On The Markov-Bernstein Inequalities with Weight Functions in L_p Spaces

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Abstract

In this paper, we study of the Markov- Bernstein inequality of a complex polynomial with exponential weight functions $e^{-r^{2\frac{z}{2}}}$ on the domain $]-\infty,+\infty[$, we also study the Integral Markov − Bernstein inequality for the algebraic polynomials of degree 2m and degree m with algebraic weight functions on the domain $[1,+\infty[$ *of type* $\left(\frac{1}{x^2}\right)^{-n}$, and on the domain $]0,+\infty[$ *of type* $\left(\frac{1}{t}\right)^{-n}$.

Keywords: Markov – Bernstein inequality; weight functions; Weight Space; $L_p - space$

1. Introduction

Weight functions play an important role in the theory of functions, sometimes when we find a study we cannot reach a result, and when we multiply by a weight function of a certain row, we easily get the desired result.

In this paper, we have studied Markov - Bernstein regression in weighted spaces, we have used algebraic weight functions to obtain the Markov - Bernstein regression on the spaces of algebraic polynomials of degree M and degree 2m, and we have used exponential weight functions to obtain the Markov-Bernstein regression on the space of trigonometric and nodal polynomials.

We point out that the Markov - Bernstein regression was previously obtained for various weight functions, see [1-2].

This research holds great importance in many problems of mathematical analysis, especially approximation theory. In this research, we aim to obtain the following results:

1- Tensors of the Markov-Bernstein type with exponential weight functions and trigonometric and nodal polynomials on the entire real axis.

2- Tensors of the Markov-Bernstein type with different algebraic weight functions and algebraic polynomials on the entire real axis.

3- Tensors of the Markov-Bernstein type with different algebraic weight functions and algebraic polynomials on the entire positive real axis only.

2. Preliminaries

Definition of Lebsegue space [3]: it is the space of all measurable functions on the domain [a,b] that satisfy the condition:

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$$\int_a^b |f(x)|^p dx < \infty$$

And is denoted by the symbol $L_p[a,b]$, and the system is taking shape $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$ where $0 < p < \infty$ and in the case of p=∞ and throw this space with L_∞ and the system takes the Form $\|f\|_\infty = max_{x\in[a,b]}|f(x)|$.

Definition of the weighted Lebsegue Space [3]:

it is denoted by the symbol $L_p(V,W)$ and is defined as the space of all functions f defined on the domain $V = [a,b]$, then:

$$\int_a^b |f|^p . w(x)dx < \infty$$

Where: W is a function different from zero and measurable on the domain V and is called the weight function and in other words, we can say:

$f \in L_p[v,w]$ On the field [a, b] if $f.w \in L_p[a,b]$

Definition of the Markov-Bernestein Tensor [1]: defined as the tensor that gives the relation between dependent systems and derivative systems of rank (n) in the same space A, so, it has the following general Form $\|f^{(n)}\|_A \leq c\|f\|_A$.

Definition of the Space $E_{(m,n)}$[4]: it is called the space of trigonometric-nodal functions of degree m and is defined as the space of weighted trigonometric-nodal polynomials consisting of the elements of the space $E_{(m)}$ with the weight function.

$$w(z) = e^{-r^2\frac{z}{2}} \quad ; n = m + r$$

Where r is an absolutely positive integer, M is a degree polynomial and where each element of the space $E_{(m,n)}$ takes the form:

$$E_{(m,n)}(z) = e^{-r^2\frac{z}{2}} E_{(m)}(z)$$

And the regularity in this space takes the form:

$$\|E_{(m,n)}(z)\| = \left(\int_{-\infty}^{+\infty} |E_{m,n}(z)|^p dz\right)^{\frac{1}{p}}$$

Definition of the space $H_{(m,n)}$ [4]: is the space of algebraic functions of degree 2m with the weight function $w(x) = \left(\frac{1}{x^2}\right)^{-n}$ defined on the domain $[1,+\infty[$, since each element of the space $H_{(m,n)}$ takes the form:

$$H_{(m,n)}(x) = H_m(x)\left(\frac{1}{x^2}\right)^{-r}$$

Where $H_m(x) = \frac{g(x)}{\left(\frac{1}{x^2}\right)^m}$ so that g(x) is an algebraic polynomial of degree 2m and from it

We have:

$$H_{(m,n)}(x) = \frac{g(x)}{\left(\frac{1}{x^2}\right)^m}\left(\frac{1}{x^2}\right)^{-r}$$

$$= g(x)\left(\frac{1}{x^2}\right)^{-(r+m)}$$

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$$= g(x)\left(\frac{1}{x^2}\right)^{-n}$$

Definition of the Space $B_{(m,n)}$: is the space of algebraic functions of degree m with the weight function $w(t) = \left(\frac{1}{t}\right)^{-n}$ defined on the domain $]0, +\infty[$, which is considered a subspace of $H_{(m,n)}$.

