NeutroAlgebra of Substructures of the Semigroups built using Z_n and Z^+

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Abstract

For the first-time authors study the NeutroAlgebraic structures of the substructures of the semigroups, $\{Z_n, \times\}$, $\{Z^+, \times\}$ and $\{Z^+, +\}$ where $Z^+ = \{1, 2, \ldots, \infty\}$. The three substructures of the semigroup studied in the context of NeutroAlgebra are subsemigroups, ideals and groups. The substructure group has meaning only if the semigroup under consideration is a Smarandache semigroup. Further in this paper, all semigroups are only commutative. It is proved the NeutroAlgebraic structure of ideals (and subsemigroups) of a semigroup can be AntiAlgebra or NeutroAlgebra in the case of infinite semigroups built on Z^+ or $Z^* = Z^+ \cup \{0\}$. However, in the case of $S = \{Z_n, \times\}$; n a composite number, S is always a Smarandache semigroup. The substructures of them are completely analyzed. Further groups of Smarandache semigroups can only be a NeutroAlgebra and never an AntiAlgebra. Open problems are proposed in the final section for researchers interested in this field of study.

Keywords: NeutroAlgebra; AntiAlgebra; groupring; NeutrosubAlgebra; Partial Algebra; groups; ideals; Smarandache semigroup.

1. Introduction

Authors in this paper study and research on the NeutroAlgebra of substructures of semigroups constructed using Z_n and Z^+. Study of this type is very new and innovative. In fact, we study the three substructures of the above said semigroups viz., subsemigroups, ideals and groups. The substructure groups in semigroups have meaning only if the semigroup under analysis is a Smarandache semigroup [1]. We study only semigroups which are commutative.

In [2] Smarandache proposed a new theory called neutrosophic theory which is a powerful tool to analyze the uncertainty, inconsistency and indeterminacy present in the data of real-world problems. The new notion of NeutroAlgebra and AntiAlgebra was introduced in 2019 by Smarandache [3]. Several researchers, in [4 -15] have studied these new algebraic structures, [5] have introduced and analyzed Neutro BE-Algebras and Anti BE-Algebras and [6] studied NeutroAlgebra and AntiAlgebra in classical number systems. Elaborate research has been carried out on Neutrosophic triplets, duplets and extended Neutrosophic and their applications in [15-34]; using these concepts, NeutroAlgebra of neutrosophic triplets in $\{Z_n, \times\}$ have been analyzed in [10].

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The groups proposed in the literature on Z_n in [8] were expanded using the automatic generation of NeutroGroups of finite or infinite cardinality, the generation of NeutroGroups was done from uninorms in [9]. The in-depth study of the NeutroAlgebra which is generated by the combining function in prospector was done in [10]. In [11], NeutroAlgebra of ideals in a ring was studied. It was carried out in the case of Z and Z_n, the ring of integers and ring of modulo integers respectively. It was also extended to the case of polynomial rings R[x], Q[x], Z[x], and Z_n [x] where R is real and Q is the rational integers, respectively. The concept of NeutroAlgebra of idempotents in group rings was studied in [12].

To analyse the access barriers faced by the migrant population in health care in Chile, NeutroAlgebra was applied to generalize the prospector function into a NeutroFunction in [13]. This was done to capture the indeterminacy present. Similarly, a recent application of NeutroAlgebra [14] studied the situation of immigrants in Ecuador to analyze the ease of access to various public services.

They have proved several related results with NeutroAlgebra in the case of Neutrosophic triplets and extended Neutrosophic triplets. Recently [11] have introduced the new NeutroAlgebra of ideals of the rings Z, Z_n and polynomial rings. Here we obtain the NeutroAlgebra of the substructures of the semigroup. We further state that we have taken only commutative semigroups buts using {Z_n, ×} or {Z^+, □} or {Z^+, +} or {$Z^+ \cup \{0\}$, □} and {$Z^+ \cup \{0\}$, +}.

The main substructures which are associated with the semigroups are subsemigroups, ideals and groups. However, groups are possible only in the case of Smarandache Semigroup. For more about Smarandache semigroups refer to [1]. Since all semigroups in this study are taken as commutative we do not have the concept of a right or left ideal.

This paper has six sections. Section one is introductory in nature. Basic concepts are given in section two to make this paper a self-contained one. Section three for the first time introduces the notion of NeutroAlgebras of substructures of a semigroup. Section four obtains the conditions on substructures of ideals of the semigroup to be a NeutroAlgebra and in section five we find the NeutroAlgebra of groups of the Smarandache semigroup. The final section gives the conclusion based on our study and open problems for the researchers on this topic.

2. Basic Concepts

Most of the basic concepts essential to make this paper a self-contained one are given here. We first recall in a line or two the definition of semigroups and its substructures subsemigroups and ideals. We just give the definition of Smarandache semigroups and illustrate them by examples. All semigroups are in general not Smarandache semigroups [1]. We recall the definition of NeutroAlgebra and AntiAlgebra more elaborately and illustrate them by examples.

Definition 2.1. *A non-empty set S with a binary operation * is said to be a semigroup*

i. *For all a, b $\in$ S a * b $\in$ S*
ii. *(a * b) * c = a * (b * c) for all a, b, c $\in$ S.*

*We call {S, *} to be a semigroup if (i) and (ii) are true.*

*If S has a unique element e in S such that for all a $\in$ S, a * e = e * a = a, we define {S, *} to be a semigroup with identity or a monoid. If the operation * defined on S is such that a * b = b * a for a, b $\in$ S, we define {S, *} to be a commutative semigroup. If the order of S is finite we define S to be a semigroup S is of finite order otherwise S is said to be of infinite order.*

*Let B be a proper subset of S. B is defined as a subsemigroup if {B, *} itself is a semigroup under the operation *, defined on the semigroup S. If in the definition for every s $\in$ S and every b $\in$ B both s * b and b * s $\in$ B then we define {B, *} as an ideal of {S, *}.*

Example 2.2. Let S = {Z_{12}, ×} be a semigroup. Take B = {0, 2, 4, 6, 8, 10} $\subseteq$ S; {B, ×} is a subsemigroup of S. Let A = {4, 8, 0} $\subseteq$ S is also a subsemigroup of S. For every s $\in$ S we have s × a $\in$ A so {A, ×} is a ideal of {S, ×}.

For more refer [1].

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Definition 2.3. *[1]. Let {S, *} be a semigroup. If S contains a subset H ⊆ S such that {H, *} is a group under the operation *. We define {S, *} to be a Smarandache semigroup.*

We prove some examples of them.

