Complex cubic neutrosophic set applied to subbisemiring and its extension of bisemiring

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Abstract

We construct the concept of complex cubic neutrosophic subbisemiring (ComCNSBS). We analyze the important properties and homomorphic aspects of ComCNSBS. For bisemirings, we propose the ComCNSBS level sets. A complex neutrosophic subset Γ if and only if each non-empty level set $R^{(\wp,\varkappa)}$, where $R = (\widehat{\Re^T_\Gamma} \cdot e^{i\theta\widehat{\Im^T_\Gamma}}, \widehat{\Re^I_\Gamma} \cdot e^{i\theta\widehat{\Im^I_\Gamma}}, \widehat{\Re^F_\Gamma} \cdot e^{i\theta\widehat{\Im^F_\Gamma}}, \Re^T_\Gamma \cdot e^{i\theta\Im^T_\Gamma}, \Re^I_\Gamma \cdot e^{i\theta\Im^I_\Gamma}, \Re^F_\Gamma \cdot e^{i\theta\Im^F_\Gamma})$ is a ComCNSBS. We show that homomorphic images of all ComCNSBSs are ComCNSBSs, and homomorphic pre-images of all ComCNSBSs are ComCNSBSs. We illustrate the practical significance of our findings.

Keywords: ComCNSBS; ComCNNSBS; SBS; homomorphism

1 Introduction

Fuzzy set (FS) theory was developed by Zadeh[1] and is most effective in managing ambiguity and uncertainty. An element in an FS is considered a member if it contains a single value inside the interval. However, since resistance can exist in real-world situations, the degree of non-membership might not necessarily equal one minus the degree of membership. As FS theory advances quickly, more and more hybrid fuzzy models are being developed. Numerous uncertain theories, including FS,[1] intuitionistic FS (IFS),[2] Pythagorean FS (PFS),[3] and spherical FS (SFS),[4] have been developed as a result of the uncertainties. Sets with grades ranging from 0 to 1, referred to as MG, comprise an FS. IFS is classified as MG, despite Atanassov[2] stating that non-membership grades (NMG) can only have a value of 1. There is a chance that, in a decision-making process, the sum of MGs and NMGs will sometimes exceed 1. The generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs, was constructed by Yager[3] using

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PFS logic. These ideas are unable to explain the neutral state, which is neither positive nor negative. Cuong[5] discussed the picture fuzzy set. Three grading points were utilized by FS: positive, neutral, and negative. These grades added together could not total more than 1. For certain reasons, it is also equivalent to PFS or IFS. It is an independent generalization of three models that deals with the truth, indeterminacy, and falsity of FS and IFS. Numerous recent studies have been discussed by Palanikumar et al.[6–8]

Smarandache[?] was developed by neutrosophic set (NS) to deal with contradicting and ambiguous data. This reasoning establishes the degree to which a statement is true, ambiguous, or untrue. Ramot et al.[10] presented the notion of the complex fuzzy set (CFS) . There is a wide range of possible values for the membership functions of CFSs deals. The unit circle of the complex plane is extended to $[0, 1]$, but the unit circle of a fuzzy membership function is fixed. Rather than extending exclusively to $[0, 1]$, the membership function $\mu_X(x)$ of the CFS X extends to the unit circle in the complex plane. Thus, $\mu_X(x)$ is a complex-valued function that assigns a MG of the form $\eta_X(x) \cdot e^{i\tau_X(x)}$, where $i = \sqrt{-1}$, to any element x in the universal set. The value of $\mu_X(x)$ is defined by the two real-valued variables, $\eta_X(x)$ and $\tau_X(x)$, where $\eta_X(x) \in [0, 1]$. Golan[11] established the concept of semiring logic and its applications. Hussian and colleagues[12] talked about the concept and expansion of bisemirings. The topic of bipolar-valued FSs and related techniques is covered by Lee.[13] Ahsan et al.[14] investigated fuzzy semirings. The notion of bisemirings was first presented by Sen et al.[16] Recently, the new type of normal subbisemiring (SBS) were presented by Palanikumar et al.,[17] bipolar-valued neutrosophic normal SBS.[18] In this study, the following contributions are made:

1. Every ComCNSBS's intersection results in another ComCNSBS.
2. Let Υ be the strongest complex neutrosophic relation of S, and let Γ be a ComCNSBS of S. If and only if Υ is a ComCNSBS of S, then Γ is a ComCNSBS of bisemiring $S \times S$.
3. $R = (\widehat{\Re^T_\Gamma} \cdot e^{i\theta\widehat{\Im^T_\Gamma}}, \widehat{\Re^I_\Gamma} \cdot e^{i\theta\widehat{\Im^I_\Gamma}}, \widehat{\Re^F_\Gamma} \cdot e^{i\theta\widehat{\Im^F_\Gamma}}, \Re^T_\Gamma \cdot e^{i\theta\Im^T_\Gamma}, \Re^I_\Gamma \cdot e^{i\theta\Im^I_\Gamma}, \Re^F_\Gamma \cdot e^{i\theta\Im^F_\Gamma})$ is a SBS of S for all $\wp, \varkappa \in \mathbb{D}[0, 1]$.
4. Every ComCNSBS has a homomorphic image that is also a ComCNSBS, and each ComCNSBS has a homomorphic preimage that is also a ComCNSBS.

We will look at certain aspects of the SBS and ComCNSBS concepts and make some conclusions. The following five sections make up the article. In Section 1, semirings and SBS are introduced. Section 2 contains information on semiring and SBS preparation. Listing of ComCNSBS properties is done in Section 3. Utilizing numerical examples are recommended for ComCNSBS evaluation. Section 4 indicates the outcome and future direction.

2 Preliminaries

Definition 2.1. [16] The structure $(S, \uplus, \ominus, \odot)$ is a bisemiring, if the conditions such as $(S, \uplus, \ominus)$ and $(S, \ominus, \odot)$ both semirings. That is, $(S, \uplus)$, $(S, \ominus)$ and $(S, \odot)$ both semigroups and

1. $\ell_v \ominus (\ell_\zeta \uplus \ell_\eta) = (\ell_v \ominus \ell_\zeta) \uplus (\ell_v \ominus \ell_\eta)$,
2. $(\ell_\zeta \uplus \ell_\eta) \ominus \ell_v = (\ell_\zeta \ominus \ell_v) \uplus (\ell_\eta \ominus \ell_v)$,
3. $\ell_v \odot (\ell_\zeta \ominus \ell_\eta) = (\ell_v \odot \ell_\zeta) \ominus (\ell_v \odot \ell_\eta)$,
4. $(\ell_\zeta \ominus \ell_\eta) \odot \ell_v = (\ell_\zeta \odot \ell_v) \ominus (\ell_\eta \odot \ell_v)$, $\forall\ \ell_v, \ell_\zeta, \ell_\eta \in S$.

Definition 2.2. [9] A NS v in the universe $\mathcal{U}$ is $v = \{\jmath, u^T_v(x), u^I_v(x), u^F_v(x) | x \in \mathcal{U}\}$,where $u^T_v(x)$,$u^I_v(x)$ $u^F_v(x)$ represents the truth, intermediate and false grade of v respectively, and $u^T_v : \mathcal{U} \to [0, 1], u^I_v : \mathcal{U} \to [0, 1], u^F_v : \mathcal{U} \to [0, 1]$ and $0 \leq u^T_v(x) + u^I_v(x) + u^F_v(x) \leq 3$.

Definition 2.3. [9] Let $\psi_1 = \langle u^T_{\psi_1}, u^I_{\psi_1}, u^F_{\psi_1} \rangle$,$\psi_2 = \langle u^T_{\psi_2}, u^I_{\psi_2}, u^F_{\psi_2} \rangle$ and $\psi_3 = \langle u^T_{\psi_3}, u^I_{\psi_3}, u^F_{\psi_3} \rangle$ are the neutrosophic numbers of $\mathcal{U}$. Then

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1. $\psi_2 \Cup \psi_3 = \Big\langle \max(\jmath^T_{\psi_2}, u^T_{\psi_3}), \min(\jmath^I_{\psi_2}, u^I_{\psi_3}), \min(\jmath^F_{\psi_2}, u^F_{\psi_3}) \Big\rangle$,
2. $\psi_2 \Cap \psi_3 = \Big\langle \min(\jmath^T_{\psi_2}, u^T_{\psi_3}), \max(\jmath^I_{\psi_2}, u^I_{\psi_3}), \max(\jmath^F_{\psi_2}, u^F_{\psi_3}) \Big\rangle$,
3. $\psi_2 \geq \psi_3$ iff $u^T_{\psi_2} \geq u^T_{\psi_3}$ and $u^I_{\psi_2} \leq u^I_{\psi_3}$ and $u^F_{\psi_2} \leq u^F_{\psi_3}$,
4. $\psi_2 = \psi_3$ iff $u^T_{\psi_2} = u^T_{\psi_3}$ and $u^I_{\psi_2} = u^I_{\psi_3}$ and $u^F_{\psi_2} = u^F_{\psi_3}$.

Definition 2.4. [9] If $\psi = \{x, \jmath^T_v(x), \jmath^I_v(x), \jmath^F_v(x)\}$ of $\mathcal{U}$ be the NS, then (ζ,τ)-cut is described as $\{x \in U | \jmath^T_v(x) \geq \zeta, \jmath^I_v(x) \geq \zeta, \jmath^F_v(x) \leq \tau\}$.

Definition 2.5. [9] If V and W be NSs of S, then Cartesian product is described as $V \times W = \{\jmath^{\mathcal{T}}_{V\times W}(\jmath,\psi), \jmath^I_{V\times W}(\jmath,\psi), \jmath^F_{V\times W}(\jmath,\psi$ $S\}$, where
$\jmath^{\mathcal{T}}_{V\times W}(\jmath,\psi) = \min\{\jmath^{\mathcal{T}}_V(x), \jmath^{\mathcal{T}}_W(\psi)\}, \jmath^I_{V\times W}(\jmath,\psi) = \frac{\jmath^I_V(x)+\jmath^I_W(\psi)}{2}, \jmath^F_{V\times W}(\jmath,\psi) = \max\{\jmath^F_V(x), \jmath^F_W(\psi)\}$.

3 Complex cubic neutrosophic SBS

Unless otherwise indicated, S denotes the bisemiring, $\Re$ means the real part, and $\Im$ refers the imaginary part and $\theta = 2\pi$.

Definition 3.1. The complex cubic NS (ComCNS) Γ in universal set $\eth$,
$\Gamma = \{\jmath, \widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma} : \jmath \in \eth\}$, where $\widehat{\Re^T_\Gamma}(\jmath) = [\Re^{TL}_\Gamma, \Re^{TU}_\Gamma], \widehat{\Re^I_\Gamma}(\jmath) = [\Re^{IL}_\Gamma, \Re^{IU}_\Gamma], \widehat{\Re^F_\Gamma}(\jmath) = [\Re^{FL}_\Gamma, \Re^{FU}_\Gamma]$ and $\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}} : \eth \to D[0,1]$ and $\Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma} : \eth \to [0,1]$ represents the degree of truth, indeterminacy and false, respectively. For simplicity the symbol $\widehat{\Re^T_\Gamma}, \widehat{\Re^I_\Gamma}, \widehat{\Re^F_\Gamma}$ is ComCNS $\Gamma = \{\jmath, \widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma} : \jmath \in \eth\}$.

