On Orthogonal Reverse α-Derivations of Semi-prime Γ-Ring

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Abstract

In this paper, the concept of reverse α-derivation on Γ-ring M, and some results concerning two reverse α-derivations on M that are related to classical result of E.Ponser are presented. These results shows that if M is α2-torsion free semiprime Γ-ring, d and g are non-zero reverse derivations of M, then dg are not a non-zero derivation. The new main results present and study notion of orthogonally reverse α-derivations d, g of M.

Keywords: Prime, semiprime; Γ-ring, derivation; α-derivati;n, reverse α-derivati;n, orthogonal reverse α-derivation.

1. Introduction

Notion of Γ-ring M is a more general concept than the ring which was defined by Nobusawa [8], but this definition of Γ-ring has been weakened by Barnes [2]. Following [2] means a Γ-ring is an order pair (M,Γ) where M and Γ are additive group such that there exists a function

$M\times\Gamma\times M \longrightarrow M$ satisfying for all $a, b \in M$ and $\gamma, \beta \in \Gamma$:

(i) $(a + b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma + \beta)b = a\gamma b + a\beta b$, $a\gamma(b + c) = a\gamma b + a\gamma c$
(ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$

M is 2-torsion free if $2a = 0$ implies $a = 0$, and it is prime if $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$.

Recall that M is semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$. It is known that every ring is a Γ-ring, but generally the converse is not true. F. J.Jing in [6] defined a derivation on M by an additive mapping $d: M \longrightarrow M$ is called derivation if for all $a, b \in M$ and $\gamma \in \Gamma$, then

$d(a\gamma b) = d(a)\gamma b + a\gamma d(b)$.

J.C. Change defined α-derivation on ring R [5], as follows :

Let α be a non zero mapping of R, an additive mapping $d: R \longrightarrow R$ is called α-derivation, if for $a, b \in R$, then $d(ab) = d(a)\alpha(b) + ad(b)$.

S.M.Salih and H.J.Thahab in [10] expanded definition on Γ-ring M in [5] as follows:

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$d(a\gamma b) = d(a)\gamma\alpha(b) + a\gamma d(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

It is obvious that every α-derivation is derivation (where $\alpha = id_M$), but generally the converse is not true. Breser and Vukman [4] introduced the notion of reverse derivation on R as an additive mapping $d:R \longrightarrow R$ satisfying $d(ab) = d(b)a + bd(a)$, for all $a, b \in R$.

In this paper, the notion of reverse derivation on Γ-ring M is presented .

Bresar and Vukman [3] introduced notion of orthogonality for two derivations d, g on a semiprime ring as follows:

Two derivations d and g on a ring R are called orthogonal, if for all $a, b \in R$, then

$d(x)Rg(y) = (0) = g(y)Rd(x)$.

Hence, A.H.Majeed introduced the notion of orthogonality for two reverse derivations d, g on semiprime ring [7] as follows:

Two reverse derivations d and g of R are called orthogonal if $d(x)Rg(y) = (0) = g(y)Rd(x)$,

for all $x, y \in R$.

It is obvious that a non-zero reverse derivation can not be orthogonal on itself.

The main aim for this paper is to introduce notion of orthogonal for two reverse α-derivations d and g on a semiprime Γ-ring, and present necessary and sufficient conditions for d and g to be an orthogonal such that these results equivalent to those earlier obtained by M.Ashraf and M.R.Jamal in [1]. First let us suppose that M is a 2-torsion free simeprime

Γ-ring.

2. Reverse Derivation and α-Derivation of Γ-Ring M

In this section, notion of reverse derivation and reverse α-derivation are defined, furthermore concepts of Jordan and triple reverse α-derivations on M.

Definition (2.1):

An additive function $d:M \longrightarrow M$ is called reverse derivation on M

if for all $a, b \in M$ and $\gamma \in \Gamma$,

$d(a\gamma b) = d(b)\gamma a + b\gamma d(a)$

Definition (2.2):

Let α be endomorphism mapping of Γ-ring M. An additive mapping $d:M \longrightarrow M$ is called a reverse α-derivation of M,

if for all $a, b \in M$ and $\gamma \in \Gamma$ satisfying:

$d(a\gamma b)= d(b)\gamma\alpha(a) + b\gamma\alpha(a)$ …(1)

d is called Jordan reverse α-derivation of M if

$d(a\gamma a= d(a)\gamma\alpha(a) + a\gamma d(a)$ …(2)

Also, d is called Jordan triple reverse α-derivation of M**,**

if for all $a, b \in M$ and $\gamma, \beta \in \Gamma$ then

$d(a\gamma b\beta a)=d(a)\beta\alpha(a)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(a) + a\beta b\gamma d(a)$ …(3)

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The following example is about reverse α-derivation of Γ-ring M.

