Two Inclusive Subfamilies of bi-univalent Functions

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Abstract

The aim of this article is to establish two new and qualitative subfamilies $\mathcal{F}(\varepsilon, \kappa, \aleph)$ and $\mathcal{G}(\varepsilon, \kappa, \aleph)$ of bi-univalent functions. For functions in these subfamilies, we determine the first two Maclaurin coefficient estimations $|C_2|$ and $|C_3|$, and address the Fekete–Szegö problem. Additionally, we mention some corollaries related to the main results.

Keywords: Analytic function; Univalent and bi-univalent functions; Fekete-Szegö problem.

1 introduction

The study of bi-univalent functions has garnered significant attention in the field of geometric function theory due to their rich mathematical structure and diverse applications. Introducing new classes of these functions not only deepens our theoretical understanding but also opens up avenues for practical problem-solving in fields such as engineering, physics, and complex dynamics. Recently, many researchers have introduced and investigated several interesting subfamilies of bi-univalent functions and they have obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients, see.[2,4,6,7,10–16]

Motivated by the aforementioned works, we propose two innovative subfamilies of bi-univalent functions, denoted as $\mathcal{F}(\varepsilon, \kappa, \aleph)$ and $\mathcal{G}(\varepsilon, \kappa, \aleph)$. These subfamilies are defined through specific constraints and conditions that extend existing frameworks, providing a broader and more nuanced perspective on the behavior and characteristics of bi-univalent functions. By focusing on these new classes, we aim to determine the first two Taylor-Maclaurin coefficient estimations $|C_2|$ and $|C_3|$, which are crucial for understanding the geometric properties and growth rates of these functions. Moreover, we address the Fekete-Szegö problem within the context of these subfamilies, a classical problem in the theory of univalent functions that involves estimating a certain functional of the second and third coefficients. This problem is not only mathematically significant but also essential for applications requiring precise control over the geometric distortion of mappings.

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2 Preliminaries and Definitions

Let Ψ be the family of all analytic and univalent functions within the open unit disk $\Lambda = \{\vartheta \in \mathbb{C} : |\vartheta| < 1\}$ of the form:

$$\ell(\vartheta) = \vartheta + \sum_{n=2}^{\infty} C_n \vartheta^n, \ \ (\vartheta \in \Lambda) \tag{1}$$

which are normalized by $\ell(0) = \ell'(0) - 1 = 0$.

The function $\ell \in \Psi$ has an inverse ℓ^{-1}, that is described by:

$$\ell^{-1}(\ell(\vartheta)) = \vartheta \qquad (\vartheta \in \Lambda)$$

and

$$\varpi = \ell(\ell^{-1}(\varpi)) \qquad \left(|\varpi| < r_0(\ell);\ r_0(\ell) \geq \frac{1}{4}\right)$$

where

$$\ell^{-1}(\varpi) = \varpi - C_2\varpi^2 + (2C_2^2 - C_3)\varpi^3 - (C_4 + 5C_2^3 - 5C_3C_2)\varpi^4 + \cdots . \tag{2}$$

A function ℓ is referred to as bi-univalent in Λ if both ℓ and its inverse function ℓ^{-1} are univalent within the domain Λ. Let Υ denote the family of all bi-univalent functions within the domain Λ and given by (1). For example, the function $\ell(\vartheta) = \frac{\vartheta}{1-\vartheta} \in \Upsilon$ but the function $\ell(\vartheta) = \frac{\vartheta}{1-\vartheta^2} \notin \Upsilon$. For more details regarding to the family Υ, see ([5] and[11]).