Definition 3.2. Let $\Gamma = \Big\{\jmath, \widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma}\Big\}$ and $\Lambda = \Big\{\jmath, \widehat{\Re^T_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Lambda}}, \widehat{\Re^I_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Lambda}}, \widehat{\Re^F_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Lambda}}, \Re^T_\Lambda(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Lambda}, \Re^I_\Lambda(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Lambda}, \Re^F_\Lambda(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Lambda}\Big\}$ be two ComCNSs of $\eth$. Then we define the intersection and union operation is defined as
(i) $\Gamma \cap \Lambda = \Big\{\Big(\jmath, \min\{\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^T_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Lambda}}\}, \min\{\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^I_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Lambda}}\}, \max\{\widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \widehat{\Re^F_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Lambda}}\}$,
$\min\{\Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^T_\Lambda(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Lambda}\}, \min\{\Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^I_\Lambda(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Lambda}\}, \max\{\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma}, \Re^F_\Lambda(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Lambda}\}\Big)\Big| \jmath \in \eth\Big\}$.
(ii) $\Gamma \cup \Lambda = \Big\{\Big(\jmath, \max\{\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^T_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Lambda}}\}, \max\{\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^I_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Lambda}}\}, \min\{\widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \widehat{\Re^F_\Lambda}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Lambda}}\}$,
$\max\{\Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^T_\Lambda(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Lambda}\}, \max\{\Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^I_\Lambda(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Lambda}\}, \min\{\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma}, \Re^F_\Lambda(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Lambda}\}\Big)\Big| \jmath \in \eth\Big\}$.

Definition 3.3. The ComCNS $\Gamma = \Big\{\jmath, \widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}}, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}}, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}}, \Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma}, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma}, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma}\Big\}$ of $\eth$. Then $(\wp, \varkappa)$-cut is described as $\Big\{\jmath \in \eth | \widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{T(\jmath)}_\Gamma}} \geq \wp, \widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{I(\jmath)}_\Gamma}} \geq \wp, \widehat{\Re^F_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^{F(\jmath)}_\Gamma}} \leq \varkappa, \Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_\Gamma} \geq \wp, \Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_\Gamma} \geq \wp, \Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_\Gamma} \leq \varkappa\Big\}$.

Definition 3.4. The Cartesian product of Γ and Λ is described as
$\Gamma \times \Lambda = \Big\{\widehat{\Re^T_{\Gamma\times\Lambda}((\jmath,\psi))}\cdot e^{i\theta\widehat{\Im^{T((\jmath,\psi))}_{\Gamma\times\Lambda}}}, \widehat{\Re^I_{\Gamma\times\Lambda}((\jmath,\psi))}\cdot e^{i\theta\widehat{\Im^{I((\jmath,\psi))}_{\Gamma\times\Lambda}}}, \widehat{\Re^F_{\Gamma\times\Lambda}(\jmath,\psi)}\cdot e^{i\theta\widehat{\Im^{F((\jmath,\psi))}_{\Gamma\times\Lambda}}}, \Re^T_{\Gamma\times\Lambda}((\jmath,\psi))\cdot$

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$e^{i\theta\Im^{T((\jmath,\psi))}_{\Gamma\times\Lambda}}, \Re^{I}_{\Gamma\times\Lambda}((\jmath,\psi))\cdot e^{i\theta\Im^{I((\jmath,\psi))}_{\Gamma\times\Lambda}}, \Re^{F}_{\Gamma\times\Lambda}(\jmath,\psi)\cdot e^{i\theta\Im^{F((\jmath,\psi))}_{\Gamma\times\Lambda}} \mid for\ all\ \jmath,\psi \in S\Big\}$,where Γ and Λ be the ComCNS of $\eth$, where

$$\left\{\begin{array}{l} \widehat{\Re^{T}_{\Gamma\times\Lambda}}((\jmath,\psi))\cdot e^{i\theta\Im^{\widehat{T((\jmath,\psi))}}_{\Gamma\times\Lambda}} = \min\left\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\Im^{\widehat{T(\jmath)}}_{\Gamma}}, \widehat{\Re^{T}_{\Lambda}}(\psi)\cdot e^{i\theta\Im^{\widehat{T(\psi)}}_{\Lambda}}\right\} \\ \widehat{\Re^{I}_{\Gamma\times\Lambda}}((\jmath,\psi))\cdot e^{i\theta\Im^{\widehat{I((\jmath,\psi))}}_{\Gamma\times\Lambda}} = \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\Im^{\widehat{I(\jmath)}}_{\Gamma}} + \widehat{\Re^{I}_{\Lambda}}(\psi)\cdot e^{i\theta\Im^{\widehat{I(\psi)}}_{\Lambda}}}{2} \\ \widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath,\psi))\cdot e^{i\theta\Im^{\widehat{F((\jmath,\psi))}}_{\Gamma\times\Lambda}} = \max\left\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\Im^{\widehat{F(\jmath)}}_{\Gamma}}, \widehat{\Re^{F}_{\Lambda}}(\psi)\cdot e^{i\theta\Im^{\widehat{F(\psi)}}_{\Lambda}}\right\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re^{T}_{\Gamma\times\Lambda}((\jmath,\psi))\cdot e^{i\theta\Im^{T((\jmath,\psi))}_{\Gamma\times\Lambda}} = \min\left\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T(\jmath)}_{\Gamma}}, \Re^{T}_{\Lambda}(\psi)\cdot e^{i\theta\Im^{T(\psi)}_{\Lambda}}\right\} \\ \Re^{I}_{\Gamma\times\Lambda}((\jmath,\psi))\cdot e^{i\theta\Im^{I((\jmath,\psi))}_{\Gamma\times\Lambda}} = \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I(\jmath)}_{\Gamma}} + \Re^{I}_{\Lambda}(\psi)\cdot e^{i\theta\Im^{I(\psi)}_{\Lambda}}}{2} \\ \Re^{F}_{\Gamma\times\Lambda}((\jmath,\psi))\cdot e^{i\theta\Im^{F((\jmath,\psi))}_{\Gamma\times\Lambda}} = \max\left\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F(\jmath)}_{\Gamma}}, \Re^{F}_{\Lambda}(\psi)\cdot e^{i\theta\Im^{F(\psi)}_{\Lambda}}\right\} \end{array}\right\}$$

Definition 3.5. Let Γ be the ComCNS of S is called as ComCNSBS if

$$\left\{\begin{array}{l} \widehat{\Re^{T}_{\Gamma}}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\boxplus_1\psi))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath))}, \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\psi))}\} \\ \widehat{\Re^{T}_{\Gamma}}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\boxplus_2\psi))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath))}, \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\psi))}\} \\ \widehat{\Re^{T}_{\Gamma}}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\boxplus_3\psi))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath))}, \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\psi))}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_1\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath))} + \widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\psi))}}{2} \\ \qquad\qquad \text{OR} \\ \widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\boxplus_2\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath))} + \widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\psi))}}{2} \\ \qquad\qquad \text{OR} \\ \widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_3\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath))} + \widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\psi))}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_1\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath))}, \widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\psi))}\} \\ \widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_2\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath))}, \widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\psi))}\} \\ \widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_3\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath))}, \widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\psi))}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re^{T}_{\Gamma}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_1\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath))}, \Re^{T}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}((\psi))}\} \\ \Re^{T}_{\Gamma}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_2\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath))}, \Re^{T}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}((\psi))}\} \\ \Re^{T}_{\Gamma}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_3\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath))}, \Re^{T}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}((\psi))}\} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re^{I}_{\Gamma}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\boxplus_1\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath))} + \Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{I}_{\Gamma}((\psi))}}{2} \\ \qquad\qquad \text{OR} \\ \Re^{I}_{\Gamma}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_2\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath))} + \Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{I}_{\Gamma}((\psi))}}{2} \\ \qquad\qquad \text{OR} \\ \Re^{I}_{\Gamma}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\boxplus_3\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath))} + \Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}((\psi))}}{2} \end{array}\right\}$$

$$\left\{\begin{array}{l} \Re^{F}_{\Gamma}((\jmath \boxplus_1 \psi))\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\boxplus_1\psi))} \leq \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath))}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}((\psi))}\} \\ \Re^{F}_{\Gamma}((\jmath \boxplus_2 \psi))\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\boxplus_2\psi))} \leq \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath))}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}((\psi))}\} \\ \Re^{F}_{\Gamma}((\jmath \boxplus_3 \psi))\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\boxplus_3\psi))} \leq \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath))}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}((\psi))}\} \end{array}\right\}$$

for all $\jmath,\psi \in S$.

Example 3.6. Consider $S = \{\tau, \upsilon, \zeta, \eta\}$:

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$\boxplus_1$	τ	υ	ζ	η
τ	τ	τ	τ	τ
υ	τ	υ	τ	υ
ζ	τ	τ	ζ	ζ
η	τ	υ	ζ	η

$\boxplus_2$	τ	υ	ζ	η
τ	τ	υ	ζ	η
υ	υ	υ	η	η
ζ	ζ	η	ζ	η
η	η	η	η	η

$\boxplus_3$	τ	υ	ζ	η
τ	τ	τ	τ	τ
υ	τ	υ	ζ	η
ζ	η	η	η	η
η	η	η	η	η

	$(w)=\tau$	$(w)=\upsilon$
$(\widehat{\Re^T_\Gamma},\widehat{\Im^T_\Gamma})(w)$	$[0.8e^{i2\pi(0.65)},0.9e^{i2\pi(0.75)}]$	$[0.7e^{i2\pi(0.55)},0.8e^{i2\pi(0.65)}]$
$(\widehat{\Re^I_\Gamma},\widehat{\Im^I_\Gamma})(w)$	$[0.5e^{i2\pi(0.35)},0.6e^{i2\pi(0.45)}]$	$[0.4e^{i2\pi(0.25)},0.5e^{i2\pi(0.35)}]$
$(\widehat{\Re^F_\Gamma},\widehat{\Im^F_\Gamma})(w)$	$[0.6e^{i2\pi(0.45)},0.7e^{i2\pi(0.55)}]$	$[0.7e^{i2\pi(0.55)},0.8e^{i2\pi(0.6)}]$

