Certain Determinants for New Subclasses of $\mu - Fold\ bi$-Univalent Functions

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Abstract

This paper introduces and investigate a new subclasses of the class Σ_μ of analytic functions that both g and $g^{-1}$are μ-fold symmetric bi-univalent functions in the open unit disk Ω and get the estimates of the initial coefficients $|\alpha_{t+1}|, |\alpha_{2t+1}|$ and $|\alpha_{3t+1}|$ for functions in each of these new subclasses. After this, the work will be discussing the Hankel determinant and a Fekete-Szegö functional.

Keywords: Analytic function; univalent function; bi-univalent functions; $\mu - fold$ symmetric bi-univalent functions; Hankel determinant; Fekete-Szegö functional.

1. Introduction and Preliminaries

The motivation for this paper has come from two directions: firstly, from a wish to introduce two new subclasses of the function class $\Sigma_\mu$of bi-univalent μ-fold analytic functions in Ω and get the estimates of the initial coefficients $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for functions in each of these new subclasses. Secondly to discuss the Hankel determinant and a Fekete-Szegö functional for functions belonging in these new subclasses.

Note: In this paper we will represent by $\Omega = \{\gamma \in C: |\gamma| < 1\}$ to the open unit disk.

Now, we recall some background information on analytic functions g in Ω which satisfy normalization condition $g(0) = g'(0) - 1 = 0$, where:

$$g(\gamma) = \gamma + \sum_{n=2}^{\infty} \alpha_n \gamma^n, \tag{1}$$

In addition to investigate some background information on the class Σ of all bi-univalent in Ω given by (1), and some of their properties.

Definition 1.1. Let $\mathcal{H}$denote the family of functions g satisfying (1) which analytic in Ω such that $g(0) = g'(0) - 1 = 0$. Let $\hbar$ be the subclass of $\mathcal{H}$
consisting of functions of the form (1) which are also univalent in Ω. It is well known that every $g \in \hbar$ has an inverse g^{-1}, defined by $g^{-1}(g(\gamma)) = \gamma,\ \ (\gamma \in \Omega)$, and $g^{-1}(g(w)) = w, \left(|w| < k_0(g), k_0(g) \geq \frac{1}{4}\right)$, where

$$g^{-1}(w) = w - \alpha_2 w^2 + (2\alpha_2^2 - \alpha_3)w^3 - (5\alpha_2^3 - 5\alpha_2\alpha_3 + \alpha_4)w^4 + \cdots. \tag{2}$$

Then g, is bi-univalent in Ω, when both g and g^{-1} are univalent in Ω.

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Denoted by Σ, the class of all bi-univalen in Ω, given by equation 1. To indeed further details, and examples of functions in the class Σ, can be found in [24], [3], [11], [12] and [17].

Lemma 1.2. For each $\mathcal{g} \in \mathcal{H}$, satisfy that:

$$\left|\frac{\gamma^2 \mathcal{g}'(\gamma)}{\left(\mathcal{g}(\gamma)\right)^2} - 1\right| < 1, (\gamma \in \Omega)$$

is univalent in Ω, and hence $\mathcal{g} \in \hbar$. The proof can be found in [12].

Definition 1.3. Denoted by $\mathcal{T}(\tau)$, the class containing the functions $\mathcal{g} \in \mathcal{H}$, such that

$$\left|\frac{\gamma^2 \mathcal{g}'(\gamma)}{\left(\mathcal{g}(\gamma)\right)^2} - 1\right| < \tau, \gamma \in \Omega, 0 < \tau \leq 1$$

and $\mathcal{T}(1) = \mathcal{T}$, so we have $\mathcal{T}(\tau) \subset \mathcal{T} \subset \hbar$ from (lemma1.2.). And

$Re\left(\frac{\gamma^2 \mathcal{g}'(\gamma)}{\left(\mathcal{g}(\gamma)\right)^2} - 1\right) > 1 - \tau$. Kuroki et. al. [10].

A function is said to be $\mu - fold$ symmetric see ([15] and [9]), if it has the following normalized form:

$$h(\gamma) = \gamma + \sum_{k=1}^{\infty} \alpha_{\mu k+1} \gamma^{\mu k+1} \quad (\gamma \in \Omega; \mu \in \mathbb{N}). \tag{3}$$

Lemma 1.4. For each function $h \in \hbar$, the function

$$g(\gamma) = \sqrt[\mu]{h(\gamma^{\mu})} \quad (\gamma \in \Omega; \mu \in \mathbb{N}), \tag{4}$$

is univalent and maps the unit disk Ω into a region with $\mu - fold$ symmetric.

