A Normalized Weighted Bonferroni Mean Aggregation Operator in Neutrosophic Vague Multi-Criteria Decision-Making

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Abstract

Decision-making problems involve uncertain and incomplete information, which can be well represented by the Neutrosophic set (NS). Various extensions of NS are available in the literature for solving such problems. However, the published extensions of NS have some restrictions such as single based membership degree. Neutrosophic vague set (NVS) is a newly developed theory to address the shortcomings of previous set theory. NVS is structured based on interval membership in the context of dependent membership functions. Beside uncertainty information, aggregation operators (AOs) are critical components in real-world decision-making issues. As a generalization to the conventional aggregation functions defined on the set of real numbers, numerous AOs have been presented in the literature. Each AO provides a distinct purpose in effectively resolving decision-making problems. Recently, Bonferroni meant (BM) operator has received great attention among scholars because of its ability to consider interrelationship among criteria available in decision-making problems. Based on the advantages of the NV and BM operator, we would like to fill in the gaps by developing a Neutrosophic vague normalized weighted Bonferroni mean (NV-NWBM). In addition, five mathematical properties related to proposed AO are also examined. Besides that, a three-phase decision-making framework is presented to clarify that the proposed AO can be applied to real world decision-making issues. The NV-NWBM operator along with decision-making framework is applied to the example of investment selection under NV environment. The finding shows a computer company is the best alternative for investment. Finally, influence of parameter is performed to validate the effect of parameter towards ranking order.

Keywords: Neutrosophic Vague; Aggregation Operator; Bonferroni mean.

1. Introduction

Neutrosophic sets (NSs) have been the subject of research since their inception in 1998. Its theories have sparked considerable interest in the field of decision-making analysis. NS is proposed by Smarandache [1] and can be identified by three independently function that express the truth, indeterminacy and falsity membership degree. One of the main obstacles of NSs is its inability to be applied to real life application including science and engineering problems. Thus, Wang et al. [2] proposed single-valued neutrosophic set (SVNS) in form of real standard interval. Researchers have integrated NS with various set theories to improve its efficiency such as Pythagorean neutrosophic set

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(PNS) [3], complex neutrosophic set (CNS) [4], farmatean neutrosphic sets (FNSs) [5], bipolar neutrosophic set (BNS) [6] and just to name a few. Despite numerous notions have been extended under NS, however there is insufficient strength in describing the uncertainty and incompleteness of information. Thus, in the context of dependent and independent function AlKhazaleh [7] introduced neutrosophic vague sets (NVSs). It consists of truth, indeterminacy, and falsity membership degree of each element. Particularly, the notion of vague set (VS) is used in generalizing NVS where the truth and falsity are dependent function while indeterminacy is assigned as independent function. It is designed to handle uncertainty and incompleteness of information. Due to its importance, several scholars have made their efforts to enrich the concept of NVS in the decision-making process and some theories are presented such as NV incidence graph [8], NV binary set [9] and NV distance measure [10] and so on. However, there has been limited research in information aggregation under NVS.

In multi-criteria decision-making (MCDM), the choice of aggregation operators (AOs) can significantly influence the final decision. Basically, different AOs perform distinct functions and have different effects towards ranking order. The common AO used in decision-making theme are arithmetic mean (AM) and geometric mean (GM). The shortcoming of these operators is that they do not account for the weight of criteria in the aggregating process. The AO was then improved by several researchers by introducing weighted average (WA) based on algebraic product and algebraic sum. Research on WA and weighted geometric (WG) can be found in works by Dong and Wong [11], Chiclana [12]. The research to date has tended to enhance the efficiency of AO in solving complicated decision-making problems involving multi criteria, multi decision-makers and different weight of clusters. Based on published literature there are several acknowledged AOs such as Power average (PA) AO, Choquet integral AO, Heronian mean (HM) AO, Hybrid AO, Hamacher AO, and many more. Based on the above review, the existing AOs are unsuitable to aggregate the neutrosophic vague information on the basis to capture the interrelationship of input arguments.

Another well-known AO is Bonferroni mean (BM) proposed by Bonferroni [13]. It is designed to capture the interrelationship of two input data in aggregating process. The introduction of NS and its combination has heightened interest in the study of BM operator. Based on recent literature, there are various BM have been combined with NS thus far. This includes the study by Liu et al. [14]. The authors proposed multi-valued neutrosophic set with BM and solving investment selection. On the other hand, Ajay et al. [15] have successfully integrated neutrosophic cubic fuzzy set with geometric Bonferroni mean operator and solving real life problems. On the other hand, Awang et al. [16] proposed an integrated Shapley fuzzy measure and normalized weighted Bonferroni mean under hesitant bipolar-valued neutrosophic environment. In recent literature, Nagarajan et al. [17] proposed two aggregation operators namely trapezoidal and triangular neutrosophic Bonferroni mean (TITRNBM) and the trapezoidal and triangular neutrosophic weighted Bonferroni mean (TITRNWBM) operator. Yet, until now there have been less research on aggregating NV information. Nevertheless, the existing AO cannot be used to aggregate NV information. Thus, in this study, we want to develop normalized weighted Bonferroni mean operator to process NV information. The normalized weighted Bonferroni mean is an enhanced version of BM that overcomes the limitations in classical BM.

The following are the key contributions in the neutrosophic field:

1. In the form of NVs, new operational rules like addition, multiplication, scalar multiplication, and power are defined. They serve as the foundation for extending the BM AO to NVs and proving significant properties.
2. A decision-making method are constructed based on proposed NV-NWBM AO to solve the investment decision. The algebraic operational in the proposed operator has a good function in terms of providing a unique optimal solution.
3. The proposed method considers the two-input data to have a relationship.
4. The selection of investment possibilities is investigated in order to validate the feasibility of our proposed method. Furthermore, sensitivity analysis is used to highlight the suggested method's strengths.

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The objectives of this research are listed as follows:

1. To introduce operational laws for NVS.
2. To develop aggregation operator (AO) for neutrosphic vague normalized weighted Bonferroni mean (NV-NWBM) by combining neutrosophic vague and Bonferroni mean.
3. To develop an algorithm that used proposed NV-NWBM operators in solving MCDM problems.
4. To present a numerical example explaining the algorithm's effectiveness.

The study is structured as follows: Section 2 presents the definition of NVS, and score functions. Subsequently, the basic concept and desirable properties related to BM, weighted Bonferroni mean (WBM) and normalized weighted Bonferroni mean (NWBM) are discussed. Section 3 presents the operational laws of INVS. In Section 4, we develop neutrosophic vague normalized weighted Bonferroni mean (NV-NWBM) and some desirable properties as well as special cases of this operator is discussed. A decision-making method based on NV-NWBM for MCDM problems is presented in Section 5. In Section 6, we give numerical example to illustrate the application of proposed method. Finally, In Section 7 gives concluding remarks and recommendations for future study.

2. Preliminaries

In this section, we present brief definition and basic concept related to the proposed NV-NWBM operator.

2.1 Neutrosophic Vague sets

Definition 1 [7] Let X be a universe discourse and the neutrosophic vague set (NVS) denoted as p is defined as:

$$p_{NV} = \left\{ \left\langle x : T_{p_{NV}}(x), I_{p_{NV}}(x), F_{p_{NV}}(x) \right\rangle \middle| x \in X \right\}. \tag{1}$$

The notations $T_{p_{NV}}(x)$, $I_{p_{NV}}(x)$, $F_{p_{NV}}(x)$ denote as truth-membership, indeterminacy-membership, and falsity-membership functions are defined as $T_{p_{NV}}(x) = \left[T^-, T^+\right]$, $I_{p_{NV}}(x) = \left[I^-, I^+\right]$ and $F_{p_{NV}}(x) = \left[F^-, F^+\right]$ where $T^+ = 1 - F^-$, $F^+ = 1 - T^-$, $0 \le T^- + I^- + F^- \le 2$ and $0 \le T^+ + I^+ + F^+ \le 2$.