In the subsequent analysis, we assume that the function $\ell \in \Upsilon$ is given by (1), with $\hbar = \ell^{-1}$ given by (2), and that $\aleph(\vartheta)$ is an analytic function with a positive real part in Λ, satisfying $\aleph(0) = 1$ and $\aleph'(0) > 0$. Additionally, $\aleph(\Lambda)$ is symmetric with respect to the real axis. Such a function can be expressed as:

$$\aleph(\vartheta) = 1 + U_1\vartheta + U_2\vartheta^2 + U_3\vartheta^3 + \cdots, \ \ U_1 > 0. \tag{3}$$

Definition 2.1. A function ℓ is said to be in the family $\mathcal{F}(\varepsilon, \kappa, \aleph)$ if and only if

$$\left(\frac{\ell(\vartheta)}{\vartheta}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\vartheta\ell''(\vartheta)}{\ell'(\vartheta)}\right) \prec \aleph(\vartheta) \tag{4}$$

and

$$\left(\frac{\hbar(\varpi)}{\varpi}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\varpi\hbar''(\varpi)}{\hbar'(\varpi)}\right) \prec \aleph(\varpi), \tag{5}$$

where $-\pi < \varepsilon \leq \pi$ and $\kappa \geq 1$.

Definition 2.2. A function ℓ is said to be in the family $\mathcal{G}(\varepsilon, \kappa, \aleph)$ if and only if

$$(\ell'(\vartheta))^{\kappa} + \frac{1+e^{i\varepsilon}}{2}(\vartheta\ell''(\vartheta)) \prec \aleph(\vartheta) \tag{6}$$

and

$$(\hbar'(\varpi))^{\kappa} + \frac{1+e^{i\varepsilon}}{2}(\varpi\hbar''(\varpi)) \prec \aleph(\varpi), \tag{7}$$

where $-\pi < \varepsilon \leq \pi$ and $\kappa \geq 1$.

By selecting special values for the parameters ε and κ, numerous subfamilies can be identified, we consider few special cases as follows:

1. For $\varepsilon = 0$ and $\kappa \geq 1$, a function ℓ is said to be in the subfamily $\mathcal{F}_0(\kappa, \aleph) := \mathcal{F}(0, \kappa, \aleph)$ if and only if

$$\left(\frac{\ell(\vartheta)}{\vartheta}\right)^{\kappa} + \frac{\vartheta\ell''(\vartheta)}{\ell'(\vartheta)} \prec \aleph(\vartheta) \ \text{ and } \ \left(\frac{\hbar(\varpi)}{\varpi}\right)^{\kappa} + \frac{\varpi\hbar''(\varpi)}{\hbar'(\varpi)} \prec \aleph(\varpi).$$

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Furthermore, a function ℓ is said to be in the subfamily $\mathcal{G}_0(\kappa,\aleph) := \mathcal{G}(0,\kappa,\aleph)$ if and only if

$$(\ell'(\vartheta))^{\kappa} + \vartheta\ell''(\vartheta) \prec \aleph(\vartheta) \text{ and } (\hbar'(\varpi))^{\kappa} + \varpi\hbar''(\varpi) \prec \aleph(\varpi).$$

2. For $\varepsilon = \pi$ and $\kappa \geq 1$, a function ℓ is said to be in the subfamily $\mathcal{F}_\pi(\kappa,\aleph) := \mathcal{F}(\pi,\kappa,\aleph)$ if and only if

$$\left(\frac{\ell(\vartheta)}{\vartheta}\right)^{\kappa} \prec \aleph(\vartheta) \text{ and } \left(\frac{\hbar(\varpi)}{\varpi}\right)^{\kappa} \prec \aleph(\varpi).$$

Furthermore, a function ℓ is said to be in the subfamily $\mathcal{G}_\pi(\kappa,\aleph) := \mathcal{G}(\pi,\kappa,\aleph)$ if and only if

$$(\ell'(\vartheta))^{\kappa} \prec \aleph(\vartheta) \text{ and } (\hbar'(\varpi))^{\kappa} \prec \aleph(\varpi).$$

3. For $-\pi < \varepsilon \leq \pi$ and $\kappa = 1$, a function ℓ is said to be in the subfamily $\mathcal{F}_1(\varepsilon,\aleph) := \mathcal{F}(\varepsilon,1,\aleph)$ if and only if

$$\frac{\ell(\vartheta)}{\vartheta} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\vartheta\ell''(\vartheta)}{\ell'(\vartheta)}\right) \prec \aleph(\vartheta) \text{ and } \frac{\hbar(\varpi)}{\varpi} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\varpi\hbar''(\varpi)}{\hbar'(\varpi)}\right) \prec \aleph(\varpi).$$