Let $\hbar_{\mu}$ be a class of $\mu - fold$ symmetric univalent functions on Ω, which are normalized depending on the series expansion (4). That is, the functions $\hbar$ are 1-fold symmetric. A concept $\mu - fold$ symmetric bi-univalent functions is introduced, in this paper, similarly the concept of $\mu - fold$ symmetric univalent functions. Then for each integer $\mu \in \mathbb{N}$, the bi-univalent function $\mathcal{g}$ of Σ generates an $\mu - fold$ symmetric bi-univalent function. The normalized form of the function $\mathcal{g}$ is given as shown above (4). The next equation represents the series expansion for its inverse $\mathcal{g}^{-1}$ which has been proved by Srivastava et al. [23].

$$k(w) = w - \alpha_{\mu+1} w^{\mu+1} + \left[(\mu+1)\alpha_{\mu+1}^2 - \alpha_{2\mu+1}\right] w^{2\mu+1}$$
$$- \left[\frac{1}{2}(\mu+1)(3\mu+2)\alpha_{\mu+1}^3 - (3\mu+2)\alpha_{\mu+1}\alpha_{2\mu+1} + \alpha_{3\mu+1}\right] w^{3\mu+1} + \cdots, \tag{5}$$

where $h^{-1} = k$. Denoted by Σ_{μ}, the class of $\mu - fold$ symmetric univalent functions in Ω, we note that for $t = 1$, formula (5) coincide with formula (2).

Noonan and Thomas [13], introduced and studied the j^{th}Hankel determinant for $j \geq 1, n \in \mathbb{N}_0$,

$$H_j(n) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\ \alpha_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n+q-1} & \cdots & \cdots & \alpha_{n+2(q-1)} \end{vmatrix}$$

where $\alpha_1 = 1$. Many researchers have studied $H_j(n)$, Specifically, the Hankel determinant $H_2(2)$, is equivalent to $|\alpha_2\alpha_4 - \alpha_3^2|$, were obtained in ([8], [7], and [20]), for many classes. A Fekete-Szegö functional $|\alpha_3 - \alpha_2^2|$, is equivalent to the Hankel determinant $H_2(1)$. Also, the Hankel determinant $H_3(1)$, studied and investigated by Babalola [4].

Some other authors have studied various classes of $\mu - fold$ symmetric bi-univalent functions, such as, see Refs. ([1], [2], [25], [26], [21], [22], [6], [5], [18] and [19]). Such works encouraged us to create the two new subclasses of the function class Σ_{μ} of analytic functions which both h and h^{-1} are $\mu - fold$ symmetric

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bi −univalent functions and we obtain estimates on the initial coefficients $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|, |\alpha_{3\mu+1}|$, Hankel determinant and Fekete-Szegö functional for functions in each of these new subclasses.

for explaining main results, we need the Remark below [26].

Remark 1.5. Let $\phi = \{g\colon\ g\ analytic\ in\ \Omega\ satisfy\ that\ Re\{g(\gamma)\} > 0\ \}$, so, $\forall \ell$, and $g \in \phi$ we have $|t_\ell| \leq 2$, and

$$g(\gamma) = 1 + t_1\gamma + t_2\gamma^2 + t_3\gamma^3 + \cdots, \quad \gamma \in \Omega.$$

2. Estimates on the Initial Coefficients

Definition 2.1. A function $\boldsymbol{\zeta} \in \Sigma_\mu$ given by (3) is said to be in the class $\mathfrak{T}^{\partial}_{\Sigma_\mu}$, if the following conditions are satisfied

$$\left|arg\left(\frac{\gamma^2\zeta'(\gamma)}{(\zeta(\gamma))^2}\right)\right| < \frac{\partial\pi}{2}, \quad (0 < \partial \leq 1, \gamma \in \Omega) \qquad (6)$$

and

$$\left|arg\left(\frac{w^2k'(w)}{(k(w))^2}\right)\right| < \frac{\partial\pi}{2}, \quad (0 < \partial \leq 1, w \in \Omega) \qquad (7)$$

where the function k is the invers of $\boldsymbol{\zeta}$, given by Equation 5. Denoted by $\mathfrak{T}^{\partial}_{\Sigma_\mu}$, the class of all $\mu - fold$ symmetric bi-univalent functions of order ∂.