Definition 2 [7] Let $A = \langle [T_A^-, T_A^+], [I_A^-, I_A^+], [F_A^-, F_A^+] \rangle$ be a NVS, the score function $s(A)$ is defined as:

$$s(A) = \left(\frac{T_A^- + T_A^+}{2} + \left(1 - \left(\frac{I_A^- + I_A^+}{2}\right)(I_A^+)\right) - \left(\frac{F_A^- + F_A^+}{2}\right)(1 - F_A^+) \right) \tag{2}$$

2.2 Bonferroni mean (BM)

The Bonferroni mean (BM) is a well-known aggregation operator (AO) that can take into account correlation among criteria in decision-making problems. The definition of BM is provided as below:

Definition 3 [13] Let $p, q \ge 0$ and $b_i = (i = 1, 2, \ldots, n)$ be a collection of non-negative numbers, then the BM operator is defined as follows:

$$BM^{p.q}(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \ne j}^{n} b_i^p b_j^q \right)^{\frac{1}{p+q}} \tag{3}$$

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Clearly, the BM has satisfied properties of commutativity, idempotency, monotonicity and boundedness. However the BM operator only can consider the interaction between b_i and other data b_j, and do not take into account their own weights of the aggregated arguments. Difference weights such be assigned in the aggregated arguments since the importance of each argument is not same. Therefore, Xu and Yager (2011) improved BM operator by introducing weight vectors of input criteria.

Definition 4 [18] Let $p,q \geq 0$ and $b_i = (i = 1,2,\ldots,n)$ be a collection of non-negative numbers. $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $b_i = (i = 1,2,\ldots,n)$, where w_i represents the importance degree of b_i satisfying $w_i > 0$ and $\sum_{i=1}^{n} 1$. Then WBM is defined as follows:

$$WBM^{p,q}(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} (w_i a_i)^q (w_j b_j)^q \right)^{\frac{1}{p+q}} \tag{4}$$

Obviously, WBM operator has satisfied the desirable properties such as commutativity, boundedness, monotonicity. However, the WBM do not satisfy the idempotency property. Thus, Xu and Yager (2011) introduced the normalized weighted Bonferroni mean (NWBM)

Definition 5 [8] Let $p,q \geq 0$ and $b_i = (i = 1,2,\ldots,n)$ be a collection of non-negative numbers. $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $b_i = (i = 1,2,\ldots,n)$, where w_i represents the importance degree of b_i satisfying $w_i > 0$ and $\sum_{i=1}^{n} 1$. Then WBM is defined as follows:

$$NWBM^{p,q}(b_1, b_2, \ldots, b_n) = \left(\sum_{i,j=1, i \neq j}^{n} \frac{w_i w_j}{(1 - w_i)} b_i^p b_j^q \right)^{\frac{1}{p+q}} \tag{5}$$

In the next sections, these definitions are frequently used in the proposed operator.

3. The operational laws of NVSs

Here, we introduce some new operation over the NVSs. The operation on NVSs such as addition, multiplication, scalar multiplication and power operations are defined as follows:

Definition 6: Let $A = \left\langle [T_1^-, T_1^+][I_1^-, I_1^+][F_1^-, F_1^+] \right\rangle$ and $B = \left\langle [T_2^-, T_2^+][I_2^-, I_2^+][F_2^-, F_2^+] \right\rangle$ be two sets of NVSs where $\lambda > 0$ is the scalar.

(i) Addition

$$A \oplus B = \left\langle [T_1^- + T_2^- - T_1^- T_2^-, T_1^+ + T_2^+ - T_1^+ T_2^+], [I_1^- I_2^-, I_1^+ I_2^+], [F_1^- F_2^-, F_1^+ F_2^+] \right\rangle \tag{6}$$

(ii) Multiplication

$$A \otimes B = \left\langle [T_1^- T_2^-, T_1^+ T_2^+], [I_1^- + I_2^- - I_1^- I_2^-, I_1^+ + I_2^+ - I_1^+ I_2^+], [F_1^- + F_2^- - F_1^- F_2^-, F_1^+ + F_2^+ - F_1^+ F_2^+] \right\rangle \tag{7}$$

(iii) Scalar multiplication

$$\lambda A = \left\langle [1 - (1 - T_1^-)^\lambda, 1 - (1 - T_1^+)^\lambda], [I_1^{-\lambda}, I_1^{+\lambda}], [F_1^{-\lambda}, F_1^{+\lambda}] \right\rangle \tag{8}$$

(iv) Power

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$$A^{\lambda}=\left\langle\left[T_1^{-\lambda},T_1^{+\lambda}\right],\left[1-(1-I_1^-)^{\lambda},1-(1-I_1^+)^{\lambda}\right],\left[1-(1-F_1^-)^{\lambda},1-(1-F_1^+)^{\lambda}\right]\right\rangle \tag{9}$$

Now, we prove the basic properties of operations as follows:

Theorem 1: Let A and B be two NVSs and $\alpha,\alpha_1,\alpha_2>0$ are the scalars, then we have

(i) $A\oplus B=B\oplus A$

(ii) $A\otimes B=B\otimes A$

(iii) $\alpha(A\oplus B)=\alpha A\oplus\alpha B$

(iv) $\alpha_1 A\oplus\alpha_2 A=(\alpha_1\oplus\alpha_2)A$

(v) $A\otimes(B\oplus C)=(A\otimes B)\oplus(A\otimes C)$

(vi) $A^{(\alpha_1+\alpha_2)}=A^{\alpha_1}A^{\alpha_2}$

(vii) $(AB)^{\alpha}=A^{\alpha}b^{\alpha}$

Proof of (i), (ii), (v), (vi) and (vii) are obvious, therefore we omitted.

Proof (iii)

We have

$$\alpha(A\oplus B)=\left[1-\left(1-T_1^--T_2^-+T_1^-T_2^-\right)^{\alpha},1-\left(1-T_1^+-T_2^++T_1^+T_2^+\right)^{\alpha}\right],$$

$$\left[\left(I_1^-I_2^-\right)^{\alpha},\left(I_1^+I_2^+\right)^{\alpha}\right],\left[\left(F_1^-F_2^-\right)^{\alpha},\left(F_1^+F_2^+\right)^{\alpha}\right]$$

$$=\left[1-\left[\left(1-T_1^-\right)-T_2^-\left(1-T_1^-\right)\right]^{\alpha},1-\left[\left(1-T_1^+\right)-T_2^+\left(1-T_1^+\right)\right]\right],$$

$$\left[\left(I_1^-I_2^-\right)^{\alpha},\left(I_1^+I_2^+\right)^{\alpha}\right],\left[\left(F_1^-F_2^-\right)^{\alpha},\left(F_1^+F_2^+\right)^{\alpha}\right]$$

$$=\left[1-\left[\left(\left(1-T_1^-\right)\left(1-T_2^-\right)\right)^{\alpha}\right],1-\left[\left(\left(1-T_1^+\right)\left(1-T_2^+\right)\right)^{\alpha}\right]\right],$$

$$\left[\left(I_1^-I_2^-\right)^{\alpha},\left(I_1^+I_2^+\right)^{\alpha}\right],\left[\left(F_1^-F_2^-\right)^{\alpha},\left(F_1^+F_2^+\right)^{\alpha}\right]$$

Now,

$$\alpha A\oplus\alpha B=\left[1-\left(1-T_1^-\right)^{\alpha},1-\left(1-T_1^+\right)^{\alpha}\right],\left[I_1^{-\alpha},I_1^{+\alpha}\right],\left[F_1^{-\alpha},F_1^{+\alpha}\right]+$$

$$\left[1-\left(1-T_2^-\right)^{\alpha},1-\left(1-T_2^+\right)^{\alpha}\right],\left[I_2^{-\alpha},I_2^{+\alpha}\right],\left[F_2^{-\alpha},F_2^{+\alpha}\right]$$

$$=\left[1-\left(1-T_1^-\right)^{\alpha}+1-\left(1-T_2^-\right)^{\alpha}-\left(1-\left(1-T_1^-\right)^{\alpha}\right)\left(1-\left(1-T_2^-\right)^{\alpha}\right),1-\left(1-T_1^+\right)^{\alpha}+1-\left(1-T_2^+\right)^{\alpha}\right.$$

$$\left.-\left(1-\left(1-T_1^+\right)^{\alpha}\right)\left(1-\left(1-T_2^+\right)^{\alpha}\right)\right],\left[I_1^{-\alpha}I_2^{-\alpha},I_1^{+\alpha}I_2^{+\alpha}\right],\left[F_1^{-\alpha}F_2^{-\alpha},F_1^{+\alpha}F_2^{+\alpha}\right]$$

It is proven for I and T membership functions as follows:

$\left[I_1^{-\alpha}I_2^{-\alpha},I_1^{+\alpha}I_2^{+\alpha}\right]=\left[\left(I_1^-I_2^-\right)^{\alpha},\left(I_1^+I_2^+\right)^{\alpha}\right]$ and $\left[F_1^{-\alpha}F_2^{-\alpha},F_1^{+\alpha}F_2^{+\alpha}\right]=\left[\left(F_1^-F_2^-\right)^{\alpha},\left(F_1^+F_2^+\right)^{\alpha}\right]$.