Example 2.4. Let S = { Z_{10} , □} be the semigroup under product. In fact, S is a monoid. For 1 □ x = x □ 1 = x for all x ∈ Z_{10} . Consider H = {1, 3, 7, 9} ⊆ Z_{10} . The Cayley Table of Z_{10} is as follows.

Table 1: Table of H under □

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

Clearly {H, □} is a group with 1 as identity. So, S = { Z_{10} , □} is a Smarandache semigroup. Further the set {P = {1} is not a subgroup as it is trivial. For this is important for otherwise all monoids will be Smarandache semigroups Further we see { Z^+ , +} { Z^+ , □} { Z^+ ∪ {0}, +} and { Z^+ ∪ {0}, □} are never Smarandache semigroups.

Example 2.5. Let S = {Z, □} be the semigroup S is a Smarandache semigroup as P = {1, – 1} ⊆ S is such that {P, □} is a group seen from the Cayley Table.

Table 2: Table of P under □

×	1	–1
1	1	–1
–1	–1	1

1 ∈ P is identity of under product in {P, ×}.

We for the first time study the NeutroAlgebra of these three structures. Now we proceed onto recall the definition of NeutroAlgebra.

A NeutroAlgebra is an algebra which has at least one Neutro-operation or one Neutro axiom (axiom that is true for some elements, indeterminate or false for other elements. A partial algebra has at the minimum one partial operation and all its axioms are classical [34] proved that NeutroAlgebra is a generalization of partial algebra and has given illustrations of NeutroAlgebra s that are partial algebras.

Partial algebra has some elements for which the operation is undefined or outer defined. On similar lines, a nonempty set is an AntiAlgebra if it is endowed with at least one anti operation or at least one anti axiom.

We have provided examples of NeutroAlgebra and AntiAlgebra. However, we provide some examples of them.

Example 2.6. Let { Z_{10} , ×} be the semigroup. The idempotents of Z_{10} are 6 and 5.

For

$$5 \times 5 = 5 (\text{mod } 10) \text{ and } 6 \times 6 = 6 (\text{mod } 10).$$

The Cayley table for W = {5, 6} under product is as follows.

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Table 3: Table for W under ×

×	5	6
5	5	od
6	od	6

Hence W is a NeutroAlgebra consider W = {6, 5}; the Cayley table for sum is as follows

Table 4: Table for W under +

+	6	5
6	od	od
5	od	od

W is an AntiAlgebra under product.

Suppose we take the nontrivial idempotents also so that V = {0, 1, 5, 6}. Then the Cayley table for V is as follows under +.

Table 5: Table of V under +

+	0	1	5	6
0	0	1	5	6
1	1	od	6	od
5	5	6	0	1
6	6	od	1	od

Then V under + becomes a NeutroAlgebra.
Consider the set of idempotents of V under ×. The Cayley Table of V under × is as follows.

Table 6: Table of V under □

×	0	1	5	6
0	0	0	0	0
1	0	1	5	6
5	0	5	5	0
6	0	6	0	6

V under × is not a NeutroAlgebra in fact a semigroup under ×.

3. NeutroAlgebra of substructures in a semigroup using $\{Z^+ \cup \{0\}\}$ and Z_n

In this section we define, describe, and develop the new notion of the NeutroAlgebra of substructures of a semigroup viz; ideals and subsemigroups of a semigroup built using $Z^+ \cup \{0\}$ and Z_n $(4 \leq n < \infty)$. We use the operation of + and × on these subsemigroups (or ideals) of the semigroup. First, we provide some examples of them.

Example 3.1. Let S = {Z_{10}, ×} be the semigroup. The subsemigroups of S are { v_1 = {0, 5}, v_2 = {2, 4, 6, 8}, v_3 = {5}, v_4 = {0}, v_5 = {2, 4, 6, 8, 0}, v_6 = {3, 9, 7, 1}, v_7 = {0, 1, 9, 7, 3}, v_{14} = {1, 9}, v_{13} = {0, 1, 9}, v_{11} =

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{1, 7, 3, 9, 5}, v_{12} = {0, 5, 1, 9, 7, 3}, S = v_{10}, v_8 = {4, 6}, {4, 6, 0} = v_9 } = V. We see there are 14 subsemigroups. Suppose W $\subseteq V\backslash v_4, v_{14}, v_{13}, S = v_{10}$ }; where W = $\{v_1, v_2, v_3, v_5, v_6, v_7, v_8, v_9, v_{11}, v_{12}\}$.

Now we define the operation on W the usual product × on W. The Cayley table of W under × is as follows.

Table 7: Table of W under ×

$\times$	v_1	v_2	v_3	v_5	v_6	v_7	v_8	v_9	v_{11}	v_{12}
v_1	v_1	od	v_1	od	od	v_{12}	od	od	v_{12}	v_{12}
v_2	od	v_2	od	v_5	v_2	v_5	v_2	v_2	v_5	v_5
v_3	v_1	od	v_3	od	v_{11}	v_{12}	od	od	v_{11}	v_{12}
v_5	od	v_5	od	v_5	v_5	v_5	v_5	v_5	v_{12}	v_{12}
v_6	od	v_2	v_{11}	v_5	v_6	v_7	v_2	v_5	v_5	v_5
v_7	v_{12}	v_5	v_{12}	v_5	v_7	v_7	v_5	v_5	v_{12}	v_{12}
v_8	od	v_2	od	v_5	v_2	v_5	v_8	v_9	v_2	v_5
v_9	od	v_2	od	v_5	v_5	v_5	v_9	v_9	v_5	v_5
v_{11}	v_{12}	v_5	v_{11}	v_{12}	v_5	v_{12}	v_2	v_5	v_{11}	v_5
v_{12}	v_{12}	v_5	v_{12}	v_{12}	v_5	v_{12}	v_5	v_5	v_{12}	v_{12}

This table has several undefined terms. Hence W is NeutroAlgebra of subsemigroup under usual × and is not a partial algebra under axiom of product of subsemigroups is different from the operation product defined on the semigroup S.

Now we obtain the Cayley table of W under usual sum of the subsemigroups.