	$(w)=\zeta$	$(w)=\eta$
$(\widehat{\Re^T_\Gamma},\widehat{\Im^T_\Gamma})(w)$	$[0.4e^{i2\pi(0.25)},0.5e^{i2\pi(0.35)}]$	$[0.6e^{i2\pi(0.45)},0.7e^{i2\pi(0.55)}]$
$(\widehat{\Re^I_\Gamma},\widehat{\Im^I_\Gamma})(w)$	$[0.2e^{i2\pi(0.15)},0.3e^{i2\pi(0.25)}]$	$[0.3e^{i2\pi(0.15)},0.4e^{i2\pi(0.25)}]$
$(\widehat{\Re^F_\Gamma},\widehat{\Im^F_\Gamma})(w)$	$[0.9e^{i2\pi(0.75)},0.8e^{i2\pi(0.65)}]$	$[0.85e^{i2\pi(0.7)},0.9e^{i2\pi(0.75)}]$

	$(w)=\tau$	$(w)=\upsilon$
$(\Re^T_\Gamma,\Im^T_\Gamma)(w)$	$0.7e^{i2\pi(0.55)}$	$0.6e^{i2\pi(0.45)}$
$(\Re^I_\Gamma,\Im^I_\Gamma)(w)$	$0.4e^{i2\pi(0.25)}$	$0.3e^{i2\pi(0.15)}$
$(\Re^F_\Gamma,\Im^F_\Gamma)(w)$	$0.5e^{i2\pi(0.35)}$	$0.6e^{i2\pi(0.45)}$

	$(w)=\zeta$	$(w)=\eta$
$(\Re^T_\Gamma,\Im^T_\Gamma)(w)$	$0.3e^{i2\pi(0.15)}$	$0.5e^{i2\pi(0.35)}$
$(\Re^I_\Gamma,\Im^I_\Gamma)(w)$	$0.1e^{i2\pi(0.05)}$	$0.2e^{i2\pi(0.05)}$
$(\Re^F_\Gamma,\Im^F_\Gamma)(w)$	$0.8e^{i2\pi(0.65)}$	$0.75e^{i2\pi(0.6)}$

Hence, Γ is a ComCNSBS of S as a result.

Theorem 3.7. *Every ComCNSBS's intersection results in another ComCNSBS of S.*

Proof. Let $\{\widehat{\Upsilon_i} : i \in I\}$ be the collection of ComCNSBSs and $\Gamma = \bigcap_{i\in I}\widehat{\Upsilon_i}$. Let $\jmath, \psi \in S$.

Now,

$$\begin{aligned}
\widehat{\Re^T_\Gamma}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath\boxplus_1\psi))} &= \inf_{i\in I}\ \widehat{\Re^T_{\Upsilon_i}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath\boxplus_1\psi))}\\
&\geq \inf_{i\in I}\ \min\{\widehat{\Re^T_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^T_{\Upsilon_i}}(\jmath)}, \widehat{\Re^T_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^T_{\Upsilon_i}}(\psi)}\}\\
&= \min\left\{\inf_{i\in I}\ \widehat{\Re^T_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^T_{\Upsilon_i}}(\jmath)}, \inf_{i\in I}\ \widehat{\Re^T_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^T_{\Upsilon_i}}(\psi)}\right\}\\
&= \min\{\widehat{\Re^T_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}, \widehat{\Re^T_\Gamma}(\psi) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi)}\}
\end{aligned}$$

Similarly,
$\widehat{\Re^T_\Gamma}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath\boxplus_2\psi))} \geq \min\{\widehat{\Re^T_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}, \widehat{\Re^T_\Gamma}(\psi) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi)}\}$,
$\widehat{\Re^T_\Gamma}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath\boxplus_3\psi))} \geq \min\{\widehat{\Re^T_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}, \widehat{\Re^T_\Gamma}(\psi) \cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi)}\}$.
Now,

$$
\begin{aligned}
\widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_1\psi))} &= \inf_{i\in I}\ \widehat{\Re^{I}_{\Upsilon_i}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_1\psi))} \\
&\geq \inf_{i\in I}\ \frac{\widehat{\Re^{I}_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon_i}}(\jmath)} + \widehat{\Re^{I}_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon_i}}(\psi)}}{2} \\
&= \frac{\inf_{i\in I}\ \widehat{\Re^{I}_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon_i}}(\jmath)} + \inf_{i\in I}\ \widehat{\Re^{I}_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon_i}}(\psi)}}{2} \\
&= \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} + \widehat{\Re^{I}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}
\end{aligned}
$$

Similarly,
$\widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_2\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} + \widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}$ and
$\widehat{\Re^{I}_{\Gamma}}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\boxplus_3\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} + \widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}$.
Now,

$$
\begin{aligned}
\widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_1\psi))} &= \sup_{i\in I}\ \widehat{\Re^{F}_{\Upsilon_i}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_1\psi))} \\
&\leq \sup_{i\in I}\ \max\{\widehat{\Re^{F}_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon_i}}(\jmath)}, \widehat{\Re^{F}_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon_i}}(\psi)}\} \\
&= \max\left\{\sup_{i\in I}\ \widehat{\Re^{F}_{\Upsilon_i}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon_i}}(\jmath)}, \sup_{i\in I}\ \widehat{\Re^{F}_{\Upsilon_i}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon_i}}(\psi)}\right\} \\
&= \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}
\end{aligned}
$$

Similarly,
$\widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_2\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}$ and
$\widehat{\Re^{F}_{\Gamma}}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\boxplus_3\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}$.
Let $\{\Upsilon_i : i \in I\}$ be the collection of ComCNSBSs of S and $\Gamma = \bigcap_{i\in I} \Upsilon_i$. Let $\jmath, \psi \in S$.

Now,

$$
\begin{aligned}
\Re^{T}_{\Gamma}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_1\psi))} &= \inf_{i\in I}\ \Re^{T}_{\Upsilon_i}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_1\psi))} \\
&\geq \inf_{i\in I}\ \min\{\Re^{T}_{\Upsilon_i}(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon_i}(\jmath)}, \Re^{T}_{\Upsilon_i}(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon_i}(\psi)}\} \\
&= \min\left\{\inf_{i\in I}\ \Re^{T}_{\Upsilon_i}(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon_i}(\jmath)}, \inf_{i\in I}\ \Re^{T}_{\Upsilon_i}(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon_i}(\psi)}\right\} \\
&= \min\{\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\}
\end{aligned}
$$

Similarly,
$\Re^{T}_{\Gamma}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_2\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\}$,
$\Re^{T}_{\Gamma}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\boxplus_3\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\}$.
Now,

$$
\begin{aligned}
\Re^{I}_{\Gamma}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\boxplus_1\psi))} &= \inf_{i\in I}\ \Re^{I}_{\Upsilon_i}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\boxplus_1\psi))} \\
&\geq \inf_{i\in I}\ \frac{\Re^{I}_{\Upsilon_i}(\jmath) \cdot e^{i\theta\Im^{I}_{\Upsilon_i}(\jmath)} + \Re^{I}_{\Upsilon_i}(\psi) \cdot e^{i\theta\Im^{I}_{\Upsilon_i}(\psi)}}{2} \\
&= \frac{\inf_{i\in I}\ \Re^{I}_{\Upsilon_i}(\jmath) \cdot e^{i\theta\Im^{I}_{\Upsilon_i}(\jmath)} + \inf_{i\in I}\ \Re^{I}_{\Upsilon_i}(\psi) \cdot e^{i\theta\Im^{I}_{\Upsilon_i}(\psi)}}{2} \\
&= \frac{\Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} + \Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2}
\end{aligned}
$$

Similarly,
$\Re^{I}_{\Gamma}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\boxplus_2\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} + \Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2}$ and

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$\Re^I_\Gamma((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\Im^I_\Gamma((\jmath\boxplus_3\psi))} \geq \frac{\Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^I_\Gamma(\jmath)}+\Re^I_\Gamma(\psi)\cdot e^{i\theta\Im^I_\Gamma(\psi)}}{2}$.
Now,

$$\begin{aligned}\Re^F_\Gamma((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^F_\Gamma((\jmath\boxplus_1\psi))} &= \sup_{i\in I}\ \Re^F_{\Upsilon_i}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^F_\Gamma((\jmath\boxplus_1\psi))}\\ &\leq \sup_{i\in I}\ \max\{\Re^F_{\Upsilon_i}(\jmath)\cdot e^{i\theta\Im^F_{\Upsilon_i}(\jmath)}, \Re^F_{\Upsilon_i}(\psi)\cdot e^{i\theta\Im^F_{\Upsilon_i}(\psi)}\}\\ &= \max\left\{\sup_{i\in I}\ \Re^F_{\Upsilon_i}(\jmath)\cdot e^{i\theta\Im^F_{\Upsilon_i}(\jmath)}, \sup_{i\in I}\ \Re^F_{\Upsilon_i}(\psi)\cdot e^{i\theta\Im^F_{\Upsilon_i}(\psi)}\right\}\\ &= \max\{\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^F_\Gamma(\jmath)}, \Re^F_\Gamma(\psi)\cdot e^{i\theta\Im^F_\Gamma(\psi)}\}\end{aligned}$$

Similarly,
$\Re^F_\Gamma((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\Im^F_\Gamma((\jmath\boxplus_2\psi))} \leq \max\{\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^F_\Gamma(\jmath)}, \Re^F_\Gamma(\psi)\cdot e^{i\theta\Im^F_\Gamma(\psi)}\}$ and
$\Re^F_\Gamma((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\Im^F_\Gamma((\jmath\boxplus_3\psi))} \leq \max\{\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^F_\Gamma(\jmath)}, \Re^F_\Gamma(\psi)\cdot e^{i\theta\Im^F_\Gamma(\psi)}\}$.
Thus,Γ is a ComCNSBS of S.