While for truth membership functions:

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$$\left[1-\left(1-T_1^-\right)^\alpha+1-\left(1-T_2^-\right)^\alpha-\left(1-\left(1-T_1^-\right)^\alpha\right)\left(1-\left(1-T_2^-\right)^\alpha\right),\right.$$

$$\left.1-\left(1-T_1^+\right)^\alpha+1-\left(1-T_2^+\right)^\alpha-\left(1-\left(1-T_1^+\right)^\alpha\right)\left(1-\left(1-T_2^+\right)^\alpha\right)\right],$$

$$=\left[2-\left(1-T_1^-\right)^\alpha-\left(1-T_2^-\right)^\alpha-\left(1-\left(1-T_2^-\right)^\alpha-\left(1-T_1^-\right)^\alpha+\left(1-T_1^-\right)^\alpha\left(1-T_2^-\right)^\alpha\right),\right.$$

$$\left.2-\left(1-T_1^+\right)^\alpha-\left(1-T_2^+\right)^\alpha-\left(1-\left(1-T_2^+\right)^\alpha-\left(1-T_1^+\right)^\alpha+\left(1-T_1^+\right)^\alpha\left(1-T_2^+\right)^\alpha\right)\right]$$

$$=\left[2-\left(1-T_1^-\right)^\alpha-\left(1-T_2^-\right)^\alpha-1+\left(1-T_2^-\right)^\alpha+\left(1-T_1^-\right)^\alpha-\left(1-T_1^-\right)^\alpha\left(1-T_2^-\right)^\alpha,\right.$$

$$\left.2-\left(1-T_1^+\right)^\alpha-\left(1-T_2^+\right)^\alpha-1+\left(1-T_2^+\right)^\alpha+\left(1-T_1^+\right)^\alpha-\left(1-T_1^+\right)^\alpha\left(1-T_2^+\right)^\alpha\right]$$

$$=\left[1-\left(1-T_1^-\right)^\alpha\left(1-T_2^-\right)^\alpha,1-\left(1-T_1^+\right)^\alpha\left(1-T_2^+\right)^\alpha\right]$$

$$=\left[1-\left(\left(1-T_1^-\right)\left(1-T_2^-\right)\right)^\alpha,1-\left(\left(1-T_1^+\right)\left(1-T_2^+\right)\right)^\alpha\right]$$

Therefore, we prove that $\alpha\left(A\oplus B\right)=\alpha A\oplus\alpha B$.

Proof (iv)

We have,

$$\left(\alpha_1+\alpha_2\right)A=\left[1-\left(1-T_1^-\right)^{(\alpha_1+\alpha_2)},1-\left(1-T_1^+\right)^{(\alpha_1+\alpha_2)}\right],\left[I_1^{-(\alpha_1+\alpha_2)},I_1^{-(\alpha_1+\alpha_2)}\right],\left[F_1^{-(\alpha_1+\alpha_2)},F_1^{+(\alpha_1+\alpha_2)}\right]$$

Then we have

$$\alpha_1 A\oplus\alpha_2 A=\left[1-\left(1-T_1^-\right)^{\alpha_1},1-\left(1-T_1^+\right)^{\alpha_1}\right],\left[I_1^{-\alpha_1},I_1^{+\alpha_1}\right],\left[F_1^{-\alpha_1},I_1^{+\alpha_1}\right]\oplus$$

$$\left[1-\left(1-T_1^-\right)^{\alpha_2},1-\left(1-T_1^+\right)^{\alpha_2}\right],\left[I_1^{-\alpha_2},I_1^{+\alpha_2}\right],\left[F_1^{-\alpha_2},I_1^{+\alpha_2}\right]$$

$$=\left[\left(1-\left(1-T_1^-\right)^{\alpha_1}\oplus1-\left(1-T_1^-\right)^{\alpha_2}-\left(1-\left(1-T_1^-\right)^{\alpha_1}\right)\left(1-\left(1-T_1^-\right)^{\alpha_2}\right)\right),\left(1-\left(1-T_1^+\right)^{\alpha_1}\oplus1-\left(1-T_1^+\right)^{\alpha_2}\right.\right.$$

$$\left.\left.-\left(1-\left(1-T_1^+\right)^{\alpha_1}\right)\left(1-\left(1-T_1^+\right)^{\alpha_2}\right)\right)\right],\left[\left(I_1^{-\alpha_1}\right)\left(I_1^{-\alpha_2}\right),\left(I_1^{+\alpha_1}\right)\left(I_1^{+\alpha_2}\right)\right],\left[\left(F_1^{-\alpha_1}\right)\left(F_1^{-\alpha_2}\right),\left(F_1^{+\alpha_1}\right)\left(F_1^{+\alpha_2}\right)\right]\Big]$$

For I and F we get

$$\left[\left(I_1^{-\alpha_1}\right)\left(I_1^{-\alpha_2}\right),\left(I_1^{+\alpha_1}\right)\left(I_1^{+\alpha_2}\right)\right]=\left[I_1^{-(\alpha_1+\alpha_2)},I_1^{+(\alpha_1+\alpha_2)}\right]\text{ and}$$

$$\left[\left(F_1^{-\alpha_1}\right)\left(F_1^{-\alpha_2}\right),\left(F_1^{+\alpha_1}\right)\left(F_1^{+\alpha_2}\right)\right]=\left[F_1^{-(\alpha_1+\alpha_2)},F_1^{+(\alpha_1+\alpha_2)}\right]$$

Then for T terms, the proof is as follows:

$$=\left[\left(1-\left(1-T_1^-\right)^{\alpha_1}+1-\left(1-T_1^-\right)^{\alpha_2}-\left(1-\left(1-T_1^-\right)^{\alpha_1}\right)\left(1-\left(1-T_1^-\right)^{\alpha_2}\right)\right),\left(1-\left(1-T_1^+\right)^{\alpha_1}+1-\left(1-T_1^+\right)^{\alpha_2}\right.\right.$$

$$\left.\left.-\left(1-\left(1-T_1^+\right)^{\alpha_1}\right)\left(1-\left(1-T_1^+\right)^{\alpha_2}\right)\right)\right]$$

$$=\left[\left(2-\left(1-T_1^-\right)^{\alpha_1}-\left(1-T_1^-\right)^{\alpha_2}-\left(1-\left(1-T_1^-\right)^{\alpha_1}\right)-\left(1-T_1^-\right)^{\alpha_2}-\left(\left(1-T_1^-\right)^{\alpha_1}\right)\left(\left(1-T_1^-\right)^{\alpha_2}\right)\right),\right.$$

$$\left.\left(2-\left(1-T_1^+\right)^{\alpha_1}-\left(1-T_1^+\right)^{\alpha_2}-\left(1-\left(1-T_1^+\right)^{\alpha_1}\right)-\left(1-T_1^+\right)^{\alpha_2}-\left(\left(1-T_1^+\right)^{\alpha_1}\right)\left(\left(1-T_1^+\right)^{\alpha_2}\right)\right)\right]$$

$$=\left[1-\left(1-T_1^-\right)^{\alpha_1+\alpha_2},1-\left(1-T_1^+\right)^{\alpha_1+\alpha_2}\right]=\left[1-\left(1-T_1^-\right)^{(\alpha_1+\alpha_2)},1-\left(1-T_1^+\right)^{(\alpha_1+\alpha_2)}\right]$$

Therefore, we prove that $\alpha_1 A\oplus\alpha_2 A=\left(\alpha_1\oplus\alpha_2\right)A$

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4. Proposed Neutrosophic Vague Normalized Weighted Bonferroni Mean (NV-NWBM)

In this section, we will propose the NV-NWBM operator which is a combination of Bonferroni mean (BM) and neutrosophic vague set (NVS). The enhanced version of BM operator namely normalized weighted BM (NWBM) is used to solve the previous limitation in the classical BM. Besides that, different properties of NVNWBM are also investigated. The definition of NV-NWBM is presented as below:

Definition 7: Let $p,q \geq 0$ and $A_i = \left\langle \left[T_i^-, T_i^+\right]\left[I_i^-, I_i^+\right]\left[F_i^-, F_i^+\right]\right\rangle$ $(i = 1,2,\ldots,n)$ be a collection of NVS. If

$$NVBM^{p.q}\left(A_1, A_2, \ldots, A_n\right) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} A_i^p \otimes A_j^q\right)^{\frac{1}{p+q}} \tag{10}$$

Then, $NVBM^{p.q}$ is called neutrosophic vague Bonferroni mean (NVBM).