In our first theorem, we finding the estimates on the initial coefficients $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for function in the class $\mathfrak{T}^{\partial}_{\Sigma_\mu}$.

Theorem 2.2. Let ζ be a function satisfy Eq. (4) in class $\mathfrak{T}^{\partial}_{\Sigma_\mu}$, $0 < \partial \leq 1$. Then

$$|\alpha_{\mu+1}| \leq \frac{2(2\mu-1)(2\partial-1)}{\partial(3\mu-1)},$$

$$|\alpha_{2\mu+1}| \leq \frac{2\partial}{(2\mu-1)},$$

$$|\alpha_{3\mu+1}| \leq \frac{6\partial}{(3\mu-1)}.$$

Proof. it follows from (6) and (7) that

$$\frac{\gamma^2\boldsymbol{\zeta}'(\gamma)}{(\boldsymbol{\zeta}(\gamma))^2} = [r(\gamma)]^{\partial}, \qquad (8)$$

and

$$\frac{w^2k'(w)}{(k(w))^2} = [u(w)]^{\partial}, \qquad (9)$$

where $r(\gamma)\ and\ u(w)$ in $\mathcal{P}$ and have the forms

$$r(\gamma) = 1 + r_\mu\gamma^\mu + r_{2\mu}\gamma^{2\mu} + r_{3\mu}\gamma^{3\mu} + \cdots, \qquad (10)$$

and

$$u(w) = 1 + u_\mu w^\mu + u_{2\mu}w^{2\mu} + u_{3\mu}w^{3\mu} + \cdots. \qquad (11)$$

Cleary, we have

$$[r(\gamma)]^{\partial} = 1 + \partial r_\mu\gamma^\mu + \left[\partial r_{2\mu} + \frac{1}{2}\partial(\partial-1)r_\mu^2\right]\gamma^{2\mu} +$$

$$\left[\partial r_{2\mu} + \partial(\partial-1)r_\mu r_{2\mu} + \frac{1}{6}\partial(\partial-1)(\partial-2)r_\mu^3\right]\gamma^{3\mu} + \cdots,$$

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$$[u(w)]^{\partial}=1+\partial u_{\mu}w^{\mu}+\left[\partial u_{2\mu}+\frac{1}{2}\partial(\partial-1)u_{\mu}^{2}\right]w^{2\mu}+$$

$$\left[\partial u_{2\mu+1}+\partial(\partial-1)u_{\mu}u_{2\mu}+\frac{1}{6}\partial(\partial-1)(\partial-2)u_{\mu}^{3}\right]w^{3\mu}+\cdots.$$

And,

$$\frac{\gamma^{2}f'(\gamma)}{\left(f(\gamma)\right)^{2}}=1+(\mu-1)\alpha_{\mu+1}\gamma^{\mu}+\left[(2\mu-1)\alpha_{2\mu+1}-(2\mu-1)\alpha_{\mu+1}^{2}\right]\gamma^{2\mu}$$
$$+\left[(3\mu-1)\alpha_{2\mu+1}+2(3\mu-1)\alpha_{2\mu+1}\alpha_{\mu+1}+(3\mu-1)\alpha_{\mu+1}^{3}\right]\gamma^{3\mu}+\cdots,$$

and

$$\frac{w^{2}g'(w)}{\left(g(w)\right)^{2}}=1-(\mu-1)\alpha_{\mu+1}w^{m}-\left[(2\mu-1)\alpha_{2\mu+1}-(2\mu-1)\alpha_{\mu+1}^{2}\right]w^{2\mu}$$
$$-(3\mu-1)\left[\alpha_{2\mu+1}-3\alpha_{2\mu+1}\alpha_{\mu+1}+2\alpha_{\mu+1}^{3}\right]w^{3\mu}+\cdots,$$