Example (2.3):

Let d be reverse α-derivation of Γ-ring M. We put $M' = M \oplus M$ and $\Gamma' = \Gamma \oplus \Gamma$, where $\oplus$ direct sum, α' be emdomprphism mapping on M' defined by $\alpha'((x,y)) = (\alpha(x),\alpha(y))$, for all $(x,y) \in M'$. Also we define d' on M' by $d'((x,y)) = (d(x),0)$, for all $(x,y) \in M'$. Then d' is reverse α'-derivation on M'. It is obvious that every reverse α-derivation is reverse derivation, but the converse is not true in general.

Now, we present some properties of reverse α-derivation on M.

Lemma (2.4):

Let α be endomorphism of M, and d be reverse α-derivation on M, then for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$ that following statements holds:

(i) $d(a\gamma b + b\gamma a) = d(b)\gamma\alpha(a) + b\gamma d(a) + d(a)\gamma\alpha(b) + a\gamma d(b)$

(ii) $d(a\gamma b\beta a + a\beta b\gamma a) = d(a)\beta\alpha(a)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(a) + a\beta b\gamma d(a) + d(a)\gamma\alpha(a)\beta\alpha(b) + a\gamma d(b)\beta\alpha(a) + a\gamma b\beta d(a)$ where In particular α is onto mapping.

(iii) In particular if M be 2-torsion free then
$d(a\gamma b\gamma a) = d(a)\gamma\alpha(b)\gamma\alpha(a) + a\gamma d(b)\gamma\alpha(a) + a\gamma b\gamma d(a)$

(iv) $d(a\gamma b\beta c + c\gamma b\beta a) = d(c)\beta\alpha(a)\gamma\alpha(b) + c\beta d(b)\gamma\alpha(a) + c\beta b\gamma d(a) + d(a)\beta\alpha(c)\gamma\alpha(b) + a\beta d(b)\gamma\alpha(c) + a\beta b\gamma d(c)$

(v) $d(a\gamma b\gamma c + c\gamma b\gamma a) = d(c)\gamma\alpha(a)\gamma\alpha(b) + c\gamma d(b)\gamma\alpha(a) + c\gamma b\gamma d(a) + d(a)\gamma\alpha(c)\gamma\alpha(b) + a\gamma d(b)\gamma\alpha(c) + a\gamma b\gamma d(c)$

Proof :

(i) By calculating we obtan $d((a + b)\gamma(b + a))$ and **(ii)** is also obtained by replacing $a\beta b + b\beta a$ for b in (i), moreover **(iii)** can be obtained by replacing β with γ in (ii). If we replace $a + c$ for a in definition (2.2)(3) we can get (iv). Wile if replacing β by γ in (iv) we obtained (v).

3. The Main Results

In this section, definition of orthogonal reverse α-derivation on Γ-ring M and some properties are introduced.

Now, we begin with the following definition:

Definition (3.1): Let M be Γ-ring and α be an endomorphism mapping of M. Then two reverse α-derivations d and g on M are called orthogonal if for all $x, y \in M$ then:

$d(x)\Gamma\alpha(M)\Gamma g(y) = (0) = g(y)\Gamma\alpha(M)\Gamma d(x)$.

It is obvious that a non-zero reverse α-derivation can not be orthogonal on itself.

Example (3.2):

Let d' is reverse α'-derivations on M' as in example (2.3) and g' is reverse α'-derivations on M' defined by $g'((x,y)) = (0,g(x))$ for all $(x,y) \in M$ then:

d' and g' are orthogonal reverse α'-derivations on M'.

The following are useful to prove the main results :-

Lemma (3.3): ([9], lemma 3)

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Let M be Γ-ring and a, b the elements of M. Then the following conditions are equivalent. For all γ, $\beta \in \Gamma$ then

(i) $a\gamma M\beta b = (0)$.
(ii) $b\gamma M\beta a = (0)$.
(iii) $a\gamma M\beta b + b\gamma M\beta a = (0)$.