In the following, we define the NV-NWBM operator as follows:

Definition 8: Let $p,q \geq 0$ and $A_i = \left\langle \left[T_i^-, T_i^+\right]\left[I_i^-, I_i^+\right]\left[F_i^-, F_i^+\right]\right\rangle$ $(i = 1,2,\ldots,n)$ be a collection of NVS. If

$$NV-NWBM^{p.q}\left(A_1, A_2, \ldots, A_n\right) = \left(\sum_{i,j=1, i\neq j}^{n} \frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)\right)^{\frac{1}{p+q}} \tag{11}$$

$w = \left(w_1, w_2, \ldots, w_n\right)^T$ is the weight vector of $A_i = (i = 1,2,\ldots,n)$ satisfying $w_j \in [0,1]$ and $\sum_{j=1}^{n} 1$. Then $NVNWBM^{p.q}$ is called neutrosophic vague normalized weighted Bonferroni mean (NVNWBM).

Theorem 2:
Let $p,q \geq 0$ and $A_i = \left\langle \left[T_i^-, T_i^+\right]\left[I_i^-, I_i^+\right]\left[F_i^-, F_i^+\right]\right\rangle$ $A_i = (i = 1,2,\ldots,n)$ be a collection of the NVSs, then the aggregated values using NV-NWBM operator in (12) is also NVSs and

Proof: By the operational rules of NVSs defined in Equation (9), so we have

$$A_i^p = \left\langle \left[T_i^{-p}, T_i^{+p}\right]\left[1-\left(1-I_i^-\right)^p, 1-\left(1-I_i^+\right)^p\right]\left[1-\left(1-F_i^-\right)^p, 1-\left(1-F_i^+\right)^p\right]\right\rangle$$
$$A_i^q = \left\langle \left[T_i^{-q}, T_i^{+q}\right]\left[1-\left(1-I_i^-\right)^q, 1-\left(1-I_i^+\right)^q\right]\left[1-\left(1-F_i^-\right)^q, 1-\left(1-F_i^+\right)^q\right]\right\rangle$$

Next, by using Equation (7) we have

$$A_i^p \otimes A_j^q = \left\langle \left[T_i^{-p} T_j^{-q}, T_i^{+p} T_j^{+q}\right], \left[1-\left(1-I_i^-\right)^p\left(1-I_j^-\right)^q, 1-\left(1-I_i^+\right)^p\left(1-I_j^+\right)^q\right], \right.$$
$$\left. \left[1-\left(1-F_i^-\right)^p\left(1-F_j^-\right)^q, 1-\left(1-F_i^+\right)^p\left(1-F_j^+\right)^q\right]\right\rangle$$

Then, using scalar multiplication of NVS in Equation (8), we obtain

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$$\frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)=$$

$$\left\langle \left[1-\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_iw_j}{1-w_i}}, 1-\left(1-T_i^{+p}T_j^{+q}\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\left(1-\left(1-I_i^{-}\right)^p\left(1-I_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \left(1-\left(1-I_i^{+}\right)^p\left(1-I_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\left(1-\left(1-F_i^{-}\right)^p\left(1-F_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \left(1-\left(1-F_i^{+}\right)^p\left(1-F_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right]\right\rangle$$

$$\Rightarrow \oplus_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)=$$

$$\left\langle \left[1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_iw_j}{1-w_i}}, 1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{+p}T_j^{+q}\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{+}\right)^p\left(1-I_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{-}\right)^p\left(1-F_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{+}\right)^p\left(1-F_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right]\right\rangle$$

$$\Rightarrow \oplus_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)=$$

$$\left\langle \left[1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_iw_j}{1-w_i}}, 1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{+p}T_j^{+q}\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{+}\right)^p\left(1-I_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right], \left[\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{-}\right)^p\left(1-F_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}, \prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{+}\right)^p\left(1-F_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right]\right\rangle$$

Therefore

$$NV-NWBM^{p,q}\left(A_1,A_2,\ldots,A_n\right)=\oplus_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)^{\frac{1}{p+q}}=$$

$$\left\langle \left[\left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, \left(1-\prod_{i,j=1,i\neq j}^{n}\left(1-T_i^{+p}T_j^{+q}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right], \left[\left(\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, \left(\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-I_i^{+}\right)^p\left(1-I_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right], \left[\left(\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{-}\right)^p\left(1-F_j^{-}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}, \left(\prod_{i,j=1,i\neq j}^{n}\left(1-\left(1-F_i^{+}\right)^p\left(1-F_j^{+}\right)^q\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}\right]\right\rangle \quad (12)$$

Following that, we will investigate the properties of NVNWBM operator that are essential when developing any aggregation operator (AO).

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Theorem 3: Reducibility

Let $w=\left(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)^T$, then $NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)=NVBM^{p,q}\left(A_1,A_2,\dots,A_n\right)$

Proof: Since $w=\left(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)^T$, then according to Equation (11) we have

$$NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}A_i^p\otimes A_j^q\right)^{\frac{1}{p+q}}$$

$$=\left(\sum_{i,j=1,i\neq j}^{n}\frac{1}{n(n-1)}A_i^p\otimes A_j^q\right)^{\frac{1}{p+q}}=\left(\frac{1}{n(n-1)}\sum_{i,j=1,i\neq j}^{n}A_i^p\otimes A_j^q\right)^{\frac{1}{p+q}}=NVBM^{p,q}\left(A_1,A_2,\dots,A_n\right)$$

which completes the proof of Theorem 2.

Theorem 4: Commutativity

Let $(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_3)$ be any permutation of $(A_1,A_2,\dots,A_3)$, then $NVNWBM^{p,q}\left(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n\right)=NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)$

Proof: Since $(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_3)$ be any permutation of $(A_1,A_2,\dots,A_3)$, then $NVNWBM^{p,q}\left(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n\right)=NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)$

$$\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}A_i^p\otimes A_j^q\right)^{\frac{1}{p+q}}=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}\tilde{A}_i^p\otimes\tilde{A}_j^q\right)^{\frac{1}{p+q}}$$

Thus, $NVNWBM^{p,q}\left(\tilde{A}_1,\tilde{A}_2,\dots,\tilde{A}_n\right)=NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)$.

Theorem 5: Idempotency

Let $A_i=\left(i=1,2,\dots,n\right)$ then $NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)=A$

Proof: Since $A_i=A$ for all i, we have $NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}\left(A_i^p\otimes A_j^q\right)\right)^{\frac{1}{p+q}}$

$$=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}\left(A^p\otimes A^q\right)\right)^{\frac{1}{p+q}}==\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}\left(A^{p+q}\right)\right)^{\frac{1}{p+q}}=A\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_iw_j}{1-w_i}\right)^{\frac{1}{p+q}}=A$$

Theorem 6: Monotonicity

Let $A_i=\left\langle\left[T_i^-,T_i^+\right]\left[I_i^-,I_i^+\right]\left[F_i^-,F_i^+\right]\right\rangle\left(i=1,2,\dots,n\right)$ and $B_i=\left\langle\left[\tilde{T}_i^-,\tilde{T}_i^+\right]\left[\tilde{I}_i^-,\tilde{I}_i^+\right]\left[\tilde{F}_i^-,\tilde{F}_i^+\right]\right\rangle$ $\left(i=1,2,\dots,n\right)$ be any two collections of NVSs. If $T_i^-\geq\tilde{T}_i^-$, $T_i^+\geq\tilde{T}_i^+$, $I_i^-\leq\tilde{I}_i^-$, $I_i^+\leq\tilde{I}_i^+$, $F_i^-\leq\tilde{F}_i^-$ and $F_i^+\leq\tilde{F}_i^+$, for all i, then $NVNWBM^{p,q}\left(A_1,A_2,\dots,A_n\right)\geq NVNWBM^{p,q}\left(B_1,B_2,\dots,B_n\right)$.