Notice that, equating the coefficient in (8) and (9), we get:

$$(\mu-1)\alpha_{\mu+1}=\partial r_{\mu}, \tag{12}$$

$$(2\mu-1)\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)=\partial r_{2\mu}+\frac{1}{2}\partial(\partial-1)r_{\mu}^{2}\,, \tag{13}$$

$$(3\mu-1)\left(\alpha_{\mu+1}^{3}+\alpha_{3\mu+1}-2\alpha_{\mu+1}\alpha_{2\mu+1}\right)$$
$$=\partial r_{2\mu}+\partial(\partial-1)r_{\mu}r_{2\mu}+\frac{1}{6}\partial(\partial-1)(\partial-2)r_{\mu}^{3}, \tag{14}$$

$$-(\mu-1)\alpha_{\mu+1}=\partial u_{\mu}, \tag{15}$$

$$-(2\mu-1)\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)=\partial u_{2\mu}+\frac{1}{2}\partial(\partial-1)u_{\mu}^{2}, \tag{16}$$

and

$$-(3\mu-1)\left(2\alpha_{\mu+1}^{3}+\alpha_{3\mu+1}-3\alpha_{\mu+1}\alpha_{2\mu+1}\right)=$$
$$\partial u_{2\mu+1}+\partial(\partial-1)u_{\mu}u_{2\mu}+\frac{1}{6}\partial(\partial-1)(\partial-2)u_{\mu}^{3}. \tag{17}$$

From Equations 12 and 15, we get:

$$r_{\mu}=-u_{\mu} \tag{18}$$

From Equation 13, and using Lemma 1.5, we obtain

$$\left|(2\mu-1)\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)\right|=|2\mu-1|\left|\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right|=\left|\partial r_{2\mu}+\frac{1}{2}\partial(\partial-1)r_{\mu}^{2}\right|$$
$$\leq\partial\left|r_{2\mu}\right|+\frac{1}{2}\partial(\partial-1)\left|r_{\mu}^{2}\right|\leq\frac{2\partial^{2}}{(2\mu-1)}. \tag{19}$$

By compensation 14, in 17, and use 18, we find that:

$$(3\mu-1)\alpha_{\mu+1}\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)=\partial\left(r_{2\mu}+u_{2\mu}\right)+\partial(\partial-1)\left(r_{\mu}r_{2\mu}+u_{\mu}u_{2\mu}\right), \tag{20}$$

which gives,

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$$\left|(3\mu-1)\alpha_{\mu+1}\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)\right| = (3\mu-1)\left|\alpha_{\mu+1}\right|\left|\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right|$$
$$\leq \frac{4\partial(2\partial-1)}{(3\mu-1)}. \quad (21)$$

Note that, from Equations 19 and 21, we get that:

$$\left|\alpha_{\mu+1}\right| \leq \frac{2(2\mu-1)(2\partial-1)}{\partial(3\mu-1)}.$$

Now, by subtracting Equation 17 from Equation 14, and use Equation 18, we have:

$$(3\mu-1)\left(3\alpha_{\mu+1}^{3}+2\alpha_{3\mu+1}-5\alpha_{\mu+1}\alpha_{2\mu+1}\right) = \partial\left(r_{2\mu}-u_{2\mu}\right)$$
$$+\partial(\partial-1)\left(r_{\mu}r_{2\mu}-u_{\mu}u_{2\mu}\right)+\frac{1}{3}\partial(\partial-1)(\partial-2)r_{\mu}^{3}, \quad (22)$$

By using Equations 20 and 22, removing α_{m+1}^{3}, we have the following:

$$2(3\mu-1)\left(\alpha_{3\mu+1}-\alpha_{\mu+1}\alpha_{2\mu+1}\right) = \partial\left(r_{2\mu}-u_{2\mu}\right)+3\partial\left(r_{2\mu}+u_{2\mu}\right)$$
$$= \partial\left(4r_{2\mu}+2u_{2\mu}\right), \quad (23)$$

which gives, via Lemma 1.5,

$$\left|\alpha_{3\mu+1}-\alpha_{\mu+1}\alpha_{2\mu+1}\right| \leq \frac{6\partial}{(3\mu-1)}. \quad (24)$$

Also, by using Equations 18 and 21, removing $\alpha_{\mu+1}\alpha_{2\mu+1}$, we have the following:

$$2(3\mu-1)\left(\alpha_{3\mu+1}-\alpha_{\mu+1}^{3}\right) = \partial\left(r_{2\mu+1}-u_{2\mu+1}\right)+5\partial\left(r_{2\mu+1}+u_{2\mu+1}\right)$$
$$= \partial\left(6r_{2\mu+1}+4u_{2\mu+1}\right), \quad (25)$$

which gives, via Lemma 1.5,

$$\left|\alpha_{3\mu+1}-\alpha_{\mu+1}^{3}\right| \leq \frac{10\partial}{(3\mu-1)}. \quad (26)$$