Theorem 3.8. *If Γ and Λ be the ComCNSBSs of S_1 and S_2 respectively, then $\widehat{\Gamma\times\Lambda}$is a ComCNSBS of $S_1 \times S_2$.*

Proof. Let $\jmath_1, \jmath_2 \in S_1$ and $\psi_1, \psi_2 \in S_2$. Then $(\jmath_1, \psi_1)$ and $(\jmath_2, \psi_2)$ are in $S_1 \times S_2$. Now

$$\begin{aligned}&\widehat{\Re^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1) \boxplus_1 (\jmath_2,\psi_2))] \cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\ =\ &\widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_1 \boxplus_1 \jmath_2, \psi_1 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\ =\ &\min\{\widehat{\Re^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath_1\boxplus_1\jmath_2))}, \widehat{\Re^T_\Lambda}((\psi_1 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^T_\Lambda}((\psi_1\boxplus_1\psi_2))}\}\\ \geq\ &\min\{\min\{\widehat{\Re^T_\Gamma}(\jmath_1)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath_1)}, \widehat{\Re^T_\Gamma}(\jmath_2)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath_2)}\}, \min\{\widehat{\Re^T_\Lambda}(\psi_1)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi_1)}, \widehat{\Re^T_\Lambda}(\psi_2)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi_2)}\}\}\\ =\ &\min\{\min\{\widehat{\Re^T_\Gamma}(\jmath_1)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath_1)}, \widehat{\Re^T_\Lambda}(\psi_1)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi_1)}\}, \min\{\widehat{\Re^T_\Gamma}(\jmath_2)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath_2)}, \widehat{\Re^T_\Lambda}(\psi_2)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi_2)}\}\}\\ =\ &\min\{\widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))}, \widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\}\end{aligned}$$

Also $\widehat{\Re^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1) \boxplus_2 (\jmath_2,\psi_2))] \cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]}$
$\geq \min\{\widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))}, \widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\}$

and $\widehat{\Re^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1) \boxplus_3 (\jmath_2,\psi_2))] \cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]}$
$\geq \min\{\widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))}, \widehat{\Re^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\}$.
Now,

$$\begin{aligned}&\widehat{\Re^I_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1) \boxplus_1 (\jmath_2,\psi_2))] \cdot e^{i\theta\widehat{\Im^I_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\ =\ &\widehat{\Re^I_{\Gamma\times\Lambda}}((\jmath_1 \boxplus_1 \jmath_2, \psi_1 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^I_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\ =\ &\frac{\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2))} + \widehat{\Re^I_\Lambda}((\psi_1 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\psi_1\boxplus_1\psi_2))}}{2}\\ \geq\ &\frac{1}{2}\left[\frac{\widehat{\Re^I_\Gamma}(\jmath_1)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_1)} + \widehat{\Re^I_\Gamma}(\jmath_2)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_2)}}{2} + \frac{\widehat{\Re^I_\Lambda}(\psi_1)\cdot e^{i\theta\widehat{\Im^I_\Lambda}(\psi_1)} + \widehat{\Re^I_\Lambda}(\psi_2)\cdot e^{i\theta\widehat{\Im^I_\Lambda}(\psi_2)}}{2}\right]\\ =\ &\frac{1}{2}\left[\frac{\widehat{\Re^I_\Gamma}(\jmath_1)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_1)} + \widehat{\Re^I_\Lambda}(\psi_1)\cdot e^{i\theta\widehat{\Im^I_\Lambda}(\psi_1)}}{2} + \frac{\widehat{\Re^I_\Gamma}(\jmath_2)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_2)} + \widehat{\Re^I_\Lambda}(\psi_2)\cdot e^{i\theta\widehat{\Im^I_\Lambda}(\psi_2)}}{2}\right]\\ =\ &\frac{1}{2}\left[\widehat{\Re^I_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))} + \widehat{\Re^I_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^T_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\right]\end{aligned}$$

Also
$\widehat{\Re^I_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]\cdot e^{i\theta\widehat{\Im^I_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]} \geq \frac{1}{2}\left[\widehat{\Re^I_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^I_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))} + \widehat{\Re^I_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot\right.$

$e^{i\theta\widehat{\Im^{I}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\Big]$
and
$\widehat{\Re^{I}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]}\geq\frac{1}{2}\Big[\widehat{\Re^{I}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))}+\widehat{\Re^{I}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot$
$e^{i\theta\widehat{\Im^{I}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\Big]$.
Now,

$$\begin{aligned}
&\widehat{\Re^{F}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\
=\;&\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\
=\;&\max\{\widehat{\Re^{F}_{\Gamma}}((\jmath_1\boxplus_1\jmath_2))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma\times\Lambda}}((\jmath_1\boxplus_1\jmath_2))},\widehat{\Re^{F}_{\Lambda}}((\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma\times\Lambda}}((\psi_1\boxplus_1\psi_2))}\}\\
\leq\;&\max\{\max\{\widehat{\Re^{F}_{\Gamma}}(\jmath_1)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath_1)},\widehat{\Re^{F}_{\Gamma}}(\jmath_2)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath_2)}\},\max\{\widehat{\Re^{F}_{\Lambda}}(\psi_1)\cdot e^{i\theta\widehat{\Im^{F}_{\Lambda}}(\psi_1)},\widehat{\Re^{F}_{\Lambda}}(\psi_2)\cdot e^{i\theta\widehat{\Im^{F}_{\Lambda}}(\psi_2)}\}\}\\
=\;&\max\{\max\{\widehat{\Re^{F}_{\Gamma}}(\jmath_1)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath_1)},\widehat{\Re^{F}_{\Lambda}}(\psi_1)\cdot e^{i\theta\widehat{\Im^{F}_{\Lambda}}(\psi_1)}\},\max\{\widehat{\Re^{F}_{\Gamma}}(\jmath_2)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath_2)},\widehat{\Re^{F}_{\Lambda}}(\psi_2)\cdot e^{i\theta\widehat{\Im^{F}_{\Lambda}}(\psi_2)}\}\}\\
=\;&\max\{\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))},\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\}
\end{aligned}$$

Also $\widehat{\Re^{F}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]}$
$\leq\max\{\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))},\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}}((\jmath_2,\psi_2))\}$,
$\widehat{\Re^{F}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]}$
$\leq\max\{\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}((\jmath_1,\psi_1))},\widehat{\Re^{F}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))}\}$.

Let $\jmath_1,\jmath_2\in S_1$ and $\psi_1,\psi_2\in S_2$. Then $(\jmath_1,\psi_1)$ and $(\jmath_2,\psi_2)$ are in $S_1\times S_2$. Now

$$\begin{aligned}
&\Re^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\
=\;&\Re^{T}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\
=\;&\min\{\Re^{T}_{\Gamma}((\jmath_1\boxplus_1\jmath_2))\cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath_1\boxplus_1\jmath_2))},\Re^{T}_{\Lambda}((\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{T}_{\Lambda}((\psi_1\boxplus_1\psi_2))}\}\\
\geq\;&\min\{\min\{\Re^{T}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath_1)},\Re^{T}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath_2)}\},\min\{\Re^{T}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi_1)},\Re^{T}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi_2)}\}\}\\
=\;&\min\{\min\{\Re^{T}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath_1)},\Re^{T}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi_1)}\},\min\{\Re^{T}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath_2)},\Re^{T}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi_2)}\}\}\\
=\;&\min\{\Re^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\}
\end{aligned}$$

Also $\Re^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]}$
$\geq\min\{\Re^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\}$

and $\Re^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]}$
$\geq\min\{\Re^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\}$.
Now,

$$\begin{aligned}
&\Re^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\
=\;&\Re^{I}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\
=\;&\frac{\Re^{I}_{\Gamma}((\jmath_1\boxplus_1\jmath_2))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2))}+\Re^{I}_{\Lambda}((\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\psi_1\boxplus_1\psi_2))}}{2}\\
\geq\;&\frac{1}{2}\left[\frac{\Re^{I}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath_1)}+\Re^{I}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath_2)}}{2}+\frac{\Re^{I}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{I}_{\Lambda}(\psi_1)}+\Re^{I}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{I}_{\Lambda}(\psi_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\frac{\Re^{I}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath_1)}+\Re^{I}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{I}_{\Lambda}(\psi_1)}}{2}+\frac{\Re^{I}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath_2)}+\Re^{I}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{I}_{\Lambda}(\psi_2)}}{2}\right]\\
=\;&\frac{1}{2}\left[\Re^{I}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))}+\Re^{I}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\right]
\end{aligned}$$

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Also
$\Re^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]} \geq \frac{1}{2}\Big[\Re^{I}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))}+\Re^{I}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\Big]$
and
$\Re^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]} \geq \frac{1}{2}\Big[\Re^{I}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))}+\Re^{I}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{I}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\Big]$.
Now,

$$\begin{aligned}
& \Re^{F}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{F}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_1(\jmath_2,\psi_2))]}\\
= & \Re^{F}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{F}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2,\psi_1\boxplus_1\psi_2))}\\
= & \max\{\Re^{F}_{\Gamma}((\jmath_1\boxplus_1\jmath_2))\cdot e^{i\theta\Im^{F}_{\Gamma\times\Lambda}((\jmath_1\boxplus_1\jmath_2))},\Re^{F}_{\Lambda}((\psi_1\boxplus_1\psi_2))\cdot e^{i\theta\Im^{F}_{\Gamma\times\Lambda}((\psi_1\boxplus_1\psi_2))}\}\\
\leq & \max\{\max\{\Re^{F}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath_1)},\Re^{F}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath_2)}\},\max\{\Re^{F}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{F}_{\Lambda}(\psi_1)},\Re^{F}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{F}_{\Lambda}(\psi_2)}\}\}\\
= & \max\{\max\{\Re^{F}_{\Gamma}(\jmath_1)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath_1)},\Re^{F}_{\Lambda}(\psi_1)\cdot e^{i\theta\Im^{F}_{\Lambda}(\psi_1)}\},\max\{\Re^{F}_{\Gamma}(\jmath_2)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath_2)},\Re^{F}_{\Lambda}(\psi_2)\cdot e^{i\theta\Im^{F}_{\Lambda}(\psi_2)}\}\}\\
= & \max\{\Re^{F}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{F}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\}
\end{aligned}$$

Also $\Re^{F}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_2(\jmath_2,\psi_2))]}$
$\leq \max\{\Re^{F}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{F}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}}((\jmath_2,\psi_2))\}$,
$\Re^{F}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}[((\jmath_1,\psi_1)\boxplus_3(\jmath_2,\psi_2))]}$
$\leq \max\{\Re^{F}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_1,\psi_1))},\Re^{F}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))\cdot e^{i\theta\Im^{T}_{\Gamma\times\Lambda}((\jmath_2,\psi_2))}\}$.
Thus,$\widehat{\Gamma\times\Lambda}$ is a ComCNSBS.