If one of these conditions are fulfilled then $a\delta b = b\delta a = 0$, for all $\delta \in \Gamma$.

$0 = d(x)\delta\alpha(z)\beta g(x) + d(x)\delta\alpha(z)\beta g(y) + d(y)\delta\alpha(z)\beta g(x) + d(y)\delta\alpha(z)\beta g(y)$

$0 = d(x)\delta\alpha(z)\beta g(y0 + d(y)\delta\alpha(z)\beta g(y)$

Therefore by our assumption we get

$d(x)\delta\alpha(z)\beta g(y)\gamma\alpha(t)\lambda d(x)\ \delta\alpha(z)\beta g(y) = 0$

$– d(x)\delta\alpha(z)\beta g(y)\gamma\alpha(t)\lambda d(x)\ \delta\alpha(z)\beta g(y) = 0$

Since M is semiprime, then $d(x)\delta\alpha(z)\beta g(y) = 0$.

Hence $d(x)\Gamma\alpha(M)\Gamma g(y) = (0)$, for all $x, y \in M$

Lemma (3.4):

Let α be an endomorphism of semiprime Γ-ring M. Suppose that additive mappings d and g are reverse α-derivations of M satisfy $d(x)\Gamma\alpha(M)\Gamma g(x) = (0)$, for all $x \in M$. Then $d(x)\Gamma\alpha(M)\Gamma g(y) = (0)$, for all $x, y \in M$.

Proof :

Since $d(x)\delta\alpha(z)\beta h(x) = 0$, for all $x, z \in M$ and $\delta, \beta \in \Gamma$.

On linearizing we get;

$0 = d(x + y)\delta\alpha(z)\beta g(x + y)$

Lemma (3.5):

Let α be an endomorphism of M, d and g are reverse α-derivation of M. Then d and g are orthogonal if and only if for all $x, y \in M$ and $\delta, \beta \in \Gamma$ then

$d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y) = 0$, where $\alpha^2 = \alpha$.

Proof :

Suppose that $d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y) = 0$ for all $x, y \in M$. Replacing x by $x\gamma y$ we get

$d(\alpha(x\gamma y))\delta g(y) + g(\alpha(x\gamma y))\beta d(y) = 0$, then $d(\alpha(x)\gamma\alpha(y))\delta g(y) + g(\alpha(x)\gamma\alpha(y))\beta d(y) = 0$, thus $(d(\alpha(y)\gamma\alpha^2(x)) + \alpha(y)\gamma d(\alpha(x))\delta g(y) + (g(\alpha(y))\gamma\alpha^2(x) + \alpha(y)\gamma g(\alpha(x))\beta d(y) = 0$.

$d(\alpha(y))\gamma\alpha(x)\delta g(y) + \alpha(y)\gamma d(\alpha(x))\gamma g(y) + g(\alpha(y)\gamma\alpha(x)\beta d(y) + \alpha(y)\gamma g(\alpha(x))\beta d(y) = 0$, therefore

$\alpha(y)\gamma(d(\alpha(x))\delta g(y) + g(\alpha(x))\beta d(y)) + d(\alpha(y))\gamma\alpha(x)\delta g(y) + g(\alpha(y)\gamma\alpha(x)\beta d(y) = 0$, by assumption we reduce this relation to form $d(\alpha(y)\gamma\alpha(x)\delta g(y) + g(\alpha(y)\gamma\alpha(x)\beta d(y) = 0$.

Hence by lemma (3.3) we get $d(\alpha(y))\gamma\alpha(x)\delta g(y) = 0 = g(\alpha(y))\gamma\alpha(x)\beta d(y)$ for all $x, y \in M$ and $\gamma, \delta, \beta \in \Gamma$.

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therefore d, g are orthogonal.

Conversely: suppose that d, g are orthogonal then $d(x)\gamma\alpha(z)\delta g(y) = 0 = g(y)\gamma\alpha(z)\delta d(x)$. Therefore by lemma (3.3) we have $d(x)\gamma\alpha(z)\delta g(y) + g(y)\gamma\alpha(z)\delta d(x) = 0$.

Thus $d(x)\beta g(y) = 0 = g(x)\beta d(y)$, since α onto then we can take $x = \alpha(x)$, therefore

$d(\alpha(x))\beta g(y) = 0 = g(\alpha(x))\beta d(y)$.