Now, to be discovered the bound on $\left|\alpha_{2\mu+1}\right|$. Using the inequality:

$$\left||\gamma_{1}|-|\gamma_{2}|\right| \leq |\gamma_{1}-\gamma_{2}|, \quad (27)$$

in Equation 19, we have:

$$\left|\alpha_{2\mu+1}\right|-\left|\alpha_{\mu+1}^{2}\right| \leq \left|\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right| \leq 2\partial\frac{1}{(2\mu-1)},$$

for this, the following is evident:

$$\left|\alpha_{2\mu+1}\right| \leq \frac{2\partial}{(2\mu-1)}.$$

Also, to be discovered the bound on $\left|\alpha_{3\mu+1}\right|$. By using the inequality (27) in (24) and (26), we obtain:

$$\left|\alpha_{3\mu+1}\right| \leq \frac{6\partial}{(3\mu-1)}.$$

Remark 2.3. In case of 1-fold symmetric bi-univalent functions, we obtain from Theorem 1, the following results

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$|\alpha_2| \leq 2 - \frac{1}{\partial}$, $|\alpha_3| \leq 2\partial$ and $|\alpha_4| \leq 3\partial$. $(0 < \partial \leq 1)$

Remark 2.4. In case of 1-fold symmetric bi-univalent functions and $\partial = 1$, the coefficients initial $|\alpha_{\mu+1}|, |\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$ for function in the class $\mathfrak{T}_{\Sigma_\mu}^{\partial}$, is given through

$$|\alpha_2| \leq 1 , |\alpha_3| \leq 2 \text{ and } |\alpha_4| \leq 3.$$

Corollary 2.5. Let ζ be a function satisfy 4, in class $\mathfrak{T}_{\Sigma_\mu}^{\partial}$. The Hankel determinant $H_2(2)$ and Fekete-Szegö functional are given by $|\alpha_2\alpha_4 - \alpha_3^2| \leq 7$, and $|\alpha_3 - \alpha_2^2| \leq 3$, respectively.

Definition 2.6. A function h given by Equation 4, is said to be $\mu - fold$ symmetric bi-univalent function for order τ, $(0 < \tau \leq 1)$, if

$$\mathcal{R}\text{e}\left\{\frac{z^2h'(z)}{(h(z))^2}\right\} > 1 - \tau, \ (z \in \Omega) \tag{28}$$

and

$$\mathcal{R}\text{e}\left\{\frac{\omega^2k'(\omega)}{(k(\omega))^2}\right\} > 1 - \tau , \ (\omega \in \Omega) \tag{29}$$

where the function k is the invers of h, given by Equation 5. Denoted by $\mathfrak{T}_{\Sigma_\mu}^{\tau}$, the class of all $\mu - fold$ symmetric bi-univalent functions of order τ.

Theorem 2.7. For $0 < \tau \leq 1$, if a function h given by Equation 3, be in the class $\mathfrak{T}_{\Sigma_\mu}^{\tau}$, then the coefficients $\alpha_{\mu+1}, \alpha_{2\mu+1}$ and $\alpha_{3\mu+1}$ satisfy the inequalities

$$|\alpha_{\mu+1}| \leq \frac{2(2\mu-1)}{3\mu-1}, |\alpha_{2\mu+1}| \leq \frac{2\tau}{(2\mu-1)} \text{ and } |\alpha_{3\mu+1}| \leq \frac{6\tau}{(3\mu-1)}.$$

Proof. Using the Equations 26 and 27, we can write:

$$\frac{\gamma^2h'(\gamma)}{(h(\gamma))^2} = (1-\tau) + \tau r(\gamma), \tag{30}$$

and

$$\frac{w^2k'(w)}{(k(w))^2} = (1-\tau) + \tau u(w), \tag{31}$$

where, $r(\gamma)$, and` $u(w)$ are given by Equations 10 and 11, respectively.