Definition 3.9. The strongest ComCN connection on S, given $\Gamma\subseteq S$ is

$$\left\{\begin{array}{l}
\widehat{\Re^{T}_{\Upsilon}}((\jmath,\psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\jmath,\psi))} = min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)},\widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)}\}\\
\widehat{\Re^{I}_{\Upsilon}}((\jmath,\psi))\cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}((\jmath,\psi))} = \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)}+\widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}\\
\widehat{\Re^{F}_{\Upsilon}}((\jmath,\psi))\cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((\jmath,\psi))} = max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)},\widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}
\end{array}\right\}$$

Theorem 3.10. *Let Υ be the strongest complex cubic neutrosophic relation of S, and let Γ be a ComCNSBS of S. If and only if Υ is a ComCNSBS of $S\times S$, then Γ is a ComCNSBS of $S\times S$.*

Proof. Let $\widehat{\Upsilon}$ be the strongest complex cubic neutrosophic relation of S, and suppose Γ is a ComCNSBS of $S\times S$.
For $\jmath=(\jmath_1,\jmath_2),\psi=(\psi_1,\psi_2)\in S\times S$. Now,

$$\begin{aligned}
& \widehat{\Re^{T}_{\Upsilon}}((\jmath\boxplus_1\psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\jmath\boxplus_1\psi))}\\
= & \widehat{\Re^{T}_{\Upsilon}}[((((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))]\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}[((((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))]}\\
= & \widehat{\Re^{T}_{\Upsilon}}(\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2)}\\
= & \min\{\widehat{\Re^{T}_{\Gamma}}((\jmath_1\boxplus_1\psi_1))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath_1\boxplus_1\psi_1))},\widehat{\Re^{T}_{\Gamma}}((\jmath_2\boxplus_1\psi_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath_2\boxplus_1\psi_2))}\}\\
\geq & \min\{\min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath_1)},\widehat{\Re^{T}_{\Gamma}}(\psi_1)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi_1)}\},\\
& \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_2)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath_2)},\widehat{\Re^{T}_{\Gamma}}(\psi_2)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi_2)}\}\}\\
= & \min\{\min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath_1)},\widehat{\Re^{T}_{\Gamma}}(\jmath_2)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath_2)}\},\\
& \min\{\widehat{\Re^{T}_{\Gamma}}(\psi_1)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi_1)},\widehat{\Re^{T}_{\Gamma}}(\psi_2)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi_2)}\}\}\\
= & \min\{\widehat{\Re^{T}_{\Upsilon}}((\jmath_1,\jmath_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\jmath_1,\jmath_2))},\widehat{\Re^{T}_{\Upsilon}}((\psi_1,\psi_2))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\psi_1,\psi_2))}\}\\
= & \min\{\widehat{\Re^{T}_{\Upsilon}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\jmath)},\widehat{\Re^{T}_{\Upsilon}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\psi)}\}
\end{aligned}$$

Also $\widehat{\Re^{T}_{\Upsilon}}((\jmath\boxplus_2\psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\jmath\boxplus_2\psi))}\geq\min\{\widehat{\Re^{T}_{\Upsilon}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\jmath)},\widehat{\Re^{T}_{\Upsilon}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\psi)}\}$,
$\widehat{\Re^{T}_{\Upsilon}}((\jmath\boxplus_3\psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\jmath\boxplus_3\psi))}\geq\min\{\widehat{\Re^{T}_{\Upsilon}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\jmath)},\widehat{\Re^{T}_{\Upsilon}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\psi)}\}$.

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Now, $\widehat{\Re^I_\Upsilon}(\jmath \boxplus_1 \psi) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}(\jmath\boxplus_1\psi)}$

$$
\begin{aligned}
&= \widehat{\Re^I_\Upsilon}[((((\jmath_1, \jmath_2)) \boxplus_1 (\psi_1, \psi_2))] \cdot e^{i\theta\widehat{\Im^I_\Upsilon}[((((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2)))]}\\
&= \widehat{\Re^I_\Upsilon}((\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}((\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2))}\\
&= \frac{\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}((\jmath_1\boxplus_1\psi_1))} + \widehat{\Re^I_\Gamma}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}((\jmath_2 \boxplus_1 \psi_2))}{2}\\
&\geq \frac{1}{2}\left[\frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_1)} + \widehat{\Re^I_\Gamma}(\psi_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\psi_1)}}{2} + \frac{\widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_2)} + \widehat{\Re^I_\Gamma}(\psi_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\psi_2)}}{2}\right]\\
&= \frac{1}{2}\left[\frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_1)} + \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath_2)}}{2} + \frac{\widehat{\Re^I_\Gamma}(\psi_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma(\psi_1)}} + \widehat{\Re^I_\Gamma}(\psi_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}(\psi_2)}}{2}\right]\\
&= \frac{\widehat{\Re^I_\Upsilon}((\jmath_1, \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}((\jmath_1,\jmath_2))} + \widehat{\Re^I_\Upsilon}((\psi_1, \psi_2)) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}((\psi_1,\psi_2))}}{2}\\
&= \frac{\widehat{\Re^I_\Upsilon}(\jmath) \cdot e^{i\theta\widehat{\Im^I_\Upsilon(\jmath)}} + \widehat{\Re^I_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}(\jmath)}}{2}
\end{aligned}
$$

Also $\widehat{\Re^I_\Upsilon}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}((\jmath\boxplus_2\psi))} \geq \frac{\widehat{\Re^I_\Upsilon}(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Upsilon(\jmath)}}+\widehat{\Re^I_\Upsilon}(\psi)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}(\psi)}}{2}$
and $\widehat{\Re^I_\Upsilon}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\widehat{\Im^I_\Upsilon}((\jmath\boxplus_3\psi))} \geq \frac{\widehat{\Re^I_\Upsilon}(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Upsilon(\jmath)}}+\widehat{\Re^I_\Upsilon}(\psi)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}(\psi)}}{2}$.
Similarly,$\widehat{\Re^F_\Upsilon}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\widehat{\Im^F_\Upsilon}((\jmath\boxplus_1\psi))} \leq \max\{\widehat{\Re^F_\Upsilon}(\jmath) \cdot e^{i\theta\widehat{\Im^F_\Upsilon(\jmath)\cdot}}, \widehat{\Re^F_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^F_\Upsilon(\psi)}}\}$,
$\widehat{\Re^F_\Upsilon}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\widehat{\Im^F_\Upsilon}((\jmath\boxplus_2\psi))} \leq \max\{\widehat{\Re^F_\Upsilon}(\jmath) \cdot e^{i\theta\widehat{\Im^F_\Upsilon(\jmath)}}, \widehat{\Re^F_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^F_\Upsilon(\psi)}}\}$ and
$\widehat{\Re^F_\Upsilon}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\widehat{\Im^F_\Upsilon}((\jmath\boxplus_3\psi))} \leq \max\{\widehat{\Re^F_\Upsilon}(\jmath) \cdot e^{i\theta\widehat{\Im^F_\Upsilon(\jmath)}}, \widehat{\Re^F_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^I_\Upsilon(\psi}}\}$.
For any $\jmath = (\jmath_1, \jmath_2), \psi = (\psi_1, \psi_2) \in S \times S$. Now,

$$
\begin{aligned}
& \Re^T_\Upsilon((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^T_\Upsilon((\jmath\boxplus_1\psi))}\\
=\;& \Re^T_\Upsilon[((((\jmath_1, \jmath_2)) \boxplus_1 ((\psi_1, \psi_2))] \cdot e^{i\theta\Im^T_\Upsilon[((((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))]}\\
=\;& \Re^T_\Upsilon(\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2) \cdot e^{i\theta\Im^T_\Upsilon(\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2)}\\
=\;& \min\{\Re^T_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_1\psi_1))}, \Re^T_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_1\psi_2))}\}\\
\geq\;& \min\{\min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}\},\\
& \min\{\Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma(\jmath_2)}, \Re^T_\Gamma(\psi_2) \cdot e^{i\theta\Im^T_\Gamma(\psi_2)}\}\}\\
=\;& \min\{\min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma(\jmath_2)}\},\\
& \min\{\Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}, \Re^T_\Gamma(\psi_2) \cdot e^{i\theta\Im^T_\Gamma(\psi_2)}\}\}\\
=\;& \min\{\Re^T_\Upsilon((\jmath_1, \jmath_2)) \cdot e^{i\theta\Im^T_\Upsilon((\jmath_1,\jmath_2))}, \Re^T_\Upsilon((\psi_1, \psi_2)) \cdot e^{i\theta\Im^T_\Upsilon((\psi_1,\psi_2))}\}\\
=\;& \min\{\Re^T_\Upsilon(\jmath) \cdot e^{i\theta\Im^T_\Upsilon(\jmath)}, \Re^T_\Upsilon(\psi) \cdot e^{i\theta\Im^T_\Upsilon(\psi)}\}
\end{aligned}
$$

Also $\Re^T_\Upsilon((\jmath \boxplus_2 \psi)) \cdot e^{i\theta\Im^T_\Upsilon((\jmath\boxplus_2\psi))} \geq \min\{\Re^T_\Upsilon(\jmath) \cdot e^{i\theta\Im^T_\Upsilon(\jmath)}, \Re^T_\Upsilon(\psi) \cdot e^{i\theta\Im^T_\Upsilon(\psi)}\}$,
$\Re^T_\Upsilon((\jmath \boxplus_3 \psi)) \cdot e^{i\theta\Im^T_\Upsilon((\jmath\boxplus_3\psi))} \geq \min\{\Re^T_\Upsilon(\jmath) \cdot e^{i\theta\Im^T_\Upsilon(\jmath)}, \Re^T_\Upsilon(\psi) \cdot e^{i\theta\Im^T_\Upsilon(\psi)}\}$.

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Now, $\Re^{I}_{\Upsilon}(\jmath \boxplus_1 \psi) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\jmath \boxplus_1 \psi)}$

$$
\begin{aligned}
&= \Re^{I}_{\Upsilon}[((((\jmath_1, \jmath_2)) \boxplus_1 (\psi_1, \psi_2))] \cdot e^{i\theta \Im^{I}_{\Upsilon}[((((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2)))]} \\
&= \Re^{I}_{\Upsilon}((\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta \Im^{I}_{\Upsilon}((\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2))} \\
&= \frac{\Re^{I}_{\Gamma}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta \Im^{I}_{\Gamma}((\jmath_1 \boxplus_1 \psi_1))} + \Re^{I}_{\Gamma}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta \Im^{I}_{\Gamma}}((\jmath_2 \boxplus_1 \psi_2))}{2} \\
&\geq \frac{1}{2}\left[\frac{\Re^{I}_{\Gamma}(\jmath_1) \cdot e^{i\theta \Im^{I}_{\Gamma}(\jmath_1)} + \Re^{I}_{\Gamma}(\psi_1) \cdot e^{i\theta \Im^{I}_{\Gamma}(\psi_1)}}{2} + \frac{\Re^{I}_{\Gamma}(\jmath_2) \cdot e^{i\theta \Im^{I}_{\Gamma}(\jmath_2)} + \Re^{I}_{\Gamma}(\psi_2) \cdot e^{i\theta \Im^{I}_{\Gamma}(\psi_2)}}{2}\right] \\
&= \frac{1}{2}\left[\frac{\Re^{I}_{\Gamma}(\jmath_1) \cdot e^{i\theta \Im^{I}_{\Gamma}(\jmath_1)} + \Re^{I}_{\Gamma}(\jmath_2) \cdot e^{i\theta \Im^{I}_{\Gamma}(\jmath_2)}}{2} + \frac{\Re^{I}_{\Gamma}(\psi_1) \cdot e^{i\theta \Im^{I}_{\Gamma}(\psi_1)} + \Re^{I}_{\Gamma}(\psi_2) \cdot e^{i\theta \Im^{I}_{\Gamma}(\psi_2)}}{2}\right] \\
&= \frac{\Re^{I}_{\Upsilon}((\jmath_1, \jmath_2)) \cdot e^{i\theta \Im^{I}_{\Upsilon}((\jmath_1, \jmath_2))} + \Re^{I}_{\Upsilon}((\psi_1, \psi_2)) \cdot e^{i\theta \Im^{I}_{\Upsilon}((\psi_1, \psi_2))}}{2} \\
&= \frac{\Re^{I}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\jmath)} + \Re^{I}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\jmath)}}{2}
\end{aligned}
$$