Proof:

1. For truth-membership part

Since $T_i^-\geq\tilde{T}_i^-$ for all i and $p,q>0$, then we have $T_i^{-p}\geq\tilde{T}_i^{-p}, T_i^{-q}\geq\tilde{T}_i^{-q}$ then

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$$1-T_i^{-p}T_j^{-q} \geq \tilde{T}_i^{-p}\tilde{T}_j^{-q}$$

$$\Rightarrow \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_i w_j}{1-w_\partial}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\tilde{T}_i^{-p}\tilde{T}_j^{-q}\right)^{\frac{w_i w_j}{1-w_\partial}}$$

$$\Rightarrow \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-T_i^{-p}T_j^{-q}\right)^{\frac{w_i w_j}{1-w_\partial}}\right)^{\frac{1}{p+q}} \geq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\tilde{T}_i^{-p}\tilde{T}_j^{-q}\right)^{\frac{w_i w_j}{1-w_\partial}}\right)^{\frac{1}{p+q}}$$

Similarly, for $T_i^{+} \geq \tilde{T}_i^{+}$ can be proved in similar manner and obtained as follows:

$$= \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-T_i^{+p}T_j^{+q}\right)^{\frac{w_i w_j}{1-w_\partial}}\right)^{\frac{1}{p+q}} \geq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\tilde{T}_i^{+p}\tilde{T}_j^{+q}\right)^{\frac{w_i w_j}{1-w_\partial}}\right)^{\frac{1}{p+q}}$$

2. For indeterminacy-membership part

Since $I_i^{-} \leq \tilde{I}_i^{-}$ for all i and $p,q>0$, then we have

$$\left(1-I_i^{-}\right)^p \geq \left(1-\tilde{I}_i^{-}\right)^p, \left(1-I_i^{-}\right)^q \geq \left(1-\tilde{I}_i^{-}\right)^q$$

then

$$\left(1-I_i^{-}\right)^p\left(1-I_i^{-}\right)^q \geq \left(1-\tilde{I}_i^{-}\right)^p\left(1-\tilde{I}_i^{-}\right)^q$$

$$\Rightarrow \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_i^{-}\right)^q\right)^{\frac{w_i w_j}{1-w_i}} \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\tilde{I}_i^{-}\right)^p\left(1-\tilde{I}_i^{-}\right)\right)^{q\frac{w_i w_j}{1-w_i}}$$

$$\Rightarrow \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_i^{-}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} \geq \left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\tilde{I}_i^{-}\right)^p\left(1-\tilde{I}_i^{-}\right)\right)^{q\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

$$\Rightarrow 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-I_i^{-}\right)^p\left(1-I_i^{-}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} \leq 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\tilde{I}_i^{-}\right)^p\left(1-\tilde{I}_i^{-}\right)\right)^{q\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

Similarly, for $I_i^{+} \leq \tilde{I}_i^{+}$ can be proved in similar manner and obtained as follows:

3. For Falsity-membership part

With similar proving, we have

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-F_i^{-}\right)^p\left(1-F_i^{-}\right)^q\right)^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} \leq 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\tilde{F}_i^{-}\right)^p\left(1-\tilde{F}_i^{-}\right)\right)^{q\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}$$

Similarly, for $F_i^{+} \leq \tilde{F}_i^{+}$ can be proved in similar manner.

Theorem 7: Boundedness

Let $A_i = \left\langle\left[T_i^{-},T_i^{+}\right]\left[I_i^{-},I_i^{+}\right]\left[F_i^{-},F_i^{+}\right]\right\rangle$ $(i=1,2,\ldots,n)$ be a collection of NVS and $A^{-} = \left\langle\left[\min_i T_i^{-},\min_i T_i^{+}\right],\left[\max_i I_i^{-},\max_i I_i^{+}\right],\left[\max_i F_i^{-},\max_i F_i^{+}\right]\right\rangle$ and $A^{+} = \left\langle\left[\max_i T_i^{-},\max_i T_i^{+}\right],\right.$ $\left.\left[\min_i I_i^{-},\min_i I_i^{+}\right],\left[\min_i F_i^{-},\min_i F_i^{+}\right]\right\rangle$ then $A^{-} \leq NVNWBM^{p.q}\left(A_1,A_2,\ldots,A_n\right) \leq A^{+}$.

Proof: Since $A_i \geq A^{-}$, the based on Theorem 5 and 6, we obtain

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$$NVNWBM^{p,q}\left(A_1,A_2,\ldots,A_n\right) \geq NVNWBM^{p,q}\left(A^-_1,A^-_2,\ldots,A^-_n\right) = A^-$$

Similarly, we can get $NVNWBM^{p,q}\left(A_1,A_2,\ldots,A_n\right) \leq NVNWBM^{p,q}\left(A^+_1,A^+_2,\ldots,A^+_n\right) = A^+$. Then $A^- \leq NVNWBM^{p,q}\left(A_1,A_2,\ldots,A_n\right) \leq A^+$

In the following, the proposed NVNWBM also have some special cases as shown below

Case 1: If $q=0$, then NVNWBM operator reduces to the

If $q=0$, then NVNWBM operator reduces to the

$$NVNWBM^{p,0}\left(A_1,A_2,\ldots,A_n\right) = \left(\sum_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i^p \otimes A_j^q\right)\right)^{\frac{1}{p+q}} = \left(\sum_{i=1}^{n} w_i A_i^P\right)^{\frac{1}{p}}$$

For example

$$NVNWBM^{p,0}\left(A_1,A_2,\ldots,A_n\right) = \left\langle \left[\left(1-\prod_{i=1}^{n}\left(1-T_i^{-p}\right)^{w_i}\right)^{\frac{1}{p}}, \left(1-\prod_{i=1}^{n}\left(1-T_i^{+p}\right)^{w_i}\right)^{\frac{1}{p}}\right], \right.$$

$$\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^-\right)^p\right)^{w_i}\right)^{\frac{1}{p}}, \left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-I_i^+\right)^p\right)^{w_i}\right)^{\frac{1}{p}}\right],$$

$$\left.\left[\left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^-\right)^p\right)^{w_i}\right)^{\frac{1}{p}}, \left[1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-F_i^+\right)^p\right)^{w_i}\right)^{\frac{1}{p}}\right]\right]\right\rangle$$

a. If $p=1, q=0$, then NVNWBM reduces to the

$$NVNWBM^{1,0}\left(A_1,A_2,\ldots,A_n\right) = \sum_{i=1}^{n} w_i A_i$$

$$= \left\langle \left[1-\prod_{i=1}^{n}\left(1-T_i^-\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-T_i^+\right)^{w_i}\right], \left[\left(\prod_{i=1}^{n} I_i^-\right)^{w_i}, \left(\prod_{i=1}^{n} I_i^+\right)^{w_i}\right], \right.$$

$$\left.\left[\left(\prod_{i=1}^{n} F_i^-\right)^{w_i}, \left(\prod_{i=1}^{n} F_i^+\right)^{w_i}\right]\right\rangle$$

where NVNWA refer to neutrosophic vague number weighted average aggregation operator

b. If $p=0, q=0$ then

$$NVNWBM^{0,0}\left(A_1,A_2,\ldots,A_n\right) = \sum_{i=1}^{n} w_i A_i$$

$$= \left\langle \left[\left[\left(\prod_{i=1}^{n} T_i^-\right)^{w_i}, \left[\left(\prod_{i=1}^{n} T_i^+\right)^{w_i}\right], \left[1-\prod_{i=1}^{n}\left(1-I_i^-\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-I_i^+\right)^{w_i}\right], \right.$$

$$\left.\left[1-\prod_{i=1}^{n}\left(1-F_i^-\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-F_i^+\right)^{w_i}\right]\right\rangle$$

where NVNWG refer to neutrosophic vague number weighted geometric aggregation operator.

Case 2: If $p=q$, then NVNWBM operator reduces to the following structure:

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$$NVNWBM^{p,p}\left(A_1,A_2,\ldots,A_n\right)=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i^p\otimes A_j^p\right)\right)^{\frac{1}{p+p}}=\left(\sum_{i,j=1,i\neq j}^{n}\frac{w_i w_j}{1-w_i}\left(A_i\otimes A_j\right)^p\right)^{\frac{1}{2p}}$$

For example

$$NVNWBM^{p,p}\left(A_1,A_2,\ldots,A_n\right)=\left\langle\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^{-p}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^{+p}\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}}\right],\right.$$

$$\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^-\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}},1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-I_i^+\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}}\right],$$

$$\left.\left[1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-F_i^-\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}},1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-F_i^+\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2p}}\right]\right\rangle$$

If $p=q=1$, then

$$NVNWBM^{1,1}\left(A_1,A_2,\ldots,A_n\right)=\left\langle\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^-T_j^-\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-T_i^+T_j^+\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right],\right.$$

$$\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-I_i^-I_j^-\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-I_i^+I_j^+\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right],$$

$$\left.\left[1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-F_i^-F_j^-\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}},1-\left(1-\prod_{\substack{i=1\\i\neq j}}^{n}\left(1-\left(1-F_i^+F_j^+\right)^p\right)^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{2}}\right]\right\rangle$$

where NVNWBM refer to neutrosophic vague normalized weighted Bonferroni mean aggregation operator.

