A Short Note on Some Observable Outputs of Some Linear Discrete Time Systems

Khadija Ben Othman

Umm Al-Qura Univerity, Mekka, Saudi Arabia

Emails: khagijabenothman33@gmail.com

Abstract.

In this paper, we study the problem of controllable and observable output of the type:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t+1)\,; t = 0,1,\dots,T \\ y(t) = Cx(t) \\ x(0) = x_0\,,\ x(T) = x_T \end{cases}$$

On the other hand, we present some new result about the formula of this linear discrete-time system and about controllable out put

Key words: Optimal solution; linear discrete time system; observable.

Introduction.

Observability and controllability are two essential concepts in control theory. These concepts were defined in [3-6]. Many models in the real life and science are controllable, but the corresponding mathematical models maybe not for this goal, we have to take care about choosing the variables which represent the state of the system which can be observable and controllable to build a strong control system.

The most effective tool for the study of these systems are matrices and their representations .

In this paper, we try to find conditions for the controllable of linear discrete – time systems.

Main Discussion:

Definition:

Consider the following system:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t+1)\,; t = 0,1,\dots,T \\ y(t) = Cx(t) \\ x(0) = x_0 = const \\ x(T) = x_T = const \end{cases}$$

Where $x(t) \in R^n\,, u(t) \in R^m\,, y(t) \in R^p$.

A, B, C are matrice of type $p \times n\,, n \times m\,, n \times m$ respectively

Doi : https://doi.org/10.54216/GJMSA.030104
Received: October 02, 2022 Accepted: January 12, 2023

Definition:

The system (1-4) is called out put controllable if we can transform the initial out put $y(0) = Cx_0$ to the final out put $y(t) = Cx_T$.

Remark.

We find an equivalent formula of the system (1-4), we assume that $det(A) \neq 0, det(C) \neq 0$.

For $t = 0, \dots, T-1$, we can write:

$y(0) = Cx(0) = Cx_0$

$y(1) = Cx(1) = C[A(x(0)) + Bu(1)] = CAx_0 + CBu$

$y(2) = Cx(2) = C[Ax(1) + Bu(2)] = CA^2x_0 + CABu(1) + CBu(2)$

.

.

.

.

$y(T) = CA^T x_0 + CA^{(T-1)}Bu(1) + CA^{(T-2)}Bu(1) + \cdots + CBu(T)$

So that we get,

$$\begin{cases} y(1) - CAx_0 = CBu(1) \\ y(2) - CA^2x_0 = CABu(1) + CBu(2) \\ y(3) - CA^3x_0 = CA^2Bu(1) + CABu(2) + CBu(3) \\ \vdots \\ y(T) - CA^T x_0 = CA^{(T-1)}Bu(1) + \cdots + CBu(T) \end{cases}$$

We put:

$$\begin{cases} \beta = y(T) - CA^T x_0 \\ U == \{u(1), V(2), \dots, U(T)\} \\ D = \{CA^{T-1}B, CA^{T-2}B, \dots, CB\} \end{cases}$$

Thus $DU = \beta$.

It is clear that the system (1-4) will be controllable output if and only if $rank(D) = p$.

Example.

Consider the following system:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t+1) \\ u_2(t+1) \\ u_3(t+1) \end{pmatrix}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

We have: $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$D = [CA^2B, CAB, CB] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$rank(D) = 1 < 2$, so it is not output controllable.

Remark.

We make a change on system (1-4) by assuming $|u(t)| \leq 1 \dots (I)$.

The system (1-4) with condition $(I)$is out put controllable if:

Doi : https://doi.org/10.54216/GJMSA.030104
Received: October 02, 2022 Accepted: January 12, 2023

1). $rank[A^{T-1}B, A^{T-2}B, \dots, B] = n.$

2). Eigen values of A has the following property $|\lambda_k| \leq 1$.

3). $rank(D) = p$.

Example.

Consider the following system:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t+1) \\ u_2(t+1) \\ u_3(t+1) \end{pmatrix}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

with $|u(t)| \leq 1$.

We have: $[A^2B, AB, B] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & \frac{3}{2} & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$

$rank[A^2B, AB, B] = 3.$

The eigen values of A are $\lambda_1 = \lambda_2 = 1 , \lambda_3 = \frac{1}{2}$.

Also, $D = \begin{pmatrix} 4 & \frac{5}{2} & 2 & 2 & 1 & 1 \\ 6 & \frac{7}{2} & 3 & 3 & 1 & 2 \end{pmatrix}$, $rank[D] = 2$,

so that the system is out put controllable.

Definition.

A system is called observable if and only if we have the ability to determine the state $x(0)$ by exact list of measurements $y(t)$ for an exact number of steps.

For example if we have the system:

$$\begin{cases} x(t+1) = Ax(t) \,; x(0) = x_0 \ (not\ known) \\ y(t) = Cx(t) \,;\ A_{n\times n} \,, C_{p\times n} \end{cases}$$

Theorem.

The previous system is observable if:

$$rank \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{pmatrix} = n$$

We denote by O to $O = \left[C^t, A^tC^t, \dots, \left(A^{(T-1)}\right)^t C^t\right]$.

Example.

Consider the following:

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \end{pmatrix}, A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, O = (C^t, A^tC^t) = \begin{pmatrix} 1 & 7 \\ 2 & 10 \end{pmatrix}$$

Doi : https://doi.org/10.54216/GJMSA.030104
Received: October 02, 2022 Accepted: January 12, 2023

$rank[O] = 2$, so it is observable.

Conclusion

In this paper, we have studied the problem of controllable and observable output of the type:

$$\begin{cases} x(t+1) = Ax(t) + Bu(t+1)\,;\, t = 0,1,\dots,T \\ y(t) = Cx(t) \\ x(0) = x_{0\,,}\, x(T) = x_T \end{cases}$$

On the other hand, we present some new result about the formula of this linear discrete-time system and about controllable output.

References

[1] GAJIC, Z.; LELIC, M., Modern Control Systems Engineering. Prentice Hall
[2] International, London, 1996, 241–247.
[3] HOU M.; MÜLLER P. C., Design of observers for linear systems with unknown inputs.
[4] IEEE Trans. Automat. Contr., vol. AC-37, 1992, 871-875.
[5] MURTA Y. Optimal Control Methods for Linear Discrete-Time Economic System,
[6] Nagoya Japan, 1982, 197
[7] RAVICHANDRAN, C.S.; SUBHA, R. S.; MANIKANDAN, T., Designing of PID
[8] Controller for Discrete Time Linear System Using Balanced Approach Reduced
[9] Order Model. American Journal of Applied Sciences, Vol. 4, No. 3, 2007, 155-159.
[10] RAVICHANDRAN, C.S.; SUBHA RANI, S.; SUNDARAPANDIAN, V. Simple
[11] Method for Design of Reduced Order Output Feedback Controller for Discrete Time
[12] Linear System. Gest International Transaction on Communication and Signal
[13] Processing, Vol. 9, No.1, 2006, 63-74.
[14] STEFANI ; SHAHIAN ; SAVANT ; HOSTETTER, Design of Feedback Systems,
[15] Oxford University Press, New York, 2002, 650–652.

Doi : https://doi.org/10.54216/GJMSA.030104
Received: October 02, 2022 Accepted: January 12, 2023