Table 8: Cayley table of W under +

$+$	v_1	v_2	v_3	v_5	v_6	v_7	v_8	v_9	v_{11}	v_{12}
v_1	v_1	od	v_1	od	od	od	od	od	od	od
v_2	od	od	od	v_5	od	od	v_5	v_5	v_{11}	od
v_3	v_1	od	od	od	v_2	od	od	od	v_5	od
v_5	od	v_5	od	v_5	v_{12}	od	v_5	v_5	v_{11}	od
v_6	od	od	v_2	v_{12}	v_5	od	od	v_5	v_5	od
v_7	od	od	od	od	od	od	od	od	od	od
v_8	od	v_5	od	v_5	od	od	od	v_5	v_{11}	od
v_9	od	v_5	od	v_5	v_5	od	v_5	v_5	v_{11}	od
v_{11}	od	v_{11}	v_5	v_{11}	v_5	od	v_{11}	v_{11}	v_5	od
v_{12}	od	od	od	od	od	od	od	od	od	od

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We see W is a NeutroAlgebra of subsemigroups under usual addition of subsemigroups.

Example 3.2. Let S = { Z_{15} , ×} be the semigroup of modulo integers under product.

The subsemigroups of S are V = {{0}, { Z_{15} }, {0, 6, 10}, {0, 10}, {0, 6}, {6}, {10}, {14}, {1, 14}, {1, 14, 0}, {1, 6}, {1, 10}, {1, 0, 10}, {0, 1, 6}, {{1, 0, 5, 10}, {0, 1}, {1}, {9, 3, 12, 6}, {9, 3, 12, 6, 0}, {3, 1, 9, 12, 6}, {1, 0, 3, 6, 9, 12}, {4, 1}, {4, 1, 0}, {2, 4, 8, 1}, {0, 1, 2, 4, 8}, {7, 4, 13, 1}, {1, 0, 7, 4, 13}, {1, 11} and so on}.
Now we can select from this V a subset of nontrivial subsemigroups (say) in this case; T = {{10}, {6}, {0, 5, 10}, {0, 6, 10}, {5, 10}} ⊆ V. The Cayley table of T under usual + and product is as follows.

Table 9: Table of T under □

×	{6}	{10}	{0, 6, 10}	{5, 10}	{0, 5, 10}
{6}	{6}	od	od	od	od
{10}	od	{10}	od	{5, 10}	{10, 5, 0}
{0, 6, 10}	od	od	{0, 6, 10}	{0, 5, 10}	{0, 5, 10}
{5, 10}	od	{5, 10}	{0, 5, 10}	{5,10}	{0, 5, 10}
{10, 5, 0}	od	{10, 0, 5}	{0, 5, 10}	{0,5,10}	{0, 5, 10}

T under usual product of subset of subsemigroups from V the semigroups is a NeutroAlgebra. Now we give the Cayley table of T under + in the following table.

Table 10: Table of T under +

+	{6}	{10}	{0, 6, 10}	{5, 10}	{0, 5, 10}
{6}	ud	ud	ud	ud	ud
{10}	ud	{10}	ud	od	{10, 5, 0}
{0, 6, 10}	ud	od	ud	ud	ud
{5, 10}	ud	od	ud	{10, 0,5}	{0, 5, 10}
{10, 5, 10}	ud	{0, 5, 10}	ud	{5, 0, 10}	{0, 5, 10}

Clearly T is a NeutroAlgebra of subsemigroups under the usual operation of addition of subsemigroups.
Now we define NeutroAlgebra of subsemigroups of a semigroup S under product.

Definition 3.3. *Let {S, ×} be a commutative semigroup. W be the collection of all proper subsemigroups of S, that is W does not contain the zero or identity subsemigroup and the whole subsemigroup S.*
Now define the usual product of two subsemigroups A and B of W as follows.

$$A \times B = \{a \times b \,/\, \forall\, a \in A,\ \forall\, b \in B\}.$$

If A × B ∈□ W we say the product is defined. If A × B ∉ W we declare the product is outer defined. If in W under usual product (or a proper subset of W under usual product) we have outer defined elements then we call {W, ×} (or a proper subset say {V, ×}; V ⊆ W) as a NeutroAlgebra of subsemigroups under the usual product.

Let A, B ∈ W, we define sum '+' of two subsemigroups A and B as follows:

$$A + B = \{a + b \,/\, \forall\, a \in A \text{ and } \forall\, b \in B\}.$$

If W under usual sum (or a proper subset of W under usual sum) is not defined for some subsemigroups in W then we say the product is outerdefined and if A + B ∈ W we say sum is defined. We call {W, +} as the NeutroAlgebra of subsemigroups of S under sum.

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We have provided two examples of the same.

Now we can give the existence theorem of NeutroAlgebra of subsemigroups under usual product in the semigroups S = { Z_n , ×}, n a composite integer.

Theorem 3.4. *Let S = { Z_n , ×} be a semigroup under ×, n is a composite number; $4 \leq n < \infty$. Let W be the collection of all non-trivial subsemigroups excluding {0} and Z_n .*

i. *{W, ×} is a NeutroAlgebra of subsemigroups of S.*
ii. *{W, +} is a NeutroAlgebra of subsemigroups of S.*

Proof. Given W is a collection of all nontrivial subsemigroups. That is W does not contain the {0}, the trivial subsemigroup and the whole semigroup Z_n . Also given n is a composite number.

Since W contains subsemigroups P_1, P_2 of orders p_1 and p_2 where p_1 and p_2 are primes such that n = $p_1 p_2$ where o($P_1 P_2$) = n then $P_1 \times P_2$ = {0} and {0} ∉ W; and $P_i \times P_i = P_i$ for all $P_i \in W$; so W has outer defined elements so {W, ×} is a NeutroAlgebra of subsemigroups under the usual product. Hence the claim. If say n = p_1, …, p_t then we have m and r numbers such that m × r = 0(mod n). So that whatever n be we have subsemigroups which are outer defined.

Now to prove {W, +} is the NeutroAlgebra of subsemigroups under addition +, we have to prove there are two subsemigroups I_1, I_2 ∈ W such that $I_1 + I_2$ = { Z_n }.

If we take any subsemigroup with 1 ∈ I_1, in W and any other subsemigroup I_2 ∈ W we see by sum of subsemigroups $I_1 + I_2 = Z_n$ ∉ W; hence the claim.

In the following we analyze the semigroups built using positive integers with zero.

Let $Z^* = Z^+ \cup \{0\}$, Z^* is a semigroup under +. In fact { Z^* , +} is a monoid with 0 as its additive identity; for every x ∈ Z^+, x + 0 = 0 + x = x.

We give the following example. Consider $Z^+ \cup \{0\}$ the positive integers with zero; $Z^* = Z^+ \cup \{0\}$.