Also $\Re^{I}_{\Upsilon}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta \Im^{I}_{\Upsilon}((\jmath \boxplus_2 \psi))} \geq \frac{\Re^{I}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\jmath)} + \Re^{I}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\psi)}}{2}$
and $\Re^{I}_{\Upsilon}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta \Im^{I}_{\Upsilon}((\jmath \boxplus_3 \psi))} \geq \frac{\Re^{I}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\jmath)} + \Re^{I}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\psi)}}{2}$.
Similarly,$\Re^{F}_{\Upsilon}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta \Im^{F}_{\Upsilon}((\jmath \boxplus_1 \psi))} \leq \max\{\Re^{F}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{F}_{\Upsilon}(\jmath) \cdot}, \Re^{F}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{F}_{\Upsilon}(\psi)}\}$,
$\Re^{F}_{\Upsilon}((\jmath \boxplus_2 \psi)) \cdot e^{i\theta \Im^{F}_{\Upsilon}((\jmath \boxplus_2 \psi))} \leq \max\{\Re^{F}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{F}_{\Upsilon}(\jmath)}, \Re^{F}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{F}_{\Upsilon}(\psi)}\}$ and
$\Re^{F}_{\Upsilon}((\jmath \boxplus_3 \psi)) \cdot e^{i\theta \Im^{F}_{\Upsilon}((\jmath \boxplus_3 \psi))} \leq \max\{\Re^{F}_{\Upsilon}(\jmath) \cdot e^{i\theta \Im^{F}_{\Upsilon}(\jmath)}, \Re^{F}_{\Upsilon}(\psi) \cdot e^{i\theta \Im^{I}_{\Upsilon}(\psi}\}$.
Therefore, Υ is a ComCNSBS of $S \times S$.
Conversely, Assume that Υ is a ComCNSBS of $S \times S$.
Let $\jmath = ((\jmath_1, \jmath_2)), \psi = ((\psi_1, \psi_2)) \in S \times S$. Now,

$$
\begin{aligned}
&\min\{\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_1 \psi_1)})}, \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_2 \boxplus_1 \psi_2)})}\} \\
&= \widehat{\Re^{T}_{\Upsilon}}(\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2) \cdot e^{i\theta \Im^{T}_{\Upsilon}(\jmath_1 \widehat{\boxplus_1 \psi_1, \jmath_2} \boxplus_1 \psi_2)} \\
&= \widehat{\Re^{T}_{\Upsilon}}[((\jmath_1, \jmath_2)) \boxplus_1 ((\psi_1, \psi_2))] \cdot e^{i\theta \Im^{T}_{v}[((\jmath_1,\jmath_2\widehat{))\boxplus_1}((\psi_1,\psi_2))]} \\
&= \widehat{\Re^{T}_{\Upsilon}}(\jmath \boxplus_1 \psi) \cdot e^{i\theta \widehat{\Im^{T}_{\Upsilon}(\jmath \boxplus_1} \psi)} \\
&\geq \min\{\widehat{\Re^{T}_{\Upsilon}}(\jmath) \cdot e^{i\theta \widehat{\Im^{T}_{\Upsilon}(\jmath)}}, \widehat{\Re^{T}_{\Upsilon}}(\psi) \cdot e^{i\theta \widehat{\Im^{T}_{\Upsilon}(\psi)}}\} \\
&= \min\{\widehat{\Re^{T}_{\Upsilon}}((\jmath_1, \jmath_2))\} \cdot e^{i\theta \Im^{T}_{\Upsilon}\widehat{((\jmath_1, \jmath_2))}\}}, \widehat{\Re^{T}_{\Upsilon}}((\psi_1, \psi_2)) \cdot e^{i\theta \widehat{\Im^{T}_{\Upsilon}}((\psi_1,\psi_2))}\} \\
&= \min\{\min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\jmath_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_2)}\}, \min\{\widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}}, \widehat{\Re^{T}_{\Gamma}}(\psi_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_2)}}\}\}
\end{aligned}
$$

If $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_1 \psi_1)})} \leq \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_2 \boxplus_1 \psi_2)})}$, then $\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)} \leq \widehat{\Re^{T}_{\Gamma}}(\jmath_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_2)}$ and $\widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}} \leq \widehat{\Re^{T}_{\Gamma}}(\psi_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_2)}}$. We get $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_1 \psi_1)})} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}}\}$ for all $\jmath_1, \psi_1 \in S$,and
$\min\{\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_2 \psi_1)})}, \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_2 \psi_2)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_2 \boxplus_2 \psi_2)})}\} \geq \min\{\min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\jmath_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_2)}\}, \min\{\widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}}, \widehat{\Re^{T}_{\Gamma}}(\psi_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_2)}}\}\}$
If $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_2 \psi_1)})} \leq \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_2 \psi_2)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_2 \boxplus_2 \psi_2)})}$,then $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta \Im^{T}_{\Gamma}(\widehat{(\jmath_1 \boxplus_2 \psi_1)})} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}}\}$.
$\min\{\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1)) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1))}, \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_3 \psi_2)) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}((\jmath_2 \boxplus_3 \psi_2))}\} \geq \min\{\min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\jmath_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_2)}\}, \min\{\widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\psi_1)}, \widehat{\Re^{T}_{\Gamma}}(\psi_2) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\psi_2)}\}\}$
If $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1)) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1))} \leq \widehat{\Re^{T}_{\Gamma}}((\jmath_2 \boxplus_3 \psi_2)) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}((\jmath_2 \boxplus_3 \psi_2))}$,then $\widehat{\Re^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1)) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}((\jmath_1 \boxplus_3 \psi_1))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}}(\jmath_1)}, \widehat{\Re^{T}_{\Gamma}}(\psi_1) \cdot e^{i\theta \widehat{\Im^{T}_{\Gamma}(\psi_1)}}\}$.
Now,

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$$\frac{1}{2}\Big[\widehat{\Re_{\Gamma}^{I}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{I}(\widehat{(\jmath_1\boxplus_1\psi_1)})} + \widehat{\Re_{\Gamma}^{I}}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im_{\Gamma}^{I(\widehat{(\jmath_2\boxplus_1\psi_2)})}}\Big]$$
$$= \widehat{\Re_{\Upsilon}^{I}}(\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2) \cdot e^{i\theta\Im_{\Upsilon}^{I}(\widehat{\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2})}$$
$$= \widehat{\Re_{\Upsilon}^{I}}[((\jmath_1,\jmath_2)) \boxplus_1 ((\psi_1,\psi_2))] \cdot e^{i\theta\Im_{\Upsilon}^{I}[\widehat{((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))}]}$$
$$= \widehat{\Re_{\Upsilon}^{I}}((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im_{\Upsilon}^{I}\widehat{((\jmath\boxplus_1\psi))}}$$
$$\geq \frac{\widehat{\Re_{\Upsilon}^{I}}(\jmath) \cdot e^{i\theta\widehat{\Im_{\Upsilon}^{I(\jmath)}}} + \widehat{\Re_{\Upsilon}^{I}}(\psi) \cdot e^{i\theta\widehat{\Im_{\Upsilon}^{I}(\psi)}}}{2}$$
$$= \frac{\widehat{\Re_{\Upsilon}^{I}}((\jmath_1,\jmath_2)) \cdot e^{i\theta\Im_{\Upsilon}^{I}\widehat{((\jmath_1,\jmath_2))}} + \widehat{\Re_{\Upsilon}^{I}}((\psi_1,\psi_2)) \cdot e^{i\theta\widehat{\Im_{\Upsilon}^{I}}}}{2}$$
$$= \frac{1}{2}\left[\frac{\widehat{\Re_{\Gamma}^{I}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_1)} + \widehat{\Re_{\Gamma}^{I}}(\jmath_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_2)}}{2} + \frac{\widehat{\Re_{\Gamma}^{I}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}(\psi_1)}} + \widehat{\Re_{\Gamma}^{I}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}(\psi_2)}}}{2}\right]$$

If $\widehat{\Re_{\Gamma}^{I}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}((\jmath_1\boxplus_1\psi_1))} \leq \widehat{\Re_{\Gamma}^{I}}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im_{\Gamma}^{I}(\widehat{(\jmath_2\boxplus_1\psi_2)})}$, then $\widehat{\Re_{\Gamma}^{I}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_1)} \leq \widehat{\Re_{\Gamma}^{I}}(\jmath_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_2)}$ and $\widehat{\Re_{\Gamma}^{I}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}(\psi_1)}} \leq \widehat{\Re_{\Gamma}^{I}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}(\psi_2)}}$.
We get $\widehat{\Re_{\Gamma}^{I}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{T}}((\jmath_1\boxplus_1\psi_1))} \geq \frac{\widehat{\Re_{\Gamma}^{I}}(\jmath_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_1)}+\widehat{\Re_{\Gamma}^{I}}(\psi_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{T}(\psi_1)}}}{2}$.
Similarly,$\widehat{\Re_{\Gamma}^{I}}((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{I}(\widehat{(\jmath_1\boxplus_2\psi_1)})} \geq \frac{\widehat{\Re_{\Gamma}^{I}}(\jmath_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_1)}+\widehat{\Re_{\Gamma}^{I}}(\psi_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\psi_1)}}{2}$
and $\widehat{\Re_{\Gamma}^{I}}((\jmath_1 \boxplus_3 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{I}(\widehat{(\jmath_1\boxplus_3\psi_1)})} \geq \frac{\widehat{\Re_{\Gamma}^{I}}(\jmath_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}}(\jmath_1)}+\widehat{\Re_{\Gamma}^{I}}(\psi_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{I}(\psi_1)}}}{2}$.
Similarly to prove that
$\max\{\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_1\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_1\psi_1)})}, \widehat{\Re_{\Gamma}^{F}}((\jmath_2\boxplus_1\psi_2))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_2\boxplus_1\psi_2)})}\} \leq \max\{\max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\jmath_2)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_2)}\}, \max\{\widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}}, \widehat{\Re_{\Gamma}^{F}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_2)}}\}\}$
If $\widehat{\Re_{\Gamma}^{F}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_1\psi_1)})} \geq \widehat{\Re_{\Gamma}^{F}}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_2\boxplus_1\psi_2)})}$,then $\widehat{\Re_{\Gamma}^{F}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)} \geq \widehat{\Re_{\Gamma}^{F}}(\jmath_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_2)}$ and $\widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}} \geq \widehat{\Re_{\Gamma}^{F}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_2)}}$.
We get $\widehat{\Re_{\Gamma}^{F}}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_1\psi_1)})} \leq \max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}}\}$.
$\max\{\widehat{\Re_{\Gamma}^{F}}((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_2\psi_1)})}, \widehat{\Re_{\Gamma}^{F}}((\jmath_2 \boxplus_2 \psi_2)) \cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_2\boxplus_2\psi_2)})}\}$
$\leq \max\{\max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\jmath_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_2)}\}, \max\{\widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}}, \widehat{\Re_{\Gamma}^{F}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}}\}\}$
If $\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_2\psi_1)})} \geq \widehat{\Re_{\Gamma}^{F}}((\jmath_2\boxplus_2\psi_2))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_2\boxplus_2\psi_2)})}$,then $\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_2\psi_1)})} \leq \max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}}\}$.
$\max\{\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_1\boxplus_3\psi_1)})}, \widehat{\Re_{\Gamma}^{F}}((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im_{\Gamma}^{F}(\widehat{(\jmath_2\boxplus_3\psi_2)})}\} \leq \max\{\max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\jmath_2)\cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_2)}\}, \max\{\widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_1)}}, \widehat{\Re_{\Gamma}^{F}}(\psi_2) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}(\psi_2)}}\}\}$
If $\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{T}(\widehat{(\jmath_1\boxplus_3\psi_1)})} \geq \widehat{\Re_{\Gamma}^{F}}((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im_{\Gamma}^{T}(\widehat{(\jmath_2\boxplus_3\psi_2)})}$,then $\widehat{\Re_{\Gamma}^{F}}((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im_{\Gamma}^{T}(\widehat{(\jmath_1\boxplus_3\psi_1)})} \leq \max\{\widehat{\Re_{\Gamma}^{F}}(\jmath_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{F}}(\jmath_1)}, \widehat{\Re_{\Gamma}^{F}}(\psi_1) \cdot e^{i\theta\widehat{\Im_{\Gamma}^{T}(\psi_1)}}\}$.