Therefore, we have

$$(1-\tau) + \tau r(\gamma) = 1 + \tau r_\mu \gamma^\mu + \tau r_{2\mu}\gamma^{2\mu} + \tau r_{3\mu}\gamma^{3\mu} + \cdots,$$

$$(1-\tau) + \tau u(w) = 1 + \tau u_\mu w^\mu + \tau u_{2\mu} w^{2\mu} + \tau u_{3\mu} w^{3\mu} + \cdots.$$

Using Equations 4 and 5, we obtain:

$$\frac{\gamma^2f'(\gamma)}{(f(\gamma))^2} = 1 + (\mu-1)\alpha_{\mu+1}\gamma^\mu + \left[(2\mu-1)\alpha_{2\mu+1} - (2\mu-1)\alpha_{\mu+1}^2\right]\gamma^{2\mu} + \left[(3\mu-1)\alpha_{2\mu+1} + 2(3\mu-1)\alpha_{2\mu+1}\alpha_{\mu+1} + (3\mu-1)\alpha_{\mu+1}^3\right]\gamma^{3\mu} + \cdots,$$

and

$$\frac{w^2g'(w)}{(g(w))^2} = 1 - (\mu-1)\alpha_{\mu+1}w^\mu - \left[(2\mu-1)\alpha_{2\mu+1} - (2\mu-1)\alpha_{\mu+1}^2\right]w^{2\mu} - (3\mu-1)\left[\alpha_{2\mu+1} - 3\alpha_{2\mu+1}\alpha_{\mu+1} + 2\alpha_{\mu+1}^3\right]w^{3\mu} + \cdots,$$

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Now, arrangement the coefficients in Equations 30 and 31, we get:

$$(\mu-1)\alpha_{\mu+1}=\tau r_{\mu}, \quad (32)$$

$$\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)=\tau r_{2\mu}\,, \quad (33)$$

$$2\left(\alpha_{\mu+1}^{3}+\alpha_{3\mu+1}-2\alpha_{\mu+1}\alpha_{2\mu+1}\right)=\tau r_{3\mu}, \quad (34)$$

$$-(\mu-1)\alpha_{\mu+1}=\tau u_{\mu}, \quad (35)$$

$$-\left(\alpha_{2\mu+1}-\alpha_{\mu+1}^{2}\right)=\tau u_{2\mu}\,, \quad (36)$$

and

$$-2\left(2\alpha_{\mu+1}^{3}+\alpha_{3\mu+1}-3\alpha_{\mu+1}\alpha_{2\mu+1}\right)=\tau u_{3\mu}. \quad (37)$$

We can now complete the further proof by continuing in the same way as with Theorem 2.2 □

Now, from this theorem and when $\mu=1$ we could get the following results:

Remark 2.8.

I. $|\alpha_2|\le 1$, $|\alpha_3|\le 2\tau$ and $|\alpha_4|\le 3\tau$.

II. Also, Theorem 2.7 restricts the findings to the Naik and Patil (2017), that we remember here below as Corollary 2.9.

Corollary 2.9. (Naik and Patil, 2017) Let the function h, Given through Equation 3, be in the class $\mathfrak{T}_{\Sigma_\mu}^{\tau}$. Then $|\alpha_2|\le 1$, $|\alpha_3|\le 2\tau$ and $|\alpha_4|\le 3\tau$.

Remark 2.10. Set $m=1$ *and* $\tau=1$, in results of Theorem 2.7. The Hankel determinant $H_2(2)$ and Fekete-Szegö functional are given by $|\alpha_2\alpha_4-\alpha_3^2|\le 7$, and $|\alpha_3-\alpha_2^2|\le 3$, respectively.

3. Conclusions.

i- The estimates $|\alpha_{\mu+1}|$, dependent on ∂ and independent on τ, for the subclasses $\mathfrak{T}_{\Sigma_\mu}^{\partial}$ and $\mathfrak{T}_{\Sigma_\mu}^{\tau}$, respectively.

ii- In case of 1-fold symmetric bi-univalent functions, when you look at the estimates $|\alpha_{2\mu+1}|$ and $|\alpha_{3\mu+1}|$, It's intriguing to see here that, we can expanding it to $|\alpha_{n\mu+1}|\le nm\partial, (n\ge 2)$ for the subclass $\mathfrak{T}_{\Sigma_\mu}^{\partial}$ and $|\alpha_{n\mu+1}|\le nm\tau, (n\ge 2)$ for the subclass $\mathfrak{T}_{\Sigma_\mu}^{\tau}$.

iii- The value of $H_2(2)$ and Fekete-Szegö functional for the subclasses $\mathfrak{T}_{\Sigma_\mu}^{\partial}$ and $\mathfrak{T}_{\Sigma_\mu}^{\tau}$ are similar.

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