Hence $d(\alpha(x))\beta g(y) + g(\alpha(x))\beta d(y) = 0$.

Now, we prove ([1], theorem 2.1) by using less condition, as follows:

Theorem (3.6):

Let α be an onto endomorphism mapping of Γ-ring M, suppose d and g are reverse α-derivations of M. Then the following conditions are equivalent, where $\alpha = \alpha^2$ and α commutative mapping:

(i) d *and* g *are orthogonal reverse* $\alpha -$ *derivation.*
(ii) $dg = 0$ *where* $d2 = d$.
(iii) $gd = 0$ *where* $g2 = g$.
(iv) $dg + gd = 0$.
(v) dg *is a reverse* $\alpha -$ *derivation of* M.
(vi) gd *is a reverse* $\alpha -$ *derivation of* M.
(vii) *There exists* a, b *and* $\beta, \delta \in \Gamma$ *such that* $dg(x) = a\beta\alpha(x) + \alpha(x)\delta b$, *for all* $x \in M$.

Proof :

(i) ↔ (ii) Since d and g are orthogonal, then $d(x)\delta\alpha(z)\beta g(y) = 0$, for all $x, y \in M$ and $\delta, \beta \in \Gamma$.
$0 = d(d(x)\delta\alpha(z)\beta g(y))$, therefore by definition (2.2)

$0 = d(g(y)\beta\alpha(d(x))\delta\alpha^2(z) = g(y)\beta d(\alpha(z))\delta\alpha(d(x)) + g(y)\beta\alpha(z)\delta d(d(x))$

$0 = d(g(y))\beta d(\alpha(x))\delta\alpha(z) + g(y)\beta\alpha(d(z))\delta d(\alpha(x)) + g(y)\beta\alpha(z)\delta d^2(x)$

$0 = d(g(y))\beta d(\alpha(x))\delta\alpha(z) + g(y)\beta\alpha(d(z))\delta d(\alpha(x)) + g(y)\beta\alpha(z)\delta d(x)$

The second and third summands are zero since d and g are orthogonal therefore this relation is reduced to $d(g(y)\beta d(\alpha(x))\delta\alpha(z) = 0$. Hence by concept semiprimeness we have $d(g(y)\beta d(\alpha(x))\delta\alpha(z)\delta d(g(y))\beta d(\alpha(x)) = 0$, thus $d(g(y)\beta d(\alpha(x)) = 0$, since M is semiprime where x, y are arbitrary elements in M, so take $\alpha(x) = g(y)$ in the above relation we get $d(g(y)\beta d(g(y)) = 0$, for all $y \in M$. Since M is semiprime then $(dg)(y) = 0$, for all $y \in M$.

Conversely:

Let $dg = 0$ where $d^2 = d$, therefore

$dg(x\gamma y) = d(g(x\gamma y)) = d(g(y)\gamma\alpha(x) + y\gamma g(x))$

$= d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma d(g(y)) + d(g(x))\gamma\alpha(y) + g(x)\gamma d(y)$,

then

$dg(x\gamma y) = d(\alpha(x))\gamma g(\alpha(y)) + \alpha(x)\gamma dg(y) + dg(x)\gamma\alpha(y) + g(x)\gamma d(y)$

$= d(\alpha(x))\gamma g(\alpha(y)) + g(x)\gamma d(y)$

Since α is onto then we can take $\alpha(y) = y$ and $x = \alpha(x)$.

Thus, $d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) = 0$, hence by lemma (3.5) we have d, g are orthogonal reverse α-derivation of M.

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(i)↔ (iii) Similar way used in the part of (ii).

(iv)↔ (i) Suppose that g and d are orthogonal, therefore from (ii) and (iii) theorem (3.6) we obtain $dg + gd = 0$.