Let $\jmath = ((\jmath_1,\jmath_2)), \psi = ((\psi_1,\psi_2)) \in S \times S$. Now,

$$\min\{\Re_{\Gamma}^{T}((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im_{\Gamma}^{T}((\jmath_1\boxplus_1\psi_1))}, \Re_{\Gamma}^{T}((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im_{\Gamma}^{T}((\jmath_2\boxplus_1\psi_2))}\}$$
$$= \Re_{\Upsilon}^{T}(\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2) \cdot e^{i\theta\Im_{\Upsilon}^{T}(\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2)}$$
$$= \Re_{\Upsilon}^{T}[((\jmath_1,\jmath_2)) \boxplus_1 ((\psi_1,\psi_2))] \cdot e^{i\theta\Im_{v}^{T}[((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))]}$$
$$= \Re_{\Upsilon}^{T}(\jmath \boxplus_1 \psi) \cdot e^{i\theta\Im_{\Upsilon}^{T}(\jmath\boxplus_1\psi)}$$
$$\geq \min\{\Re_{\Upsilon}^{T}(\jmath) \cdot e^{i\theta\Im_{\Upsilon}^{T}(\jmath)}, \Re_{\Upsilon}^{T}(\psi) \cdot e^{i\theta\Im_{\Upsilon}^{T}(\psi)}\}$$
$$= \min\{\Re_{\Upsilon}^{T}((\jmath_1,\jmath_2))\} \cdot e^{i\theta\Im_{\Upsilon}^{T}((\jmath_1,\jmath_2))\}}, \Re_{\Upsilon}^{T}((\psi_1,\psi_2)) \cdot e^{i\theta\Im_{\Upsilon}^{T}((\psi_1,\psi_2))}\}$$
$$= \min\{\min\{\Re_{\Gamma}^{T}(\jmath_1) \cdot e^{i\theta\Im_{\Gamma}^{T}(\jmath_1)}, \Re_{\Gamma}^{T}(\jmath_2) \cdot e^{i\theta\Im_{\Gamma}^{T}(\jmath_2)}\}, \min\{\Re_{\Gamma}^{T}(\psi_1) \cdot e^{i\theta\Im_{\Gamma}^{T}(\psi_1)}, \Re_{\Gamma}^{T}(\psi_2) \cdot e^{i\theta\Im_{\Gamma}^{T}(\psi_2)}\}\}$$

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If $\Re^T_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_1\psi_1))} \leq \Re^T_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_1\psi_2))}$, then $\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)} \leq \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma(\jmath_2)}$ and $\Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)} \leq \Re^T_\Gamma(\psi_2) \cdot e^{i\theta\Im^T_\Gamma(\psi_2)}$. We get $\Re^T_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_1\psi_1))} \geq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}\}$ for all $\jmath_1, \psi_1 \in S$,and
$\min\{\Re^T_\Gamma((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_2\psi_1))}, \Re^T_\Gamma((\jmath_2\boxplus_2\psi_2))\cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_2\psi_2))}\} \geq \min\{\min\{\Re^T_\Gamma(\jmath_1)\cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\jmath_2)\cdot e^{i\theta\Im^T_\Gamma(\jmath_2)}\}, \min\{\Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}, \Re^T_\Gamma(\psi_2) \cdot e^{i\theta\Im^T_\Gamma(\psi_2)}\}\}$
If $\Re^T_\Gamma((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_2\psi_1))} \leq \Re^T_\Gamma((\jmath_2\boxplus_2\psi_2))\cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_2\psi_2))}$,then $\Re^T_\Gamma((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_2\psi_1))} \geq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}\}$.
$\min\{\Re^T_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_3\psi_1))}, \Re^T_\Gamma((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_3\psi_2))}\} \geq \min\{\min\{\Re^T_\Gamma(\jmath_1)\cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\jmath_2)\cdot e^{i\theta\Im^T_\Gamma(\jmath_2)}\}, \min\{\Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}, \Re^T_\Gamma(\psi_2) \cdot e^{i\theta\Im^T_\Gamma(\psi_2)}\}\}$
If $\Re^T_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_3\psi_1))} \leq \Re^T_\Gamma((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_3\psi_2))}$,then $\Re^T_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_3\psi_1))} \geq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma(\jmath_1)}, \Re^T_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}\}$.
Now,

$$
\begin{aligned}
&\frac{1}{2}\Big[\Re^I_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^I_\Gamma((\jmath_1\boxplus_1\psi_1))} + \Re^I_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im_\Gamma^{I((\jmath_2\boxplus_1\psi_2))}}\Big] \\
&= \Re^I_\Upsilon(\jmath_1 \boxplus_1 \psi_1, \jmath_2 \boxplus_1 \psi_2) \cdot e^{i\theta\Im^I_\Upsilon(\jmath_1\boxplus_1\psi_1,\jmath_2\boxplus_1\psi_2)} \\
&= \Re^I_\Upsilon[((\jmath_1, \jmath_2)) \boxplus_1 ((\psi_1, \psi_2))] \cdot e^{i\theta\Im^I_\Upsilon[((\jmath_1,\jmath_2))\boxplus_1((\psi_1,\psi_2))]} \\
&= \Re^I_\Upsilon((\jmath \boxplus_1 \psi)) \cdot e^{i\theta\Im^I_\Upsilon((\jmath\boxplus_1\psi))} \\
&\geq \frac{\Re^I_\Upsilon(\jmath) \cdot e^{i\theta\Im_\Upsilon^{I(\jmath)}} + \Re^I_\Upsilon(\psi) \cdot e^{i\theta\Im^I_\Upsilon(\psi)}}{2} \\
&= \frac{\Re^I_\Upsilon((\jmath_1, \jmath_2)) \cdot e^{i\theta\Im^I_\Upsilon((\jmath_1,\jmath_2))} + \Re^I_\Upsilon((\psi_1, \psi_2)) \cdot e^{i\theta\Im^I_\Upsilon}}{2} \\
&= \frac{1}{2}\left[\frac{\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma(\jmath_1)} + \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma(\jmath_2)}}{2} + \frac{\Re^I_\Gamma(\psi_1) \cdot e^{i\theta\Im^I_\Gamma(\psi_1)} + \Re^I_\Gamma(\psi_2) \cdot e^{i\theta\Im^I_\Gamma(\psi_2)}}{2}\right]
\end{aligned}
$$

If $\Re^I_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^I_\Gamma((\jmath_1\boxplus_1\psi_1))} \leq \Re^I_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im^I_\Gamma((\jmath_2\boxplus_1\psi_2))}$, then $\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma(\jmath_1)} \leq \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma(\jmath_2)}$ and $\Re^I_\Gamma(\psi_1) \cdot e^{i\theta\Im^I_\Gamma(\psi_1)} \leq \Re^I_\Gamma(\psi_2) \cdot e^{i\theta\Im^I_\Gamma(\psi_2)}$.
We get $\Re^I_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_1\psi_1))} \geq \frac{\Re^I_\Gamma(\jmath_1)\cdot e^{i\theta\Im^I_\Gamma(\jmath_1)} + \Re^I_\Gamma(\psi_1)\cdot e^{i\theta\Im^T_\Gamma(\psi_1)}}{2}$.
Similarly,$\Re^I_\Gamma((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta\Im^I_\Gamma((\jmath_1\boxplus_2\psi_1))} \geq \frac{\Re^I_\Gamma(\jmath_1)\cdot e^{i\theta\Im^I_\Gamma(\jmath_1)} + \Re^I_\Gamma(\psi_1)\cdot e^{i\theta\Im^I_\Gamma(\psi_1)}}{2}$
and $\Re^I_\Gamma((\jmath_1 \boxplus_3 \psi_1)) \cdot e^{i\theta\Im^I_\Gamma((\jmath_1\boxplus_3\psi_1))} \geq \frac{\Re^I_\Gamma(\jmath_1)\cdot e^{i\theta\Im^I_\Gamma(\jmath_1)} + \Re^I_\Gamma(\psi_1)\cdot e^{i\theta\Im^I_\Gamma(\psi_1)}}{2}$.
Similarly to prove that
$\max\{\Re^F_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_1\psi_1))}, \Re^F_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_2\boxplus_1\psi_2))}\}$
$\leq \max\{\max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma(\jmath_2)}\}, \max\{\Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)}, \Re^F_\Gamma(\psi_2) \cdot e^{i\theta\Im^F_\Gamma(\psi_2)}\}\}$
If $\Re^F_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_1\psi_1))} \geq \Re^F_\Gamma((\jmath_2 \boxplus_1 \psi_2)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_2\boxplus_1\psi_2))}$,then $\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)} \geq \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma(\jmath_2)}$ and $\Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)} \geq \Re^F_\Gamma(\psi_2) \cdot e^{i\theta\Im^F_\Gamma(\psi_2)}$.
We get $\Re^F_\Gamma((\jmath_1 \boxplus_1 \psi_1)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_1\psi_1))} \leq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)}\}$.
$\max\{\Re^F_\Gamma((\jmath_1 \boxplus_2 \psi_1)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_2\psi_1))}, \Re^F_\Gamma((\jmath_2 \boxplus_2 \psi_2)) \cdot e^{i\theta\Im^F_\Gamma((\jmath_2\boxplus_2\psi_2))}\}$
$\leq \max\{\max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma(\jmath_2)}\}, \max\{\Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)}, \Re^F_\Gamma(\psi_2) \cdot e^{i\theta\Im^F_\Gamma}\}\}$
If $\Re^F_\Gamma((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_2\psi_1))} \geq \Re^F_\Gamma((\jmath_2\boxplus_2\psi_2))\cdot e^{i\theta\Im^F_\Gamma((\jmath_2\boxplus_2\psi_2))}$,then $\Re^F_\Gamma((\jmath_1\boxplus_2\psi_1))\cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_2\psi_1))} \leq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)}\}$.
$\max\{\Re^F_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^F_\Gamma((\jmath_1\boxplus_3\psi_1))}, \Re^F_\Gamma((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im^F_\Gamma((\jmath_2\boxplus_3\psi_2))}\} \leq \max\{\max\{\Re^F_\Gamma(\jmath_1)\cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\jmath_2)\cdot e^{i\theta\Im^F_\Gamma(\jmath_2)}\}, \max\{\Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^F_\Gamma(\psi_1)}, \Re^F_\Gamma(\psi_2) \cdot e^{i\theta\Im^F_\Gamma(\psi_2)}\}\}$
If $\Re^F_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_3\psi_1))} \geq \Re^F_\Gamma((\jmath_2\boxplus_3\psi_2))\cdot e^{i\theta\Im^T_\Gamma((\jmath_2\boxplus_3\psi_2))}$,then $\Re^F_\Gamma((\jmath_1\boxplus_3\psi_1))\cdot e^{i\theta\Im^T_\Gamma((\jmath_1\boxplus_3\psi_1))} \leq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma(\jmath_1)}, \Re^F_\Gamma(\psi_1) \cdot e^{i\theta\Im^T_\Gamma(\psi_1)}\}$.
So, Γ is a ComCNSBS of S.