Furthermore, a function ℓ is said to be in the subfamily $\mathcal{G}_1(\varepsilon,\aleph) := \mathcal{G}(\varepsilon,1,\aleph)$ if and only if

$$\ell'(\vartheta) + \frac{1+e^{i\varepsilon}}{2}\left(\vartheta\ell''(\vartheta)\right) \prec \aleph(\vartheta) \text{ and } \hbar'(\varpi) + \frac{1+e^{i\varepsilon}}{2}\left(\varpi\hbar''(\varpi)\right) \prec \aleph(\varpi).$$

Two lemmas that are employed in our proofs are as follows.

Lemma 2.3. *([3])Let the analytic function $\gamma(\vartheta) = 1 + R_1\vartheta + R_2\vartheta^2 + \cdots$ with positive real parts in Λ, then $|R_j| \leq 2$ for each $j \in \mathbb{N}$. This inequality is sharp for all $j \in \mathbb{N}$.*

Lemma 2.4. *([8]) Let $\Theta,\zeta \in \mathbb{R}$ and $\sigma_1,\sigma_2 \in \mathbb{C}$. If $|\sigma_1|, |\sigma_2| < \xi$, then*

$$|(\Theta+\zeta)\sigma_1 + (\Theta-\zeta)\sigma_2| \leq \begin{cases} 2|\Theta|\xi & for\ \ |\Theta| \geq |\zeta|, \\ 2|\zeta|\xi & for\ \ |\Theta| \leq |\zeta|. \end{cases}$$

3 Coefficient Bounds for the subfamilies $\mathcal{F}(\varepsilon,\kappa,\aleph)$ and $\mathcal{G}(\varepsilon,\kappa,\aleph)$

The coefficient estimates and the Fekete–Szegö problem for the functions in the subfamilies $\mathcal{F}(\varepsilon,\kappa,\aleph)$ and $\mathcal{G}(\varepsilon,\kappa,\aleph)$ are provided in this section.

Theorem 3.1. *If $\ell \in \mathcal{F}(\varepsilon,\kappa,\aleph)$, then*

$$|C_2| \leq \sqrt{\frac{2U_1^3}{U_1^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right) - 2\left(e^{i\varepsilon}+\kappa+1\right)^2(U_2-U_1)}}$$

and

$$|C_3| \leq U_1\left(\frac{1}{3(e^{i\varepsilon}+1)+\kappa} + \frac{U_1}{\left(e^{i\varepsilon}+\kappa+1\right)^2}\right),$$

where $-\pi < \varepsilon \leq \pi$ and $\kappa \geq 1$.

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Proof. Since $\ell(\vartheta) = \vartheta + \sum_{n=2}^{\infty} C_n \vartheta^n \in \mathcal{F}(\varepsilon, \kappa, \aleph)$, we can consider two functions $\wp_1,\ \wp_2 : \Lambda \to \Lambda$, with $\wp_1(0) = \wp_1(0) = 0$ and $|\wp_1(\vartheta)|, |\wp_2(\varpi)| < 1$ for all $\vartheta, \varpi \in \Lambda$. So from (*4*) and (*5*), we can write

$$\left(\frac{\ell(\vartheta)}{\vartheta}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\vartheta\ell''(\vartheta)}{\ell'(\vartheta)}\right) = \aleph(\wp_1(\vartheta)) \tag{8}$$

and

$$\left(\frac{\hbar(\varpi)}{\varpi}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\varpi\hbar''(\varpi)}{\hbar'(\varpi)}\right) = \aleph(\wp_2(\varpi)). \tag{9}$$