Example 3.5. Let { Z^* , +} be the monoid under +. The subsemigroups of { Z^* , +} are $H_1 = \langle 2 \rangle$ = {2, 4, 6, 8, …, ∞}, $H_2 = \langle 3 \rangle$ = {3, 6, …, ∞}, $H_3 = \langle 5 \rangle$ = {5, 10, …, ∞}, $H_4 = \langle 7 \rangle$ = {7, 14, , …, ∞}, …, $H_p = \langle p \rangle$ = {p, 2p, , …, ∞}.

Example 3.6. Let { Z^* , +} be the semigroup. Let P_1 = {2, 4, 6, 8, 10, …, 2n, …, ∞} = ⟨2⟩ and P_2 = {3, 6, 9, 12, …, 3n, …, ∞} = ⟨3⟩, be two subsemigroups of { Z^* , +} generated by 2 and 3 respectively.

Now we show { P_1, P_2 } = P is a set of subsemigroups which is an AntiAlgebra under addition of subsemigroups.

Table 11: P under +

+	P_1	P_2
P_1	od	od
P_2	od	od

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$S_1 = P_1 + P_1 = \{2, 4, 6, 8, \ldots, \infty\} + \{2, 4, 6, 8, \ldots, \infty\}$
$= \{4, 6, 8, 2, 14, \ldots, \infty\}$
$= \{2 + 2n \mid n = Z^+\}$.

So it is a subsemigroup generated differently (not cyclic) so is outer defined in P.

$S_3 = P_1 + P_2 = \{2, 4, 6, \ldots, \infty\} + \{3, 6, 9, 12, \ldots, \infty\}$
$= \{5, 7, 8, 10, 11, 12, 13, 14, 15, \ldots \infty\} \neq P_1$ or P_3 in fact $P_1 + P_1$ does not generate a sub semigroup. In fact, it is not a subsemigroup of $\{Z^*, +\}$.

Consider $P_2 + P_2 = S_2$

$S_2 = \{6, 9, 12, \ldots, \infty\} = \{6 + 3n / n = 0, 1, 2, \ldots, \infty\}$ so $P_2 + P_2$ is outer defined in P.

$S_3 = P_1 + P_2 = P_2 + P_1$
$= \{2, 4, 6, \ldots, \infty\} + \{3, 6, \ldots, \infty\} + \{3, 6, \ldots, \infty\}$
$= \{5, 7, 8, 9, 10, \ldots, \infty\}$.

S_3 is outer defined in P

Thus $P = \{P_1, P_2\}$ is a Anti Algebra of subsemigroups under +.

If S = {collection of all subsemigroups of $\{Z^*, +\}$ which are non-trivial and generated by prime elements of Z^+}. Then {S, +} is an AntiAlgebra of infinite order of subsemigroups of $\{Z^*, +\}$.

If M = {collection of all proper submonoids of $\{Z^*, +\}$ which are generated by primes in Z^+} then also {M, +} is only an a NeutroAlgebra of submonoids of $\{Z^*, +\}$.

In fact, these submoniods are also ideals of $\{Z^*, +\}$. If $P_1 = \{0, 2, 4, \ldots, \infty\}$ and $P_2 = \{0, 3, 6, 9, \ldots, \infty\} \in M$ then $P_1 + P_1 = P_1 \in M$ and $P_2 + P_2 = P_2 = P_2 \in$ □□.

In fact, for any $P_i \in M$; $P_i + P_i \in M$.

So M is a NeutroAlgebra of submonoids of $\{Z^*, +\}$.

Now we proceed onto study the subsemigroups of $\{Z^*, +\}$ under product.

Let $P = \{P_1, P_2\}$ we give the Cayley table of P under × in the following

Table 12: Table of P under ×

×	P_1	P_2
P_1	od	od
P_2	od	od

So P under product is only an AntiAlgebra of subsemigroups.

For $P_1 \times P_1 = \{2, 4, 6, \ldots, \infty\} \times \{2, 4, 6, \ldots, \infty\}$
$= \{4, 8, 12, \ldots, \infty\} \neq P_1$ or P_2 but outer defined in P.
$P_2 \times P_1 = \{2, 4, 6, ., \infty\} \times \{3, 6, 9, \ldots, \infty\}$
$= \{6, 12, 24, 18, \ldots, \infty\} \neq P_1$ or P_2

On similar lines we can prove $P_2 \times P_2 \notin P$.

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So {P, ×} is only a AntiAlgebra of subsemigroups under product.

If we take N = {collection of all subsemigroups generated by primes in Z^+, excluding {0} the trivial subsemigroup}. Then {N, ×} is only a AntiAlgebra under product.

However, if R = {collection of all subsemigroups generated by elements of Z^+ }, then R under × is a NeutroAlgebra of subsemigroups of { Z^* , +}. Further R under + is also a NeutroAlgebra of subsemigroups of { Z^* , +}.

In view of this we have the following theorem.

Theorem 3.7. *Let S = { Z^* , +} be the monoid of infinite order where $Z^* = Z^+ \cup \{0\}$*

Let M = {collection of all subsemigroups generated by a non-zero prime p $\in Z^+$ }. Then {M, +} is an AntiAlgebra of subsemigroups under + of { Z^* , +}

{M, ×} is also an AntiAlgebra of subsemigroups under + of S = { Z^* , +}.

If N = {collection of all nontrivial subsemigroups of S generated by n $\in Z^+ \setminus \{1\}$} then {N, +} is also an AntiAlgebra of subsemigroups of S.

i. *{N, ×} is only a semigroup of nontrivial subsemigroup s. But every subset of {N, ×} is a NeutroAlgebra or an AntiAlgebra under × depending on the subset chosen from N.*
ii. *If we take M ∪ {0} = M^* the improper zero subsemigroup then {M^*, +} is a NeutroAlgebra of subsemigroups of S and {M^*, ×} is also a NeutroAlgebra of subsemigroups of S.*
iii. *If $N^* = N \cup \{0\}$ then {N^*, +} is a NeutroAlgebra of subsemigroups of S under +.*
iv. *{N*, ×} is also a NeutroAlgebra of subsemigroups of S under ×.*

Proof: Proof of (i) Suppose if $P_1 = \langle p\rangle$ and $P_2 = \langle q\rangle$ be two subsemigroups of S generated by the primes p and q. Take M = { P_1, P_2 }; Clearly $P_1 + P_2 = \langle p + q, p + 2q, 2p + q \ldots \rangle \notin$ M; $P_1 + P_1 = \langle 2p, 3p, \ldots\rangle \notin$ M. Thus for any P_i, $P_j \in$ M, $P_i + P_j$ and $P_i + P_i$ and $P_j + P_j \notin$ M. Hence {M, +} is only an AntiAlgebra of subsemigroups. Hence (i) is true and clearly from Table 11 which is as follows.