In the following, three phase decision-making framework is developed to solve investment selection using proposed NVNWBM aggregation operator.

5. The Multi-Criteria Decision-Making Method based on NV-NWBM operator

In this step, we develop the decision-making method to apply the proposed NV-NWBM where the input arguments are in form of NVSs. Suppose that $A=\{A_1,A_2,\ldots,A_m\}$ is a set of alternatives $C=\{C_1,C_2,\ldots,C_n\}$ is a set of criteria and decision makers $DM=\{Z_1,Z_2,\ldots,Z_k\}$ is a set of decision-makers. This collective information is represented in NVs decision matrix denoted as $\tilde{A}=\left[a_{ij}\right]_{4x3}$. The following is a description of the decision-making algorithm based on the proposed NV-NWBM operator:

Step 1: Utilize the NV-NWBM operator to aggregate the evaluation of each decision-maker (DM) to get the collective information by the Equation 12.

Step 2: Utilize the NV-NWBM operator to get comprehensive evaluation for each alternative using Equation 12.

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Step 3: Calculate the score values of each alternative $S(a_i) = (i = 1,2,3,4)$ using Equation 2.

Step 4: Rank all the alternatives $\{A_1, A_2, \ldots, A_m\}$ based on result obtained in step 3.

6. Numerical Example

Assume an investment company decides to spend an amount of money in the best option. There is a list of four potential alternatives for investing the money which is a car company (A_1), a food company (A_2), a computer company (A_3) and an arms company (A_4) evaluated by three decision makers denoted by (Z_1, Z_2, Z_3). The weight of decision makers is $\gamma = (0.314, 0.355, 0.331)^T$. There are three measured criteria that influence the option which are risk analysis (C_1), the growth analysis (C_2) and the environment impact analysis (C_3) and the weight of each criteria is $\omega = (0.4, 0.2, 0.4)^T$. The evaluation information in form of NVSs. According to the evaluation from decision makers, we construct three decision matrices as shown in Table 1-3.

Table 1: Decision matrix Z_1

	C_1	C_2	C_3
A_1	$\langle[0.3,0.5],[0.5,0.5],[0.5,0.7]\rangle$	$\langle[0.4,0.7],[0.6,0.6],[0.3,0.6]\rangle$	$\langle[0.1,0.5],[0.5,0.5],[0.5,0.9]\rangle$
A_2	$\langle[0.3,0.5],[0.7,0.8],[0.5,0.7]\rangle$	$\langle[0.4,0.7],[0.6,0.8],[0.3,0.6]\rangle$	$\langle[0.1,0.5],[0.3,0.6],[0.5,0.9]\rangle$
A_3	$\langle[0.7,0.8],[0.3,0.5],[0.2,0.3]\rangle$	$\langle[0.2,0.4],[0.2,0.4],[0.6,0.8]\rangle$	$\langle[0.9,1],[0.6,0.7],[0,0.1]\rangle$
A_4	$\langle[0.1,0.5],[0.6,0.7],[0.5,0.9]\rangle$	$\langle[0.3,0.5],[0.7,0.8],[0.5,0.7]\rangle$	$\langle[0.2,0.4],[0.6,0.8],[0.6,0.8]\rangle$

Table 2: Decision matrix Z_2

	C_1	C_2	C_3
A_1	$\langle[0.5,0.8],[0.4,0.9],[0.2,0.5]\rangle$	$\langle[0.1,0.4],[0.3,0.7],[0.6,0.9]\rangle$	$\langle[0.4,0.5],[0.5,0.5],[0.5,0.6]\rangle$
A_2	$\langle[0.7,0.8],[0.4,0.7],[0.2,0.3]\rangle$	$\langle[0.3,0.8],[0.4,0.6],[0.2,0.7]\rangle$	$\langle[0.1,0.7],[0.1,0.3],[0.3,0.9]\rangle$
A_3	$\langle[0.5,0.6],[0.5,0.5],[0.4,0.5]\rangle$	$\langle[0.2,0.5],[0.1,0.6],[0.5,0.8]\rangle$	$\langle[0.6,0.9],[0.3,0.7],[0.1,0.4]\rangle$
A_4	$\langle[0.3,0.9],[0.7,0.9],[0.1,0.7]\rangle$	$\langle[0.3,0.3],[0.3,0.6],[0.7,0.8]\rangle$	$\langle[0.2,0.5],[0.4,0.6],[0.5,0.8]\rangle$

Table 3: Decision matrix Z_3

	C_1	C_2	C_3
A_1	$\langle[0.2,0.7],[0.3,0.5],[0.3,0.8]\rangle$	$\langle[0.4,0.7],[0.5,0.7],[0.3,0.6]\rangle$	$\langle[0.3,0.8],[0.2,0.5],[0.2,0.7]\rangle$
A_2	$\langle[0.5,0.6],[0.1,0.9],[0.4,0.5]\rangle$	$\langle[0.2,0.5],[0.2,0.7],[0.5,0.8]\rangle$	$\langle[0.1,0.9],[0.1,0.2],[0.1,0.9]\rangle$
A_3	$\langle[0.4,0.9],[0.4,0.5],[0.1,0.6]\rangle$	$\langle[0.6,0.8],[0.4,0.8],[0.2,0.4]\rangle$	$\langle[0.7,0.8],[0.8,0.9],[0.2,0.3]\rangle$
A_4	$\langle[0.3,0.7],[0.5,0.6],[0.3,0.7]\rangle$	$\langle[0.4,0.5],[0.4,0.9],[0.5,0.6]\rangle$	$\langle[0.8,0.9],[0.7,0.9],[0.1,0.2]\rangle$

6.1 The Evaluation Steps by NV-NWBM Operator

The steps are shown as follows:

Step 1: Aggregate the evaluations of each DM into collective information $\tilde{A} = [a_{ij}]_{4x3}$ using NV-NWBM operator (suppose $(p = q = 1)$.

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$a_{11} = \langle [0.224, 0.6706], [0.3998, 0.6587], [0.3294, 0.6756] \rangle,$
$a_{21} = \langle [0.4965, 0.634], [0.4028, 0.8073], [0.3657, 0.5035] \rangle,$
$a_{31} = \langle [0.5268, 0.7671], [0.4038, 0.5], [0.2329, 0.4732] \rangle,$
$a_{41} = \langle [0.2278, 0.7054], [0.6052, 0.7509], [0.2946, 0.7722] \rangle,$
$a_{12} = \langle [0.2803, 0.5935], [0.4680, 0.6693], [0.4065, 0.7197] \rangle,$
$a_{22} = \langle [0.2943, 0.6677], [0.4010, 0.7025], [0.3323, 0.7057] \rangle,$
$a_{32} = \langle [0.3065, 0.5598], [0.2282, 0.6143], [0.4402, 0.6935] \rangle,$
$a_{42} = \langle [0.2931, 0.4266], [0.4666, 0.7818], [0.5734, 0.7069] \rangle,$
$a_{13} = \langle [0.2578, 0.9545], [0.4055, 0.5], [0.4055, 0.7422] \rangle,$
$a_{23} = \langle [0.1, 0.7037], [0.1583, 0.3622], [0.2963, 0.9] \rangle,$
$a_{33} = \langle [0.7297, 0.9062], [0.5843, 0.7752], [0.0938, 0.2703] \rangle,$
$a_{33} = \langle [0.3486, 0.5888], [0.5721, 0.7818], [0.4112, 0.6514] \rangle$

Step 2: Compute the comprehensive evaluation value of alternative a_i by using NV-NWBM operator (suppose $(p = q = 1)$. We can obtain the results as follows:

$a_1 = \langle [0.2876, 0.6230], [0.4179, 0.6047], [0.3370, 0.7124] \rangle,$
$a_2 = \langle [0.2618, 0.6684], [0.3127, 0.6437], [0.3316, 0.7382] \rangle,$
$a_3 = \langle [0.5492, 0.7780], [0.4341, 0.6449], [0.222, 0.4508] \rangle,$
$a_4 = \langle [0.2861, 0.5952], [0.5607, 0.7703], [0.4048, 0.7139] \rangle$

Step 3: Calculate the score function values of each alternative using Equation 2 based on comprehensive evaluation obtained in step 2, $S(a_i) = (i = 1, 2, 3, 4)$.