Consider the functions

$$\imath_1(\vartheta) = \frac{1+\wp_1(\vartheta)}{1-\wp_1(\vartheta)} = 1 + \delta_1\vartheta + \delta_2\vartheta^2 + \delta_3\vartheta^3 + \cdots,\ |\delta_j| \le 2 \text{ for all } j \in \mathbb{N}. \tag{10}$$

and

$$\imath_2(\varpi) = \frac{1+\wp_2(\varpi)}{1-\wp_2(\varpi)} = 1 + \mu_1\varpi + \mu_2\varpi^2 + \mu_3\varpi^3 + \cdots,\ |\mu_j| \le 2 \text{ for all } j \in \mathbb{N}. \tag{11}$$

Or equivalently,

$$\wp_1(\vartheta) = \frac{\imath_1(\vartheta)-1}{\imath_1(\vartheta)+1} = \frac{\delta_1}{2}\vartheta + \frac{1}{2}\left(\delta_2 - \frac{\delta_1^2}{2}\right)\vartheta^2 + \frac{1}{2}\left(\delta_3 + \frac{\delta_1}{2}\left(\frac{\delta_1^2}{2} - \delta_2\right) - \frac{\delta_1\delta_2}{2}\right)\vartheta^3 + \cdots$$

and

$$\wp_2(\varpi) = \frac{\imath_2(\varpi)-1}{\imath_2(\varpi)+1} = \frac{\mu_1}{2}\varpi + \frac{1}{2}\left(\mu_2 - \frac{\mu_1^2}{2}\right)\varpi^2 + \frac{1}{2}\left(\mu_3 + \frac{\mu_1}{2}\left(\frac{\mu_1^2}{2} - \mu_2\right) - \frac{\mu_1\mu_2}{2}\right)\varpi^3 + \cdots.$$

Using the last two expression for $\wp_1(\vartheta)$ and $\wp_2(\varpi)$ in (8) and (9), we have

$$\begin{aligned}&\left(\frac{\ell(\vartheta)}{\vartheta}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\vartheta\ell''(\vartheta)}{\ell'(\vartheta)}\right)\\ &= 1 + \frac{1}{2}U_1\delta_1\vartheta + \left(\frac{1}{2}U_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{1}{4}U_2\delta_1^2\right)\vartheta^2 + \cdots\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\left(\frac{\hbar(\varpi)}{\varpi}\right)^{\kappa} + \frac{1+e^{i\varepsilon}}{2}\left(\frac{\varpi\hbar''(\varpi)}{\hbar'(\varpi)}\right)\\ &= 1 + \frac{1}{2}U_1\mu_1\varpi + \left(\frac{1}{2}U_1\left(\mu_2 - \frac{\mu_1^2}{2}\right) + \frac{1}{4}U_2\mu_1^2\right)\varpi^2 + \cdots.\end{aligned} \tag{13}$$

By comparing the coefficients in (12) and (13), we conclude that

$$\left(e^{i\varepsilon} + \kappa + 1\right)C_2 = \frac{1}{2}U_1\delta_1, \tag{14}$$

$$\left(\frac{\kappa(\kappa-1)}{2} - 2(e^{i\varepsilon}+1)\right)C_2^2 + \left(3(e^{i\varepsilon}+1) + \kappa\right)C_3 = \frac{1}{2}U_1\left(\delta_2 - \frac{\delta_1^2}{2}\right) + \frac{1}{4}U_2\delta_1^2, \tag{15}$$

$$-\left(e^{i\varepsilon} + \kappa + 1\right)C_2 = \frac{1}{2}U_1\mu_1, \tag{16}$$

and

$$\left(4(e^{i\varepsilon}+1) + \frac{\kappa(\kappa+3)}{2}\right)C_2^2 - \left(3(e^{i\varepsilon}+1) + \kappa\right)C_3 = \frac{1}{2}U_1\left(\mu_2 - \frac{\mu_1^2}{2}\right) + \frac{1}{4}U_2\mu_1^2. \tag{17}$$

From (14) and (16), we get

$$\delta_1 = -\mu_1 \tag{18}$$

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and

$$8\left(e^{i\varepsilon}+\kappa+1\right)^2 C_2^2 = U_1^2(\delta_1^2+\mu_1^2). \tag{19}$$