Table 13: Table of M under +

+	P_1	P_2
P_1	od	od
P_2	od	od

Thus {M, +} is only an AntiAlgebra of subsemigroups of S = { Z^* , +}.

Table 14: Table of M under □

×	P_1	P_2
P_1	od	od
P_2	od	od

Proof of (ii). Now consider M under ×, clearly every subsemigroup under product is a subsemigroup generated by an non-prime number so product of any two subsemigroups is not in M. Hence the claim (ii) (Refer Table 12).

Proof of (iii) Since {0} ∉ N we see if S_1 and S_2 are two subsemigroups in N ($S_1 \neq S_2$ or $S_1 = S_2$) we have $S_1 + S_2 \notin$ N as it is not generated by a n $\in Z^+ \setminus \{1\}$. Hence no sum in N is again in N; hence N is only a Anti Algebra.

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Proof of (iv) We see {N, ×} is a semigroup for every pair S_i, S_j ∈ N; $S_i \times S_j$ ∈ N. However no proper subset W of N is a subsemigroup in general as $S_i \times S_j \notin$ W. For if $S_1, S_2, \ldots, S_t$ are subsemigroups under □ generated by $n_1, n_2, \ldots, n_t$ with $n_1 < n_2, < \ldots < n_t$ then $n_t \times n_t \notin$ W and some product does not belong to W and in some cases no product can belong to W so {W, ×} only be a NeutroAlgebra or AntiAlgebra depending on the subset W of N. We will illustrate this by two examples.

Given – S = { Z^*, +} be the semigroup. Suppose V = { $S_1 = \langle 2\rangle$, $S_2 = \langle 4\rangle$} be two subsemigroup of S. We find V under × which is represented by the following Cayley table.

Table 15: Table of V under □

×	s_1	s_2
s_1	od	od
s_2	od	od

So {V, ×} is only a AntiAlgebra of subsemigroups of S under ×.
Suppose T = { $S_1 = \langle 3\rangle$ and $S_2 = \langle 4\rangle$} then the Cayley table of T under × is as follows.

Table 16: Table of T under +

+	s_1	s_2
s_1	od	od
s_2	od	od

Hence in this case {T, +} is a AntiAlgebra of subsemigroups of S. Thus, proof of (iv) is complete.

Proof of (v): If N^* = N ∪ {0} then clearly {N^*, +} is a NeutroAlgebra of subsemigroup of S as S_i + {0} = $S_i \in N^*$. Hence claim (v) is true.

Proof of (vi): {N^*, +} is a NeutroAlgebra of subsemigroups under + as S_1 + {0} = S_1 for all $S_1 \in N^*$ and {0} ∈ N^*. Hence proof of (vi).

Proof of (vii): {N^*, ×} will also be a NeutroAlgebra of subsemigroups under × as S_i × {0} = {0} for all $S_i \in N^*$ and {0} ∈ N^*. Hence proof of (vii).
Now we prove some results about $Z^* = Z^+ \cup \{0\}$ under ×.

We provide some examples of them first.
Example 3.8. Let P = { Z^*, ×} be the semigroup under × or a monoid (with multiplicative identity 1).

Clearly the trivial subsemigroups of P are {0}, {1} and Z^*.

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We see if T = {⟨2⟩ and ⟨7⟩} be two proper subsemigroups of P. Clearly the Cayley table of T under sum + is as follows.

Table 17: Table of T under □

+	⟨2⟩	⟨7⟩
⟨2⟩	od	od
⟨7⟩	od	od

{T, +} is a AntiAlgebra of subsemigroups under sum. Consider {T, ×}; the Cayley table of T under × is as follows.

Table 18: Table of T under □

×	⟨2⟩	⟨7⟩
⟨2⟩	od	od
⟨7⟩	od	od

However, if we take the submonoid R_1 = ⟨2⟩ ∪ {1} and R_2 = ⟨7⟩ ∪ {1}. Let M = { R_1 , R_2 } be the submonoid set M under + is represented by the following Cayley Table.

Table 19: Table of M under +

+	R_1	R_2
R_1	od	od
R_2	od	od

Thus {M, +} is only an AntiAlgebra of submonoids of P under +. Consider {M, ×}, the Cayley table of {M, ×} under × is as follows.

Table 20: Table of M under □

×	R_1	R_2
R_1	R_1	od
R_2	od	R_2

{M, ×} is only a NeutroAlgebra of submonoids of P.

Now we prove the following theorem in case of P under ×.

Theorem 3.9. *Let P = { Z^* , ×} be the semigroup (infact a monoid) under usual product.*

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S = {collection of all subsemigroups of P generated by n ∈ Z^+ \ {1} under ×}. S is an AntiAlgebra of subsemigroups under +.

i. *{S, ×} is a AntiAlgebra of subsemigroups under product.*
ii. *Let M = {collection of all subsemigroups of P including {0}, {1}}.*
   a. *Then {M, +} is a NeutroAlgebra of subsemigroups of P under +.*
   b. *{M, ×} is a NeutroAlgebra of subsemigroups of P under ×.*

Proof: Proof of (i): Given S = {collection of all subsemigroups of P generated by n ∈ Z^+ \ {1} under product}. Clearly if A = ⟨2⟩ and B = ⟨4⟩ then A + B ∉ S

Thus

A = {2, 4, 8, 16, 32, …} and B = {4, 16, 64, …}

A + B = {2, 4, 8, 16, 32, …, ∞} + {4, 16, 64, 256, …, ∞}

A + B = {6, 8, 12, 18, 20, 24, 32, 36, …}.

Clearly A + B cannot be generated by a n ∈ Z^+

Even if A = ⟨2⟩, A + A is not generated under × as a subsemigroup as

A + A = {4, 6, 8, 10, 12, 18, 20, …}. Thus {S, +} is only an AntiAlgebra of subsemigroups of P under +. Hence the result.

Proof of (ii): Consider {S, ×} if S_1, S_2 ∈ S we see $S_1 \times S_2$ ∉ S even if S_1 is a subsemigroup of S_2 or S_2 is a subsemigroup of S_1.

For if S_1 = {3, 9, 27, …} and S_2 = {27, 27^2, …} then $S_1 \times S_2$ = {81 = 27 □ 3, 27 × 3^2, …,}

so $S_1 \times S_2$ ∉ S as it cannot be generated by as integer but $S_1 \times S_2 \neq S_1$ or $S_1 \times S_2 \neq S_2$

But however if S_1 = {2, 4, 8, …} and S_2 = {3, 9, 27, …}; $S_1 \times S_2$ = {6, 18, 12, …} ∉ S; as it cannot be generated by an integer in Z^+ \ {1} under product. So {S, ×} is only a AntiAlgebra of subsemigroups under ×. Proof of (ii) is true.