Each element of the space $B_{(m,n)}$ takes the form: $B_{(m,n)}(t) = B_m(t)\left(\frac{1}{t}\right)^{-r}$ where $B_m(t) = \frac{g(t)}{\left(\frac{1}{t}\right)^m}$ so that g(t) is an algebraic polynomial of degree m, thus:

$$B_{(m,n)}(t) = \frac{g(t)}{\left(\frac{1}{t}\right)^m}\left(\frac{1}{t}\right)^{-r}$$

$=g(t)\left(\frac{1}{t}\right)^{-(r+m)}$

$$= g(t)\left(\frac{1}{t}\right)^{-n}$$

We will Code $E_{(m,n)}$ for the space of trigonometric polynomials with an exponential weight function $e^{-r^2\frac{z}{2}}$, and we will Code $H_{(m,n)}$ for the space of algebraic polynomials of degree 2m with the algebraic function $w(x) = \left(\frac{1}{x^2}\right)^{-n}$, and we will Code $B_{(m,n)}$ for the space of algebraic polynomials of degree m with the algebraic weight function $w(t) = \left(\frac{1}{t}\right)^{-n}$.

Then we find the following:

1 - We can correspond each element of the space $E_{(m,n)}$ to an element of the Space [4][5]$H_{(m,n)}$ by the following transformation:

$$\left(\frac{1}{x^2}\right) \leftrightarrow e^{r\frac{z}{2}}$$

Thus, $E_{(m)}(z)e^{-r^2\frac{z}{2}} = H_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}$

2 - We can correspond each element of the space $B_{(m,n)}$ to an element of the Space [4][5]$H_{(m,n)}$ by the following transformation: $t \leftrightarrow x^2$where $x > 0$

Thus, $B_{(m)}(t)\left(\frac{1}{t}\right)^{-r} = H_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}$

3. Results and discussion

In the following theorem, we obtain the Markov - Bernstein regression on $E_{(m,n)}$ for exponential weight functions in a weighted Liege space.

Theorem1:

If $E_{(m)}$ is a triangular polynomial of degree m and $1 < p < +\infty$and r is a completely positive integer, then the following relation is true:

$$\left\|E'_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \leq \left((m+r) + \frac{r^2}{2}\right)\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \dots\dots(1)$$

Proof:

Let $E_{(m)}(z)e^{-r^2\frac{z}{2}}$ be a triangular polynomial of degree n , and let the Markov-Bernstein Tensor for triangular polynomials of degree n be:

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$$\|T_n'\| \leq n\|T_n\| \ldots\ldots (2)$$

Taking advantage of (2), we get:

$$\left\|\left(E_{(m)}(z)e^{-r^2\frac{z}{2}}\right)'\right\|_p \leq (m+r)\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \ldots\ldots (3)$$

$$\left(E_{(m)}(z)e^{-r^2\frac{z}{2}}\right)' = E'_{(m)}(z)e^{-r^2\frac{z}{2}} + E_{(m)}(z)\left(-\frac{r^2}{2}e^{-r^2\frac{z}{2}}\right)$$

$$\left(E_{(m)}(z)e^{-r^2\frac{z}{2}}\right)' = E'_{(m)}(z)e^{-r^2\frac{z}{2}} - \frac{r^2}{2}E_{(m)}(z)e^{-r^2\frac{z}{2}}$$

$$E'_{(m)}(z)e^{-r^2\frac{z}{2}} = \left(E_{(m)}(z)e^{-r^2\frac{z}{2}}\right)' + \frac{r^2}{2}E_{(m)}(z)e^{-r^2\frac{z}{2}}$$

Taking the regularities of each of the parties, we find that:

$$\left\|E'_{(m)}e^{-r^2\frac{z}{2}}\right\|_p \leq \left\|\left(E_{(m)}(z)e^{-r^2\frac{z}{2}}\right)'\right\|_p + \frac{r}{2}\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \ldots\ldots\ldots.(4)$$

Taking advantage of (3) and compensating for (4), we find that:

$$\left\|E'_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \leq (m+r)\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p + \frac{r^2}{2}\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p$$

From it we find that:

$$\left\|E'_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p \leq \left((m+r)+\frac{r^2}{2}\right)\left\|E_{(m)}(z)e^{-r^2\frac{z}{2}}\right\|_p$$

Which is required.