$S(a_1) = 0.4664$
$S(a_2) = 0.4834$
$S(a_3) = 0.5959$
$S(a_4) = 0.4053$

Step 4: Ranking the alternatives.
This is the last step in procedure of MCDM. Based on the values of the score function in step 3 $S(a_i) = (i = 1, 2, 3, 4)$. The highest value is $A_3 = 0.5959$ followed by $A_2 = 0.4834$ and $A_1 = 0.4664$ while the lowest value is $A_4 = 0.4053$. Therefore, we can conclude the alternative A_1, A_2, A_3 and A_4 can be rank as $A_3 \succ A_2 \succ A_1 \succ A_4$. Hence, the best alternative is A_3.

6.2 Influence of Parameter

By using similar decision-making example, the two parameters p and q are used to demonstrate the influence of parameter. In order to assist calculation of parameter p and q, Microsoft Excel Spreadsheet command is used as shown in Figure 1. Similar step in 1 and 2 are performed by changing the value of p and q to rank the alternatives. The ranking results is shown in Table 4.

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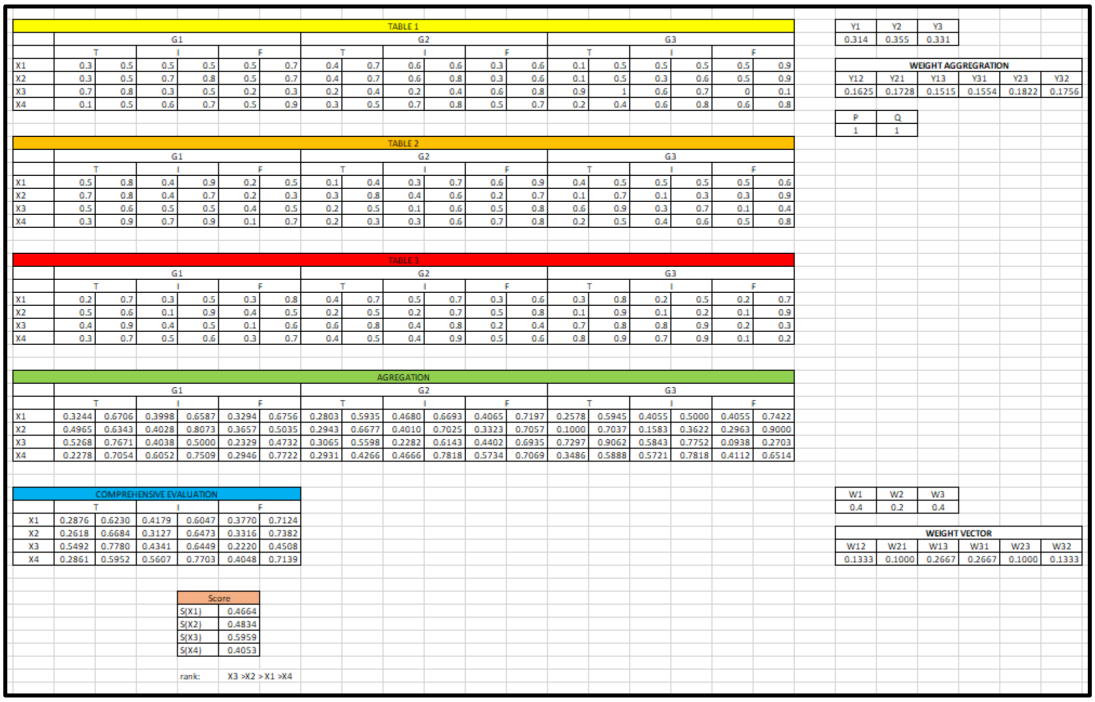

TABLE 1

	G1						G2						G3					
	T		I		F		T		I		F		T		I		F	
X1	0.3	0.5	0.5	0.5	0.5	0.7	0.4	0.7	0.6	0.6	0.3	0.6	0.1	0.5	0.5	0.5	0.5	0.9
X2	0.3	0.5	0.7	0.8	0.5	0.7	0.4	0.7	0.6	0.8	0.3	0.6	0.1	0.5	0.3	0.6	0.5	0.9
X3	0.7	0.8	0.3	0.5	0.2	0.3	0.2	0.4	0.2	0.4	0.6	0.8	0.9	1	0.6	0.7	0	0.1
X4	0.1	0.5	0.6	0.7	0.5	0.9	0.3	0.5	0.7	0.8	0.5	0.7	0.2	0.4	0.6	0.8	0.6	0.8

TABLE 2

	G1						G2						G3					
	T		I		F		T		I		F		T		I		F	
X1	0.5	0.8	0.4	0.9	0.2	0.5	0.1	0.4	0.3	0.7	0.6	0.9	0.4	0.5	0.5	0.5	0.5	0.6
X2	0.7	0.8	0.4	0.7	0.2	0.3	0.3	0.8	0.4	0.6	0.2	0.7	0.1	0.7	0.1	0.3	0.3	0.9
X3	0.5	0.6	0.5	0.5	0.4	0.5	0.2	0.5	0.1	0.6	0.5	0.8	0.6	0.9	0.3	0.7	0.1	0.4
X4	0.3	0.9	0.7	0.9	0.1	0.7	0.2	0.3	0.3	0.6	0.7	0.8	0.2	0.5	0.4	0.6	0.5	0.8

TABLE 3

	G1						G2						G3					
	T		I		F		T		I		F		T		I		F	
X1	0.2	0.7	0.3	0.5	0.3	0.8	0.4	0.7	0.5	0.7	0.3	0.6	0.3	0.8	0.2	0.5	0.2	0.7
X2	0.5	0.6	0.1	0.9	0.4	0.5	0.2	0.5	0.2	0.7	0.5	0.8	0.1	0.9	0.1	0.2	0.1	0.9
X3	0.4	0.9	0.4	0.5	0.1	0.6	0.6	0.8	0.4	0.8	0.2	0.4	0.7	0.8	0.8	0.9	0.2	0.3
X4	0.3	0.7	0.5	0.6	0.3	0.7	0.4	0.5	0.4	0.9	0.5	0.6	0.8	0.9	0.7	0.9	0.1	0.2

AGREGATION

	G1						G2						G3					
	T		I		F		T		I		F		T		I		F	
X1	0.3244	0.6706	0.3998	0.6587	0.3294	0.6756	0.2803	0.5935	0.4680	0.6693	0.4065	0.7197	0.2578	0.5945	0.4055	0.5000	0.4055	0.7422
X2	0.4965	0.6343	0.4028	0.8073	0.3657	0.5035	0.2943	0.6677	0.4010	0.7025	0.3323	0.7057	0.1000	0.7037	0.1583	0.3622	0.2963	0.9000
X3	0.5268	0.7671	0.4038	0.5000	0.2329	0.4732	0.3065	0.5598	0.2282	0.6143	0.4402	0.6935	0.7297	0.9062	0.5843	0.7752	0.0938	0.2703
X4	0.2278	0.7054	0.6052	0.7509	0.2946	0.7722	0.2931	0.4266	0.4666	0.7818	0.5734	0.7069	0.3486	0.5888	0.5721	0.7818	0.4112	0.6514

COMPREHENSIVE EVALUATION

	T		I		F	
X1	0.2876	0.6230	0.4179	0.6047	0.3770	0.7124
X2	0.2618	0.6684	0.3127	0.6473	0.3316	0.7382
X3	0.5492	0.7780	0.4341	0.6449	0.2220	0.4508
X4	0.2861	0.5952	0.5607	0.7703	0.4048	0.7139