Proof of (iii): (a) As M includes {0}, {1} and {P} we see in case of {M, +}; 0 + X = X where X is a subsemigroup. Hence {M, +} is a NeutroAlgebra of subsemigroups.

(b) Also for all X ∈ M; X □ {0} = {0} and X □ {1} = X so (iii) b is true.

Hence (iii) is true.

Now based on the above theorems we have the following results.

4. NeutroAlgebra of Ideals in the semigroups $\{Z_n, \times\}$, {Z^+, □} and {Z^+ ∪ {0}, □}

In this section we find the ideals of semigroups $\{Z_n, \times\}$, {Z^+, +}, {Z^+ ∪ {0}, +}, {Z^+, □} and {Z^+ ∪ {0}, □} and study whether they form a NeutroAlgebra of ideals or AntiAlgebra of ideals under + and □ of ideals. We give several examples in case of ideals in these semigroups.

We have recalled the definition of ideals in case of semigroups in section 2 of this paper. Further every ideal is a subsemigroup but the converse is not true. All semigroups are assumed to be commutative.

Example 4.1. Let S = {Z_{12}, □} be the semigroup of modulo integers under □. We list out all the ideals of S. We define {0} and Z_{12} as trivial ideals of S.

Consider the ideals of S.

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I_1 = {0, 4, 8} is a nontrivial ideal of S.

However, H_1 = {0, 1, 4, 8} is a nontrivial subsemigroup of S but is not an ideal of S.

I_2 = {0, 4, 6, 8} is an ideal of S.

I_3 = {0, 6} is again an ideal of S.

I_4 = {0, 6, 3, 9} is an ideal of S,

I_5 = {0, 2, 4, 8, 6, 10} is an ideal of S.

Thus I = { I_1, I_2, I_3, I_4, I_5 } are the nontrivial ideals of { Z_{12}, □}.

The Cayley table of I is as follows.

Table 21: Table of I under +

+	I_1	I_2	I_3	I_4	I_5
I_1	I_1	I_5	I_5	od	I_5
I_2	I_5	I_5	I_5	I_4	I_5
I_3	I_5	I_5	I_3	I_4	I_5
I_4	od	I_4	I_4	I_4	od
I_5	I_5	I_5	I_5	od	I_5

Clearly {I, +} is only a NeutroAlgebra of ideals of S.
We see all subsets of I will be NeutroAlgebra of ideals. We do not have an AntiAlgebra of ideals using subsets of I they are NeutroAlgebra of ideals or just a semigroup of ideals of S for $I_j + I_j = I_j$ for all $I_j \in I$.
We will illustrate this situation by the following example.

Consider the subset J = { I_2, I_4 } $\subseteq$ I. The Cayley Table is as follows.

Table 22: Table of J under +

+	I_2	I_4
I_2	od	od
I_4	od	I_4

So {J, +} is only a Neutro subalgebra of the NeutroAlgebra of {I, +}.

Let K = {I_1, I_3, I_5} be a proper subset of I. The Cayley table of K is as follows.

Table 23: Table of K under +

+	I_1	I_3	I_5

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I_1	I_1	I_5	I_5
I_3	I_5	I_3	I_5
I_5	I_5	I_5	I_5

We see {K, +} is a semigroup of ideals of S.
However, we do not get any AntiAlgebra of ideals under + in case of ideals of this semigroup $\{ Z_{12}, □\}$.

Example 4.2. Let $S = \{ Z_8, □\}$ be the semigroup of modulo integers. $H_1 = \{0, 4\}$, $H_2 = \{0, 2, 4, 6\}$ are the only ideals of S. The Cayley table of $P = \{ H_1, H_2 \}$ is as follows.

Table 24: Table of P under +

+	H_1	H_2
H_1	H_1	H_2
H_2	H_2	H_2

P is only a semigroup of ideals.

In view of all these we have the following result.

Theorem 4.3. *Let $S = \{ Z_{p^n}, □\}$ be the semigroup. (p a prime),*

i. *There are only (n – 1) number of ideals of orders $p^{n-1}, p^{n-2}, \ldots, p$.*
ii. *In $I_1, I_2, \ldots, I_{n-1}$ number of ideals of order $p^{n-1}, p^{n-2}, \ldots, p$ respectively then $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_{n-1}$.*
iii. *The collection $B = \{ I_1, \ldots, I_{n-1}\}$ forms a semigroup both under + and □.*

Proof: Consider the semigroup Z_{p^n} under □.

Let

$$I_1 = p^{n-1} □ Z_{p^n} = \{0, p^{n-1}, 2.\ p^{n-1}, 3.\ p^{n-1}, \ldots, (p-1).p^{n-1}\}$$

is an ideal of order p in $S = \{ Z_{p^n}, □\}$. Let

$$I_2 = p^{n-2} Z_{p^n} = \{0, p^{n-2}, 2p^{n-2}, 3p^{n-2}, \ldots, 1(p-1)\ p^{n-2}\ p.\ p^{n-2}\ (p^{n-1}), 2p^{n-1}, 3p^{n-1}, \ldots (p-1)\ p^{n-1}\}$$

is an ideal of S. We see $I_1 \subsetneq I_2$.

Consider

$$I_3 = p^{n-3} Z_{p^n}$$

$$= \{0, p^{n-3}, 2p^{n-3}, \ldots, (p-1)\ p^{n-3}, p^{n-2}, 2p^{n-2}, \ldots, (p-1)\ p^{n-2}, p^{n-1}, 2p^{n-1}, \ldots, (p-1)\ p^{n-1}\}$$

is an ideal of S.

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We see $I_2 \subsetneq I_3$ and so on.

$$I_{n-2}\ p^2\ Z_{p^n} = \{0,\ p^2, 2p^2, 3p^2, \ldots, (p-1)\ p^2,\ p^3, 2p^3, \ldots, (p-1)\ p^3, \ldots,\ p^{n-1}, 2p^{n-1}, \ldots,\ (p-1)p^{n-1}\}$$

is an ideal of S and

$$I_{n-1}\ p\ Z_{p^n} = \{0, p, 2p, \ldots, (p-1)\,p;\ p^2, 2p^2, \ldots, (p-1)\ p^2, \ldots,\ p^{n-1}, 2p^{n-1}, \ldots, (p-1)\,p^{n-1}\}$$

is again an ideal of S. We see $p^2\ Z_{p^n} \subseteq p\ Z_{p^n}$.

$$I_{n-2} \subsetneq I_{n-1}$$

So $I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_{n-2} \subsetneq I_{n-1}$ from the above proved (or illustrated equations).

