

$\forall x = (a, bI_1, cI_2) \in R(I_1, I_2)$, we have.

$a, a + c, a + b + c \in R$, hence $a = e_1 + l_1, a + c = e_2 + l_2, a + b + c = e_3 + l_3$; where e_i are units and l_i are idempotents.

Also, $x = (a, I_1[(a + b + c) - (a + c)], I_2[(a + c) - a]) = (e_1 + l_1, I_1[(e_3 + l_3) - (e_2 + l_2)], I_2[(e_2 + l_2) - (e_1 + l_1)]) = (e_1, [e_3 - e_2]I_1, [e_2 - e_1]I_2) + (l_1, [l_3 - l_2]I_1, [l_2 - l_1]I_2) = x_1 + x_2$.

By using theorem 6, we get.

x_1 is a unit, that is because $e_1, (e_2 - e_1) + e_1 = e_2, e_1 + (e_3 - e_2) + (e_2 - e_1) = e_3$ are units in R . Also, x_2 is an idempotent by a similar discussion.

This implies that $R(I_1, I_2)$ is clean.

Example 3.4:

$Z_2 = \{0, 1\}$ is clean ring, where $1 = 1 + 0, 0 = 1 + 1$.

$Z_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$ is clean too.

Remark that: $(0, 0, 0) = (1, 0, 0) + (1, 0, 0)$.

$$(1, 0, 0) = (1, 0, 0) + (0, 0, 0)$$

$$(0, I_1, 0) = (1, 0, 0) + (1, I_1, 0)$$

$$(0, 0, I_2) = (1, 0, 0) + (1, 0, I_2)$$

The rest can be computed by the same way.

Theorem 3.5:

Let R be any ring, $R_n(I)$ be its corresponding n -refined neutrosophic ring. Then $R_n(I)$ is clean if and only if R is clean.

Proof.

If $R_n(I)$ is clean, then R is clean as a direct result of the homomorphism between $R_n(I)$ and R .

For the converse, suppose that R is clean. We must prove that $R_n(I)$ is clean.

$$\begin{aligned} \forall x \in R_n(I); x &= a_0 + a_1I_1 + a_2I_2 + \dots + a_nI_n \\ &= a_0 + I_1 \left[\left(\sum_{i=0}^n a_i - \sum_{i \neq 1}^n a_i \right) \right] + I_1 \left[\left(\sum_{i \neq 1}^n a_i - \sum_{i \neq 1, 2}^n a_i \right) \right] + \dots \\ &\quad + [(a_{n-1} + a_n + a_0) - (a_{n-1} + a_0)]I_{n-1} + [(a_0 + a_n) - a_0]I_n \end{aligned}$$

$$x = e_0 + l_0 + I_1[(e_1 + l_1) - (e_2 + l_2)] + I_2[(e_2 + l_2) - (e_3 + l_3)] + \dots \\ + I_{n-1}[(e_{n-1} + l_{n-1}) - (e_n + l_n)] \\ + I_n[(e_n + l_n) - (e_0 + l_0)]; e_i \text{ are units, } l_i \text{ are idempotents.}$$

$$x = [e_0 + I_1[e_1 - e_2] + I_2[e_2 - e_3] + \dots + I_{n-1}[e_{n-1} - e_n] + I_n[e_n - e_0]] + [l_0 + I_1[l_1 - l_2] + \\ I_2[l_2 - l_3] + \dots + I_{n-1}[l_{n-1} - l_n] + I_n[l_n - l_0]] = x_1 + x_2$$

According to theorem, x_1 is a unit, that is because $e_0, e_0(e_1 - e_2) + \dots + (e_n - e_0) = e_1, e_0 + (e_2 - e_3) + (e_3 - e_4) + \dots + (e_{n-1} - e_n) + (e_n - e_0) = e_2$ are units in R .

By the same method, we get that x_2 is an idempotent, hence $R_n(I)$ is clean.

4. Conclusion

In this article, we have determined necessary and sufficient condition for a neutrosophic ring to be clean. Also, we have found the equivalence between classical clean ring R and the corresponding neutrosophic ring $R(I)$, refined neutrosophic ring $R(I_1, I_2)$, and n-refined neutrosophic ring $R_n(I)$.

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