Score	
S(X1)	0.4664
S(X2)	0.4834
S(X3)	0.5959
S(X4)	0.4053

rank: X3 >X2 > X1 >X4

Y1	Y2	Y3
0.314	0.355	0.331

WEIGHT AGGREGRATION

Y12	Y21	Y13	Y31	Y23	Y32
0.1625	0.1728	0.1515	0.1554	0.1822	0.1756

P	Q
1	1

W1	W2	W3
0.4	0.2	0.4

WEIGHT VECTOR

W12	W21	W13	W31	W23	W32
0.1333	0.1000	0.2667	0.2667	0.1000	0.1333

Figure 1: Excell command spreadsheet

Table 4: The order of the alternatives in NVWBM operator by different p,q

p,q	Score function $S(a_i)=(i=1,2,3,4)$	Ranking
$p=q=0$	$S(a_1)=0.4496, S(a_2)=0.4596, S(a_3)=0.5689, S(a_4)=0.3859$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=0.001, q=0$	$S(a_1)=0.4819, S(a_2)=0.5264, S(a_3)=0.6961, S(a_4)=0.4433$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=1, q=0$	$S(a_1)=0.4958, S(a_2)=0.5559, S(a_3)=0.7104, S(a_4)=0.4702$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=2, q=0$	$S(a_1)=0.5094, S(a_2)=0.5857, S(a_3)=0.7238, S(a_4)=0.5010$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=5, q=0$	$S(a_1)=0.5413, S(a_2)=0.6500, S(a_3)=0.7554, S(a_4)=0.5781$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$p=10, q=0$	$S(a_1)=0.5741, S(a_2)=0.7948, S(a_3)=0.7891, S(a_4)=0.6420$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$p=0.001, q=1$	$S(a_1)=0.4920, S(a_2)=0.5480, S(a_3)=0.6785, S(a_4)=0.4633$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=0.01, q=1$	$S(a_1)=0.4910, S(a_2)=0.5451, S(a_3)=0.6661, S(a_4)=0.4603$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=1, q=1$	$S(a_1)=0.4664, S(a_2)=0.4834, S(a_3)=0.5959, S(a_4)=0.4053$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=2, q=1$	$S(a_1)=0.4782, S(a_2)=0.5049, S(a_3)=0.6134, S(a_4)=0.4229$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=5, q=1$	$S(a_1)=0.5152, S(a_2)=0.5817, S(a_3)=0.6716, S(a_4)=0.4989$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=10, q=1$	$S(a_1)=0.5542, S(a_2)=0.6563, S(a_3)=0.7300, S(a_4)=0.5833$	$A_3 \succ A_2 \succ A_4 \succ A_1$
$p=0.001, q=1$	$S(a_1)=0.5057, S(a_2)=0.5769, S(a_3)=0.6981, S(a_4)=0.4939$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=0.001, q=2$	$S(a_1)=0.5051, S(a_2)=0.5751, S(a_3)=0.6879, S(a_4)=0.4919$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=1, q=2$	$S(a_1)=0.4770, S(a_2)=0.5017, S(a_3)=0.6124, S(a_4)=0.4216$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=q=2$	$S(a_1)=0.4783, S(a_2)=0.4998, S(a_3)=0.6095, S(a_4)=0.4183$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=5, q=2$	$S(a_1)=0.5046, S(a_2)=0.5533, S(a_3)=0.6497, S(a_4)=0.4665$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$p=10, q=2$	$S(a_1)=0.5417, S(a_2)=0.6257, S(a_3)=0.7085, S(a_4)=0.5466$	$A_3 \succ A_2 \succ A_1 \succ A_4$

Based on Table 4, it can be noticed that the ranking order is consistent with any parameter values p,q where the best alternative is A_3 and the worst alternative A_4 except for parameter $p=5, q=0$,

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$p=10, q=0$ and $p=10, q=1$. However, the best alternative is always A_3 regardless of any changes of parameters. Therefore, it can be concluded that the proposed NV-NWBM can influence the decision results. The overall effect of parameter $p, q \in (0,10]$ using NV-NWBM operator is graphically presented in Figure 2-3.

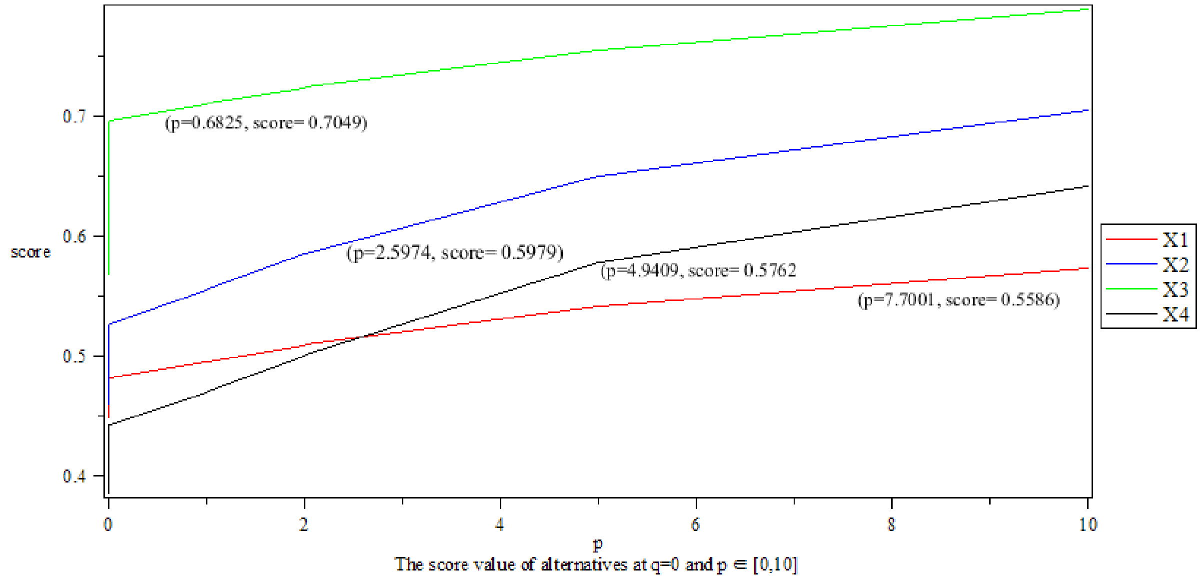

Figure 2: Graph of score function when $q=0$ and $p \in (0,10]$

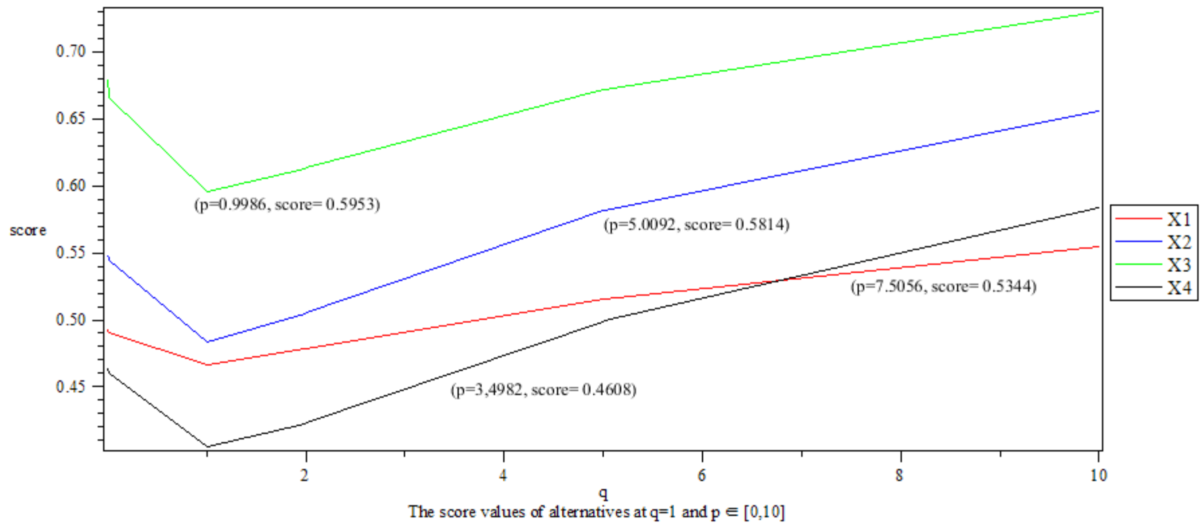

Figure 3: Graph of score function when $q=1$ and $p \in (0,10]$

Based on Figure 2 and 3, we can conclude that the pattern of graph is almost consistent and the best alternative is A_3. Besides that, as the parameter p increased the score values obtained by proposed NV-NWBM operator also increased.

7. Conclusion

In this study, we have developed WNBM operator to aggregate neutrosophic vague information. The proposed operator is satisfying five properties such as reducibility, idempotency, commutativity, monotonicity and boundedness. Some special cases of the operator also investigated in detail. Besides that, to verify the applicability and practicality of the proposed operator, a decision-making method is developed. An illustrative example is presented to verify the developed method based on NV-NWBM operator. Finally, the influence of parameter for different values of p and q is conducted to show the

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influence of p and q towards ranking order of decision-making. The key benefit of the proposed NV-NWBM operator is a practical tool that can take inconsistent and incomplete information into account when examining interrelationship of two input arguments. In future, the proposed NVBM can be extended to other neutrosophic environments such as Pythagorean neutrsophic set and farmatean neutrosophic sets. The proposed decision-making method also can be applied in food security problems and medical diagnosis.

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