* In the following theorem we obtain the Markov retracement.Bernstein for algebraic polynomials using algebraic weight functions.

Theorem2:

Let $H_{(m,n)}$ be the space of algebraic functions of degree 2m with the weight function $w(x) = \left(\frac{1}{x^2}\right)^{-r}$ defined on the domain $[1, +\infty[$ and r is a completely positive integer and $1 < p < +\infty$, then the following relationship is true:

$$\left\|H'_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}\right\|_p \leq \frac{4}{r}\left((m+r)+\frac{r^2}{2}\right)\left\|H_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}\right\|_p \ldots\ldots (5)$$

Proof:

We know that every element of the space $H_{(m,n)}$ can be written as $H_m(x)\left(\frac{1}{x^2}\right)^{-r}$

Taking advantage of the transformation $\left(\frac{1}{x^2}\right) \leftrightarrow e^{+r\frac{z}{2}}$, we find that $H_m(x) = E_m(z)$,then :

$$\frac{-2}{x^3} = \frac{re^{r\frac{z}{2}}}{2}$$

$\frac{dz}{dx} = \frac{-2}{x^3}.\frac{2}{r}e^{-r\frac{z}{2}}$ ……(6)

$$\frac{dH_m(x)}{dx} = \frac{dE_m(z)}{dz}.\ \frac{dz}{dx} \quad \ldots\ldots (7)$$

$$H'_m(x) = E'_m(z).\frac{-4}{x^3.r}e^{-r\frac{z}{2}}$$

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$$H'_m(x) = -4E'_m(z).\frac{e^{-r\frac{z}{2}}}{x.e^{-r\frac{z}{2}}.r}$$

$$H'_m(x) \text{ x } e^{-r^2\frac{z}{2}} = \frac{-4}{r}E'_m(z)e^{-r^2\frac{z}{2}}$$

$$\left\|H'_{(m)}(x)e^{-r^2\frac{z}{2}}\right\|_p \leq \|x\|_p \left\|H'_m(x)e^{-r^2\frac{z}{2}}\right\|_p = \frac{4}{r}\left\|E'_m(z)e^{-r^2\frac{z}{2}}\right\|_p \dots\dots\dots (8)$$

Taking advantage of (1) and compensating for (8), we find that:

$$\left\|H'_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}\right\|_p \leq \frac{4}{r}\left((m+r)+\frac{r^2}{2}\right)\left\|E_m(z)e^{-r^2\frac{z}{2}}\right\|_p = \frac{4}{r}\left((m+r)+\frac{r^2}{2}\right)\left\|H_{(m)}(x)\left(\frac{1}{x^2}\right)^{-r}\right\|_p$$

Which is required.

Let $B_{(m,n)}$be a space of algebraic functions of degree m with the weight function $w(t) = \left(\frac{1}{t}\right)^{-r}$ defined on the domain $]0, +\infty[$ and r is a completely positive integer and $1 < p < +\infty$ then the following relationship is fulfilled:

$$\left\|\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r}\right\|_p \leq \frac{2}{r}\left((m+r)+\frac{r^2}{2}\right)\left\|B_{(m)}(t)\left(\frac{1}{t}\right)^{-r}\right\|_p \dots\dots\dots (9)$$

Proof:

We know that each element of the space $B_{(m,n)}(t)$ can be written as:

$$B_{(m,n)}(t) = B_m(t)\left(\frac{1}{t}\right)^{-r}$$

using the transformation $t \leftrightarrow x^2$where $x > 0$ we find that $B_m(t) = H_m(x)$, then:

$$\frac{dt}{dx} = 2x = 2\sqrt{t} \Rightarrow \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \dots . (10)$$

$$B_m(t) = \frac{g(t)}{\left(\frac{1}{t}\right)m} = H_m(x) \quad \dots (11)$$

$$\frac{dB_m(t)}{dt} = \frac{dH_m(x)}{dx}.\frac{dx}{dt} \Rightarrow B'_m(t) = H'_m(x).\frac{1}{2\sqrt{t}}$$

$$2\sqrt{t}B'_m(t) = H'_m(x)$$

$$2\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r} = H'_m(x)\left(\frac{1}{t}\right)^{-r}$$

$$\left\|2\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r}\right\|_p = 2\left\|\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r}\right\|_p = \left\|H'_m(x)\left(\frac{1}{t}\right)^{-r}\right\|_p \dots . (12)$$