Conversely: Suppose that $dg + gd = 0$, then

$0 = (dg + gd)(x\gamma y) = (dg)(x\gamma y) + (gd)(x\gamma y)$. Therefore

$$0 = d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma dg(y) + dg(x)\gamma\alpha(y) + g(x)\gamma d(y) + g(\alpha(x))\gamma\alpha(d(y)) + \alpha(x)\gamma gd(y) + gd(x)\gamma\alpha(y) + d(x)\gamma g(y)$$

$$0 = (d(\alpha(x))\gamma g(\alpha(y)) + g(\alpha(x))\gamma d(\alpha(y)) + \alpha(x)\gamma(dg(y) + gd(y)) + (dg(x)+gd(x))\gamma\alpha(y) + g(x)\gamma d(y) + d(x)\gamma g(y)$$

$$0 = d(\alpha(x))\gamma g(\alpha(y)) + g(\alpha(x))\gamma d(\alpha(y)) + g(x)\gamma d(y) + d(x)\gamma g(y).$$

Hence since α is onto then we can take $x = \alpha(x)$, $\alpha = y$. $0 = d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) + g(x)\gamma d(y) + d(\alpha(x))\gamma g(y)$.

$0 = 2(d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y))$.

Since M be 2-torsion free then $d(\alpha(x))\gamma g(y) + g(\alpha(x))\gamma d(y) = 0$, be lemma (3.5) we get d, g are orthogonal.

(v) ↔ (i) Suppose that dg is a reverse α-derivation of M.

$$(dg)(x\gamma y) = (dg)(y)\alpha(x) + y\gamma(dg)(x) \qquad \ldots(1)$$

But,

$(dg)(x\gamma y) = d(\alpha(x))\gamma\alpha(g(y)) + \alpha(x)\gamma d(g(y)) + d(g(x))\gamma\alpha(y) + g(x)\gamma d(y)$.

Since α is commutative mapping then

$$(dg)(x\gamma y) = d(\alpha(x))\gamma\alpha(g(y)) + dg(y)\gamma\alpha(x) + \alpha(y)\gamma dg(x) + g(x)\gamma d(y) \qquad \ldots(2)$$

α be onto, then replacing $\alpha(y)$ by y and comparing (1) with (2) we get

$d(\alpha(x))\gamma\alpha(g(y)) + g(x)\gamma d(y) = 0$. Replace x by $\alpha(x)$ then

$d(\alpha(x))\gamma\alpha(g(y)) + g(\alpha(x))\gamma d(y) = 0$, therefore by lemma (3.5) we have d, g are orthogonal.

Conversely: Suppose d, g are orthogonal, then $d(x)\gamma\alpha(z)\beta g(y) = 0$.

Thus by lemma (3.3)

$$d(x)\delta g(y)=0= g(y)\delta d(x) \quad \ldots(1)$$

Now, $(dg)(x\gamma y) = d(\alpha(x))\gamma g(\alpha(y)) + (dg)(y)\gamma\alpha(x) + \alpha(y)\gamma(dg)(x) + g(x)\gamma d(y)$.

α be onto, then replacing $\alpha(y)$ by y

$(dg)(x\gamma y) = d(\alpha(x))\gamma g(y) + dg(y)\gamma\alpha(x) + y\gamma dg(x) + g(x)\gamma d(y)$.

By (1) we can reduce above relation to $(dg)(x\gamma y) = dg(y)\gamma\alpha(x) + y\gamma dg(x)$.

Hence dg is a reverse α-derivation.

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(vi)↔ (i) Similar way use in the proof (v). (i) ↔ (iii) Let d and g are orthogonal,

then by (ii) we have dg = 0, so we can choose $a = b = 0$ and $\beta, \delta \in \Gamma$.

So that $dg(x) = a\beta\alpha(x) + \alpha(x)\delta b$.

Theorem (3.7):

Let M be a Γ-ring, and d, g be reverse α-derivation of M and α be onto endomorphism mapping of M. Suppose $d^2 = g^2$. Then d + g and d – g are orthogonal.

Proof :

Suppose that $d^2 = g^2$, now for all $x \in M$

$\{(d-g)(d+g) + (d+g)(d-g)\}(x)$

$= (d-g)(d+g)(x) + (d+g)(d-g)(x)$

$= d^2(x) + dg(x) - gd(x) - g^2(x) + d^2(x) - dg(x) + gd(x) - g^2(x) = 2(d^2(x) - g^2(x)).$

$= 2(d^2(x) - g^2(x)).$

Since M be a 2-torsion free, therefore $d^2(x)$ –= $2(d^2(x) - g^2(x))$. $g^2(x) = 0$, thus by using assumption we get

$(d-g)(d+g) + (d+g)(d-g) = 0.$

Hence by (iv), (i) of theorem (3.6) we d + g and d – g are orthogonal reverse α-derivation.

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