Hence (i) and (ii) of the theorem is proved.
Because of the containment relation we see if B = $\{I_1, I_2, \ldots, I_n\}$.
It is easily verified {B, +} and {B, □} are just ideals of semigroups.

Thus, from this result we do not take semigroup $\{Z_{p^n}, □\}$ to study NeutroAlgebra structure of ideals as the set of ideals is a semigroup under + and □. That is semigroups formed by modulo integers Z_{p^n}, p a prime under × does not give way to NeutroAlgebra of ideals.

Next, we study or analyze the case of Z_{pq} (q and p district primes) first by examples. Then we will generalize to the case of Z_n, where n is a composite number of the form n = $p_1^{t_1}, p_2^{t_2}, \ldots p_m^{t_m}$ where each p_i is an odd prime and $t_i \geq 1$; $1 \leq i \leq m$.

***Example 4.4*:** Let S = $\{Z_{15}, □\}$ be the semigroup under □ modulo 15. The ideals of S are I_1 = {0, 5, 10} and I_2 = {0, 3, 6, 9, 12} are the only nontrivial ideals of S.
Let B = $\{I_1, I_2\}$ be the Cayley table of B is as follows.

Table 25: Table of B under +

+	I_1	I_2
I_1	I_1	od
I_2	od	I_2

Clearly {B, +} is a NeutroAlgebra of ideals of S. However {B, □} is not a NeutroAlgebra only a semigroup of ideals of S.

Example 4.5. Let S = $\{Z_{30}, □\}$ be the semigroup of modulo integers.
The nontrivial ideals of S are B = { I_1 = {0, 15}, I_2 = {0, 5, 10, 15, 20, 25}, I_3 = {0, 3, 6, 9, 12, 15, 18, 21, 24, 27}, I_4 = {0, 6, 12, 18, 24}, I_5 = {0, 10, 20} and I_6 = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28}}. The Cayley table of B = $\{I_1, I_2, \ldots, I_6\}$ under sum is only a NeutroAlgebra of ideals.

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Table 26: Table of B under +

+	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_2	I_3	I_3	I_2	od
I_2	I_2	I_2	od	od	I_2	od
I_3	I_3	od	I_3	I_3	od	od
I_4	I_3	od	I_3	I_4	I_3	I_6
I_5	I_2	I_2	od	I_6	I_5	od
I_6	od	od	od	I_6	od	I_6

Clearly {B, +} is only a NeutroAlgebra of ideals of S.
But {B, □} is only a semigroup of ideals of S.
In view of all these we suggest a few open problems in the last section of this paper.
Next we study the NeutroAlgebraic structure of ideals in $\{Z^+, □\}$, $\{Z^+ \cup \{0\}, □\}$ $\{Z^+, +\}$ and $\{Z^+ \cup \{0\}, +\}$.

Result 4.6. Let $S_1 = \{Z^+, □\}$ be the semigroup under □.
The ideals of S_1 are $I_1 = \langle 2\rangle = \{2, 4, 6, 8, \ldots, \infty\}$, $I_2 = \langle 3\rangle = \{3, 6, 9, 12, \ldots,\infty\}$ and so on; $I_p = \langle p\rangle = \{p, 2p, \ldots, \infty\}$ and so on, where p is a prime in Z^+.
If $W_1 = \{I_1, I_2, \ldots, I_p, \ldots, \infty\}$ be the ideals of S_1 we see $\{I_1, +\}$ under + is only an AntiAlgebra of ideals of S_1.
On the contrary if we take the ideals of $S_1 = \{Z^+, □\}$ generated by every $n \in Z^+ \setminus \{1\}$ and if W_2 = {collection of all ideals of S generated by any $n \in Z^+ \setminus \{1\}$}, then if we consider + as the operation on W_2 we see if J_1 and J_2 are ideals of W_2 then $J_1 + J \notin W_2$.
Likewise, $J_1 + J_2 \notin W_2$. Hence, $\{W_2, +\}$ is only an AntiAlgebra of ideals of S_1.

Result 4.7. Let $S_2 = \{Z^+, □\}$ be the semigroup. Now consider W_1 under □ is also an AntiAlgebra.
If on the other than we include the element zero in all the ideals together with the zero ideal by taking $\{Z^+ \cup \{0\}, +\}$ and $\{Z^+ \cup \{0\}, □\}$ we can easily prove the set of ideals {0}. $W_1^* = \{\langle p\rangle$ / p a prime and $\{0\}\}$ and $W_2^* = W_2 \cup \{0\} = \{\langle n\rangle$ / n any number in $Z^+ \setminus \{1\}\}$ are only NeutroAlgebras under both + and □. Thus, we have both $\{W_1, □\}$ and $\{W_1, +\} \subseteq S$ are only AntiAlgebra of ideals of S.
Only inclusion of the zero ideal {0} and S will make them a NeutroAlgebra of ideals.

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5. NeutroAlgebra of Groups in the Smarandache Semigroups.

In this section we for the first time study the NeutroAlgebraic structure of the groups contained in a Smarandache semigroup. We obtain several interesting results in this direction.
In the first place we wish to keep on record that all semigroups are not in general Smarandache semigroups. However, all Smarandache semigroups are semigroups. We illustrate this situation of semigroups which are not Smarandache groups by examples.

Example 5.1. Consider the semigroups. Let $Z^* = Z^+ \cup \{0\}$. S = { Z^*, +} and S′ = { Z^*, □} be two semigroups of infinite order. Clearly both S and S′ are not Smarandache semigroups though both S and S′ are monoids. For if P = {⟨s⟩} where s ∈ S, P can never be a group under + as S has no additive inverse though zero '0' is the additive identity of S. For no a ∈ S (a ≠ 0) we have a, b in S such that a + b = 0. Similarly, S has no groups under □. For 1 ∈ S′ is the multiplicative identity but no element x; x ≠ 1 in S′ has an inverse x′ in S′ such that x □ x′ = 1.
Hence the claim; in general, all semigroups are not Smarandache Semigroups.

We will give examples of Smarandache semigroups.

Example 5.2. Let S = { Z_{16} , □} be a semigroup under □ modulo 16. We see T = {1, 15} ⊆ S is clearly a group given by the following Cayley table.