If we add (15) to (17), we get

$$\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right) C_2^2 = \frac{1}{2}U_1\left(\delta_2+\mu_2\right)+\frac{1}{4}(U_2-U_1)\left(\delta_1^2+\mu_1^2\right). \tag{20}$$

From equations (19) and (20), we have

$$C_2^2 = \frac{U_1^3\left(\delta_2+\mu_2\right)}{2\left[U_1^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right)-2\left(e^{i\varepsilon}+\kappa+1\right)^2(U_2-U_1)\right]}. \tag{21}$$

Furthermore, subtracting (17) from (15) and by assistance of (18), we obtain

$$C_3 = \frac{U_1\left(\delta_2-\mu_2\right)}{4\left(3(e^{i\varepsilon}+1)+\kappa\right)}+\frac{U_1^2\delta_1^2}{4\left(e^{i\varepsilon}+\kappa+1\right)^2}. \tag{22}$$

Finally, from equations (21), (22) and by assistance of (10) and (11), we get

$$|C_2| \leq \sqrt{\frac{2U_1^3}{U_1^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right)-2\left(e^{i\varepsilon}+\kappa+1\right)^2(U_2-U_1)}}$$

and

$$|C_3| \leq U_1\left(\frac{1}{3(e^{i\varepsilon}+1)+\kappa}+\frac{U_1}{\left(e^{i\varepsilon}+\kappa+1\right)^2}\right).$$

Hence, the proof of Theorem 3.1 is completed. □

Using the values of C_2^2 and C_3 for the function $f \in \mathcal{F}(\varepsilon,\kappa,\aleph)$, we will address in the next result the Fekete and Szegö problem.[9]

Theorem 3.2. *If $\ell \in \mathcal{F}(\varepsilon,\kappa,\aleph)$, then*

$$\left|C_3-\rho C_2^2\right| \leq \begin{cases} \frac{U_1}{3(e^{i\varepsilon}+1)+\kappa}, & |\daleth(\rho)| \leq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \\ 2U_1\left|\daleth(\rho)\right|, & |\daleth(\rho)| \geq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \end{cases}$$

where

$$\daleth(\rho) = \frac{U_1^2\left(1-\rho\right)}{U_1^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right)-2\left(e^{i\varepsilon}+\kappa+1\right)^2(U_2-U_1)}.$$

Proof. Subtracting (17) from (15) and by assistance of (18), we get

$$C_3 = \frac{U_1\left(\delta_2-\mu_2\right)}{4\left(3(e^{i\varepsilon}+1)+\kappa\right)}+C_2^2,$$

which prompts writing

$$\begin{aligned} C_3-\rho C_2^2 &= \frac{U_1\left(\delta_2-\mu_2\right)}{4\left(3(e^{i\varepsilon}+1)+\kappa\right)}+(1-\rho)C_2^2 \\ &= \frac{U_1}{2}\left(\left[\daleth(\rho)+\frac{1}{2\left(3(e^{i\varepsilon}+1)+\kappa\right)}\right]\delta_2+\left[\daleth(\rho)-\frac{1}{2\left(3(e^{i\varepsilon}+1)+\kappa\right)}\right]\mu_2\right), \end{aligned} \tag{23}$$

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where

$$\beth(\rho)=\frac{U_1^2\left(1-\rho\right)}{U_1^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right)-2\left(e^{i\varepsilon}+\kappa+1\right)^2\left(U_1-U_2\right)}.$$

Then, in view (23) and using Lemma 2.4, we get

$$\left|C_3-\rho C_2^2\right|\leq\begin{cases}\frac{U_1}{3(e^{i\varepsilon}+1)+\kappa}, & |\beth(\rho)|\leq\frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)},\\ \\ 2U_1\left|\beth(\rho)\right|, & |\beth(\rho)|\geq\frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)},\end{cases}$$

which completes the proof of Theorem 3.2. □

Likewise, the next two theorems for the family $\mathcal{G}(\varepsilon,\kappa,\aleph)$ can be proved using the same method and technique that used in the previous theorems.