Taking advantage of (5) and substituting in (12), we find that:

$$2\left\|\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r}\right\|_p = \left\|H'_m(x)\left(\frac{1}{t}\right)^{-r}\right\|_p \leq \frac{4}{r}\left((m+r)+\frac{r^2}{2}\right)\left\|H_{(m)}(x)\left(\frac{1}{t}\right)^{-r}\right\|_p$$

From it we find that:

$$\left\|\sqrt{t}B'_m(t)\left(\frac{1}{t}\right)^{-r}\right\|_p \leq \left((m+r)+\frac{r^2}{2}\right)\left\|B_{(m)}(t)\left(\frac{1}{t}\right)^{-r}\right\|_p$$

Which is required.

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At the end of our research, we obtained the following theorem, in which we obtained the Markov-Bernstein regression with algebraic weight functions.

Theorem 4:

Let $B_{(m,n)}(t)$ be the space of algebraic functions of degree m with the weight function $w(t) = \left(\frac{1}{t}\right)^{-r}$ defined on the domain $]0, +\infty[$, and r is a completely positive number, and $1 < p < +\infty$ then the following relationship is fulfilled:

$$\left\| B'_m(t)\left(\frac{1}{t}\right)^{-r} \right\|_p \leq \left((m+r) + \frac{r^2}{2}\right) \left\| B_{(m)}(t)\left(\frac{1}{t}\right)^{-r} \right\|_p \dots (13)$$

Proof:

We know that each element of the space $B_{(m,n)}(t)$ can be written as: $B_{(m,n)}(t) = B_m(t)\left(\frac{1}{t}\right)^{-r}$

Taking advantage of the transformation $t \leftrightarrow x^2$, we find that $B_m(t) = H_m(x)$, so let $x > 0$ and $x > 0$, then:

$$\frac{1}{t} = \frac{1}{x^2} = e^{r\frac{z}{2}}$$

$$\frac{1}{-t^2}dt = \frac{r}{2}e^{r\frac{z}{2}}dz$$

$$\frac{dz}{dt} = -\frac{2}{t^2.re^{r\frac{z}{2}}}$$

We have $B_m(t) = E_m(z)$

Thus, we find $\frac{dB_m(t)}{dt} = \frac{dE_m(z)}{dz}.\frac{dz}{dt}$

Using the conversion $\frac{1}{x^2} \leftrightarrow e^{r\frac{z}{2}}$and the conversion $t \leftrightarrow x^2$.

$$B'_m(t) = E'_m(z).\left(\frac{-2}{t^2.re^{r\frac{z}{2}}}\right) = -2E'_m(z)\left(\frac{1}{e^{-r\frac{z}{2}}.re^{-r\frac{z}{2}}}e^{-r\frac{z}{2}}\right)$$

$$= -2E'_m(z)\left(\frac{1}{t.r}\right)$$

$$B'_m(t).t = -\frac{2}{r}E'_m(z)$$

$$B'_m(t) \quad te^{-r^2\frac{z}{2}} = -\frac{2}{z}E'_m(z)e^{-r^2\frac{z}{2}}$$

$$\left\|B'_m(t)e^{-r^2}\right\|_p \leq \|t\|_p \left\|B'_m(t)e^{-r^2\frac{z}{2}}\right\|_p = \frac{2}{r}\left\|E'_m(z)e^{-r^2\frac{z}{2}}\right\|_p$$

Taking advantage of (1) and substituting in (14) we find that:

$$\left\| B'_m(t)\left(\frac{1}{t}\right)^{-r} \right\|_p \leq \frac{2}{r}\left((m+r) + \frac{r^2}{2}\right)\left\|E_m(z)e^{-r^2\frac{z}{2}}\right\|_p =$$

$$= \frac{2}{r}\left((m+r) + \frac{r^2}{2}\right)\left\| B_{(m)}(t)\left(\frac{1}{t}\right)^{-r} \right\|_p$$

Which is required.

4. Conclusion

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From the above, we have come from this research to the study of Markov-Bernstein regression with algebraic and exponential weight functions, which is useful in solving many problems of mathematical analysis, especially approximation theory, using appropriate mathematical methods to reach the required regression. We recommend that you study this topic for various other weight functions.

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