Table 27: Table of T under □

□	1	15
1	1	15
15	15	1

We have the following result which clearly shows the class of Smarandache semigroups is nonempty.

Theorem 5.3. *Let { Z_n , □} be a semigroup of modulo integers (n a composite number) { Z_n , □} is a Smarandache semigroup.*

Proof: Clearly B = {1, (n – 1)} ⊆ Z_n is a group under □ so; { Z_n , □} is a Smarandache semigroup.

Now our study is whether the collection of all groups in a Smarandache semigroup forms a NeutroAlgebra of groups under □ (and +).

First, we illustrate this situation by some examples.

Example 5.4. Let S = { Z_{20} , □} be a semigroup of order 20. The groups of S are { I_1 = {1, 19}, I_2 = {1, 3, 9, 7}, I_3 = {11, 1}, I_4 = {1, 3, 9, 7, 11, 13, 17, 19}}. Thus, B = { I_1 , I_2 , I_3 , I_4 } is the collection of non-trivial groups under □ modulo 20.
We see if A = { I_1 , I_2 , I_3 } ⊆ B, {A, □} is only a NeutroAlgebra of groups of S. The Cayley table of A is as follows.

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Table 28: Table of A under □

□	I_1	I_2	I_3
I_1	I_1	od	od
I_2	od	I_2	od
I_3	od	od	I_3

Clearly {A, □} is only a NeutroAlgebra groups of the Smarandache semigroup S.

However, it is easily verified {B, □} is a group of semigroups under □ of the Smarandache semigroup S.

Example 5.5. Let S = { Z_{24} , □} be the semigroup under □. The groups of S under the operation □ is given by B = { I_1 = {1, 23}, I_2 = {1, 5}, I_3 = {1, 7}, I_4 = {1, 11}, I_5 = {1, 13}, I_6 = {1, 17}, I_7 = {1, 19}, I_{18} = {1, 5, 7, 11}, I_9 = {1, 5, 13, 17}, I_{10} = {1, 5, 19, 23}, I_{11} = {1, 7, 13, 19}, I_{12} = {1, 7, 17, 23} and so on. I_K = {1, 5, 7, 11, 13, 17, 19, 23}}. We see there are subsets of B which are only NeutroAlgebra of groups of S. However, no subset of B is an AntiAlgebra of groups of S.

In view of all these we can prove the following result.

Theorem 5.6. *Let S = { Z_n , □} be a semigroup (n a composite number)*

i. *S is a Smarandache semigroup.*
ii. *The collection of groups H in S or any subset of H is never an AntiAlgebra of groups of S.*
iii. *Every subset of H or H is a NeutroAlgebra of groups of S or a semigroup of groups of S.*

Proof. Proof of (i) Given S = { Z_n , □} is semigroup. A = {1, (n – 1)} $\subseteq$ S is group given by the following Cayley Table.

Table 29: Table of A under □

□	1	n – 1
1	1	n – 1
n – 1	n – 1	1

Thus S is a Smarandache semigroup. Hence the claim of (i).

Proof of (ii) Given H = { $G_1, G_2, \ldots, G_p$) be the collection of groups of S.

We know $G_i \times G_i = G_i$ for every $G_i \in$ H; $1 \le i \le p$.

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Table 30: Table of H under □

+	G_1	G_2	⋯	G_p
G_1	G_1	…	⋯	⋯
G_2	od	G_2	od	od
⋮	⋮	⋮	⋮	⋮
G_p	…	…	⋯	G_p

Clearly as all the diagonal elements are defined, we see (H, □) cannot be an AntiAlgebra of groups of S.
Hence the proof of (ii).
Proof of (iii). Either H can be semigroup of groups of S or a NeutroAlgebra of groups of S.
Note If S = { Z_p , □}, p a prime then we see the only group of $G_1 = Z_p \setminus \{0\}$; $G_2 = \{p - 1, 1\}$. Clearly H = { G_1, G_2 } is a semigroup of group of the semigroup S.

6. Conclusions

Here we proceed on to record the outcomes of the research carried out in this paper. Here we have found the NeutroAlgebra or AntiAlgebra of the subsemigroups of the semigroup built using { $Z^+ \cup \{0\}, \times$}, { $Z^+ \cup \{0\}, +$} and { Z_n , ×}; (n a composite number). The NeutroAlgebra / AntiAlgebra of ideals of a semigroup is studied. Conditions for the existence of the same is obtained.
Finally, only in the case of Smarandache semigroup; the groups of the Smarandache semigroup are only the NeutroAlgebra of groups (under × only) of a Smarandache semigroup. No collection of groups of a Smarandache semigroup is an Anti Algebra.
If S is the set of subsemigroups of the semigroup generated by $n \in Z^+$ then {S, ×} is not a NeutroAlgebra. If N is the set of subsemigroups where the subsemigroups are generated by $p \in Z^+ \setminus \{1\}$, p a prime then we see {N, +} and {N, ×} are only Anti Algebras of subsemigroups; if we take {N ∪ {0}, +} and {N ∪ {0}, ×} then they are NeutroAlgebra of subsemigroups of the semigroup mentioned above.
We proved similar results in the case of Z_n , n a composite number.
For future study we will be taking up the problem of finding the NeutroAlgebra in case of the subsemigroups and ideals in finite and infinite noncommutative semigroups.
Finally, we propose the following open problems.

Problem 6.1. Let M be any semigroup and S be the collection of subsemigroups of M. Suppose {S, +} ad {S, ×} are NeutroAlgebras. Can these NeutroAlgebras be such that every proper subset of M is a NeutrosubAlgebra of S?
Can every proper subset of M be only an AntiAlgebra of M? Justify your claim not for $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$ under + or $Z^+ \cup \{0\}$ and Z_n under ×. But for some other different class of semigroups.
Now in case of ideals of a semigroup we propose the following problem.

Problem 6.2. Let S = { Z_n , □} be a semigroup, where n is a composite number. M= {Collection of non-trivial ideals of S} so that {M, □} is a NeutroAlgebra of ideals under product in S or M has proper subset P such that {P, □} is a NeutroAlgebra of ideals under □ of S. Study the same in case of {M, +} and {P, +}.

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Next, we propose an open problem in case of collection of T groups of a Smarandache semigroup S to be a NeutroAlgebra under product of S.

Problem 6.3. Let S = { Z_n , □} be a Smarandache semigroup, n a composite number. Let T = {collection of all groups under □ in S}. Obtain conditions on n so that {T, □} is a NeutroAlgebra of groups of S under □.

For future study authors will research on the 3 open problems suggested and study these 3 substructures in case of non-commutative semigroups.

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