Theorem 3.3. *If $\ell\in\mathcal{G}(\varepsilon,\kappa,\aleph)$, then*

$$|C_2|\leq\sqrt{\frac{U_1^3}{U_1^2\left(3(e^{i\varepsilon}+1)+\kappa(2\kappa+1)\right)-\left(e^{i\varepsilon}+2\kappa+1\right)^2\left(U_2-U_1\right)}},$$

and

$$|C_3|\leq U_1\left(\frac{1}{3(e^{i\varepsilon}+\kappa+1)}+\frac{U_1}{\left(e^{i\varepsilon}+2\kappa+1\right)^2}\right).$$

Theorem 3.4. *If $\ell\in\mathcal{G}(\varepsilon,\kappa,\aleph)$, then*

$$\left|C_3-\rho C_2^2\right|\leq\begin{cases}\frac{U_1}{3(e^{i\varepsilon}+\kappa+1)}, & |\beth(\rho)|\leq\frac{1}{3(e^{i\varepsilon}+\kappa+1)},\\ \\ U_1\left|\beth(\hbar)\right|, & |\beth(\rho)|\geq\frac{1}{3(e^{i\varepsilon}+\kappa+1)},\end{cases}$$

where

$$\beth(\rho)=\frac{U_1^2\left(1-\rho\right)}{U_1^2(3(e^{i\varepsilon}+1)+\kappa(2\kappa+1))-\left(e^{i\varepsilon}+2\kappa+1\right)^2\left(U_2-U_1\right)}.$$

4 Corollaries

A multitude of functions $\aleph(\vartheta)$ exist that could yield intriguing subfamilies within function family Υ. For illustration, if we let

$$\aleph(\vartheta)=\frac{1+(1-2\upsilon)\vartheta}{1-\vartheta}=1+2(1-\upsilon)\vartheta+2(1-\upsilon)\vartheta^2+\cdots\ (0\leq\upsilon<1)$$

or

$$\aleph(\vartheta)=\left(\frac{1+\vartheta}{1-\vartheta}\right)^\eta=1+2\eta\vartheta+2\eta^2\vartheta^2+\cdots\ (0<\eta\leq1).$$

By the previous special functions we derive the following corollaries for our main findings in the previous section.

I) If we set $\aleph(\vartheta)=\frac{1+(1-2\upsilon)\vartheta}{1-\vartheta}=1+2(1-\upsilon)\vartheta+2(1-\upsilon)\vartheta^2+\cdots\quad(0\leq\upsilon<1)$, which gives $U_1=U_2=2(1-\upsilon)$, in Theorems 3.1, 3.2, 3.3 and 3.4, respectively, we obtain the subsequent two corollaries.

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Corollary 4.1. *If $\ell \in \mathcal{F}(\varepsilon, \kappa, \frac{1+(1-2v)\vartheta}{1-\vartheta})$, then*

$$|C_2| \leq 2\sqrt{\frac{1-v}{(2(e^{i\varepsilon}+1)+\kappa(\kappa+1))}},$$

$$|C_3| \leq \frac{2(1-v)}{3(e^{i\varepsilon}+1)+\kappa} + \frac{4(1-v)^2}{(e^{i\varepsilon}+\kappa+1)^2}$$

and

$$\left|C_3 - \rho C_2^2\right| \leq \begin{cases} \frac{2(1-v)}{3(e^{i\varepsilon}+1)+\kappa}, & |\beth(\varrho)| \leq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \\ 4(1-v)\left|\beth(\rho)\right|, & |\beth(\rho)| \geq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \end{cases}$$

where

$$\beth(\rho) = \frac{1-\rho}{2(e^{i\varepsilon}+1)+\kappa(\kappa+1)}.$$

Corollary 4.2. *If $\ell \in \mathcal{G}(\varepsilon, \kappa, \frac{1+(1-2v)\vartheta}{1-\vartheta})$, then*

$$|C_2| \leq \sqrt{\frac{2(1-v)}{3(e^{i\varepsilon}+1)+\kappa(2\kappa+1)}},$$

$$|C_3| \leq \frac{2(1-v)}{3(e^{i\varepsilon}+\kappa+1)} + \frac{4(1-v)^2}{(e^{i\varepsilon}+2\kappa+1)^2}$$

and

$$\left|C_3 - \rho C_2^2\right| \leq \begin{cases} \frac{2(1-v)}{3(e^{i\varepsilon}+\kappa+1)}, & |\beth(\rho)| \leq \frac{1}{3(e^{i\varepsilon}+\kappa+1)}, \\ 2(1-v)\left|\beth(\rho)\right|, & |\beth(\rho)| \geq \frac{1}{3(e^{i\varepsilon}+\kappa+1)}, \end{cases}$$

where

$$\beth(\rho) = \frac{1-\rho}{3(e^{i\varepsilon}+3)+\kappa(2\kappa+1)}.$$

II) If we set $\aleph(\vartheta) = \left(\frac{1+\vartheta}{1-\vartheta}\right)^\eta = 1 + 2\eta\vartheta + 2\eta^2\vartheta^2 + \cdots \quad (0 < \eta \leq 1)$, which gives $U_1 = 2\eta$ and $U_2 = 2\eta^2$, in Theorems 3.1, 3.2, 3.3 and 3.4, respectively, we obtain the subsequent two corollaries.

Corollary 4.3. *If $\ell \in \mathcal{F}(\varepsilon, \kappa, \left(\frac{1+\vartheta}{1-\vartheta}\right)^\eta)$, then*

$$|C_2| \leq 2\sqrt{\frac{\eta^3}{\eta^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right) - (e^{i\varepsilon}+\kappa+1)^2(\eta^2-\eta)}},$$

$$|C_3| \leq \frac{2\eta}{3(e^{i\varepsilon}+1)+\kappa} + \frac{4\eta^2}{(e^{i\varepsilon}+\kappa+1)^2}$$

and

$$\left|C_3 - \rho C_2^2\right| \leq \begin{cases} \frac{2\eta}{3(e^{i\varepsilon}+1)+\kappa}, & |\beth(\rho)| \leq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \\ 4\eta\left|\beth(\rho)\right|, & |\beth(\rho)| \geq \frac{1}{2(3(e^{i\varepsilon}+1)+\kappa)}, \end{cases}$$

where

$$\beth(\rho) = \frac{\eta^2(1-\rho)}{\eta^2\left(2(e^{i\varepsilon}+1)+\kappa(\kappa+1)\right) - (e^{i\varepsilon}+\kappa+1)^2(\eta^2-\eta)}.$$

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Corollary 4.4. *If* $\ell \in \mathcal{G}(\varepsilon, \kappa, \left(\frac{1+\vartheta}{1-\vartheta}\right)^{\eta})$, *then*

$$|C_2| \leq 2\sqrt{\frac{\eta^3}{2\eta^2\left(3(e^{i\varepsilon}+1)+\kappa(2\kappa+1)\right)-\left(e^{i\varepsilon}+2\kappa+1\right)^2\left(\eta^2-\eta\right)}},$$

$$|C_3| \leq \frac{2\eta}{3(e^{i\varepsilon}+\kappa+1)} + \frac{4\eta^2}{\left(e^{i\varepsilon}+2\kappa+1\right)^2}$$

and

$$\left|C_3-\rho C_2^2\right| \leq \begin{cases} \frac{2\eta}{3(e^{i\varepsilon}+\kappa+1)}, & |\beth(\rho)| \leq \frac{1}{3(e^{i\varepsilon}+\kappa+1)}, \\ 2\eta\left|\beth(\rho)\right|, & |\beth(\rho)| \geq \frac{1}{3(e^{i\varepsilon}+\kappa+1)}, \end{cases}$$

where

$$\beth(\rho) = \frac{2\eta^2\left(1-\rho\right)}{2\eta^2(3(e^{i\varepsilon}+1)+\kappa(2\kappa+1))-\left(e^{i\varepsilon}+2\kappa+1\right)^2\left(\eta^2-\eta\right)}.$$

Remark 4.5. Putting $\kappa = 1$ and $\varepsilon = \pi$ in Theorem 3.1, we get a result previously provided by Ali et al.[1]

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