Theorem 3.11. *Suppose* Γ *is a subset of* S. *Then* $R = (\widehat{\Re^T_\Gamma} \cdot e^{i\theta\widehat{\Im^T_\Gamma}}, \widehat{\Re^I_\Gamma} \cdot e^{i\theta\widehat{\Im^I_\Gamma}}, \widehat{\Re^F_\Gamma} \cdot e^{i\theta\widehat{\Im^F_\Gamma}}, \Re^T_\Gamma \cdot e^{i\theta\Im^T_\Gamma}, \Re^I_\Gamma \cdot e^{i\theta\Im^I_\Gamma}, \Re^F_\Gamma \cdot e^{i\theta\Im^F_\Gamma})$ *is a ComCNSBS of* S *if and only if* $\Re^{(\wp,\varkappa)}$ *is a SBS, where* $\wp, \varkappa \in D[0,1]$.

Proof. Assume that S has a ComCNSBS in $\widehat{\Re}$. For all $\wp, \varkappa \in D[0,1]$ and $\jmath_1, \jmath_2 \in \widehat{\Re^{(\wp,\varkappa)}}$. Now,$\widehat{\Re^T_\Gamma}(\jmath_1) \cdot$

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$e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_1) \geq \wp, \widehat{\Re^T_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_2) \geq \wp$ and $\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_1) \geq \wp, \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_2) \geq \wp$ and $\widehat{\Re^F_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_1) \leq \varkappa, \widehat{\Re^F_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_2) \leq \varkappa$. Now,$\widehat{\Re^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \geq \min\{\widehat{\Re^T_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_1), \widehat{\Re^T_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_2)\} \geq \wp$ and $\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \geq \frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_1) + \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_2)}{2} \geq \frac{\wp+\wp}{2} = \wp$ and $\widehat{\Re^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \leq \max\{\widehat{\Re^F_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_1), \widehat{\Re^F_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_2)\} \leq \varkappa$. This implies that $\jmath_1 \boxplus_1 \jmath_2 \in \widehat{\Re^{(\wp,\varkappa)}}$. Similarly,$\jmath_1 \boxplus_2 \jmath_2 \in \widehat{\Re^{(\wp,\varkappa)}}$ and $\jmath_1 \boxplus_3 \jmath_2 \in \widehat{\Re^{(\wp,\varkappa)}}$. Hence,$\widehat{\Re^{(\wp,\varkappa)}}$ is a SBS of S,for all $\wp, \varkappa \in D[0,1]$.
For each $\wp, \varkappa \in [0,1]$ and $\jmath_1, \jmath_2 \in \Re^{(\wp,\varkappa)}$. Now,$\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_1) \geq \wp, \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_2) \geq \wp$ and $\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_1) \geq \wp, \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_2) \geq \wp$ and $\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_1) \leq \varkappa, \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_2) \leq \varkappa$. Now,$\Re^T_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \geq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_1), \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_2)\} \geq \wp$ and $\Re^I_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \geq \frac{\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_1) + \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_2)}{2} \geq \frac{\wp+\wp}{2} = \wp$ and $\Re^F_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \leq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_1), \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_2)\} \leq \varkappa$. This implies that $\jmath_1 \boxplus_1 \jmath_2 \in \Re^{(\wp,\varkappa)}$. Similarly,$\jmath_1 \boxplus_2 \jmath_2 \in \Re^{(\wp,\varkappa)}$ and $\jmath_1 \boxplus_3 \jmath_2 \in \Re^{(\wp,\varkappa)}$. Hence,$\Re^{(\wp,\varkappa)}$ is a SBS of S,for all $\wp, \varkappa \in D[0,1]$.

Conversely, assume that $\widehat{\Re^{(\wp,\varkappa)}}$ is a SBS of S and $\wp, \varkappa \in D[0,1]$. Suppose if there exist $\jmath_1, \jmath_2 \in S$ such that $\widehat{\Re^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) < \min\{\widehat{\Re^T_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_1), \widehat{\Re^T_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_2)\}$,$\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) < \frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_1) + \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_2)}{2}$ and $\widehat{\Re^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) > \max\{\widehat{\Re^F_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_1), \widehat{\Re^F_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_2)\}$. For $\wp, \varkappa \in D[0,1]$ such that $\widehat{\Re^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) < \wp \leq \min\{\widehat{\Re^T_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_1), \widehat{\Re^T_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_2)\}$ and $\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) < \wp \leq \frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_1) + \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_2)}{2}$ and $\widehat{\Re^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) > \varkappa \geq \max\{\widehat{\Re^F_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_1), \widehat{\Re^F_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_2)\}$. Thus,$\jmath_1, \jmath_2 \in \widehat{\Re^{(\wp,\varkappa)}}$,but $\jmath_1 \boxplus_1 \jmath_2 \notin \widehat{\Re^{(\wp,\varkappa)}}$. This contradicts,$\widehat{\Re^{(\wp,\varkappa)}}$ is a SBS of S. Therefore $\widehat{\Re^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \geq \min\{\widehat{\Re^T_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_1), \widehat{\Re^T_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath_2)\}$, $\widehat{\Re^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \geq \frac{\widehat{\Re^I_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_1) + \widehat{\Re^I_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath_2)}{2}$ and $\widehat{\Re^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}((\jmath_1 \boxplus_1 \jmath_2)) \leq \max\{\widehat{\Re^F_\Gamma}(\jmath_1) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_1), \widehat{\Re^F_\Gamma}(\jmath_2) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath_2)\}$. Similarly,$\boxplus_2$ and $\boxplus_3$ cases. Hence $\widehat{\Re} = (\widehat{\Re^T_\Gamma} \cdot e^{i\theta\widehat{\Im^T_\Gamma}}, \widehat{\Re^I_\Gamma} \cdot e^{i\theta\widehat{\Im^I_\Gamma}}, \widehat{\Re^F_\Gamma} \cdot e^{i\theta\widehat{\Im^F_\Gamma}})$ is a ComCNSBS of S.
Let us assume that $\Re^{(\wp,\varkappa)}$ is a SBS of S and $\wp, \varkappa \in [0,1]$. Suppose if there exist $\jmath_1, \jmath_2 \in S$ such that $\Re^T_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) < \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_1), \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_2)\}$,$\Re^I_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) < \frac{\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_1) + \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_2)}{2}$ and $\Re^F_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) > \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_1), \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_2)\}$. For $\wp, \varkappa \in D[0,1]$ such that $\Re^T_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) < \wp \leq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_1), \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_2)\}$ and $\Re^I_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) < \wp \leq \frac{\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_1) + \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_2)}{2}$ and $\Re^F_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) > \varkappa \geq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_1), \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_2)\}$. Thus,$\jmath_1, \jmath_2 \in \Re^{(\wp,\varkappa)}$,but $\jmath_1 \boxplus_1 \jmath_2 \notin \Re^{(\wp,\varkappa)}$. This contradicts,$\Re^{(\wp,\varkappa)}$ is a SBS of S. Therefore $\Re^T_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^T_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \geq \min\{\Re^T_\Gamma(\jmath_1) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_1), \Re^T_\Gamma(\jmath_2) \cdot e^{i\theta\Im^T_\Gamma}(\jmath_2)\}$, $\Re^I_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^I_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \geq \frac{\Re^I_\Gamma(\jmath_1) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_1) + \Re^I_\Gamma(\jmath_2) \cdot e^{i\theta\Im^I_\Gamma}(\jmath_2)}{2}$ and $\Re^F_\Gamma((\jmath_1 \boxplus_1 \jmath_2)) \cdot e^{i\theta\Im^F_\Gamma}((\jmath_1 \boxplus_1 \jmath_2)) \leq \max\{\Re^F_\Gamma(\jmath_1) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_1), \Re^F_\Gamma(\jmath_2) \cdot e^{i\theta\Im^F_\Gamma}(\jmath_2)\}$. Similarly,$\boxplus_2$ and $\boxplus_3$ cases. Hence $\Re = (\Re^T_\Gamma \cdot e^{i\theta\Im^T_\Gamma}, \Re^I_\Gamma \cdot e^{i\theta\Im^I_\Gamma}, \Re^F_\Gamma \cdot e^{i\theta\Im^F_\Gamma})$ is a ComCNSBS of S.

Definition 3.12. Let $(S_1, \Cup_1, \Cup_2, \Cup_3)$ and $(S_2, \Cap_1, \Cap_2, \Cap_3)$ be bisemirings and $F : S_1 \to S_2$ and Γ be the ComCNSBS of S_1, Υ be the ComCNSBS of $F(S_1) = S_2$. If $\Re_\Gamma \cdot e^{i\theta\Im_\Gamma} = [\widehat{\Re^T_\Gamma} \cdot e^{i\theta\widehat{\Im^T_\Gamma}}, \widehat{\Re^I_\Gamma} \cdot e^{i\theta\widehat{\Im^I_\Gamma}}, \widehat{\Re^F_\Gamma} \cdot e^{i\theta\widehat{\Im^F_\Gamma}}, \Re_\Gamma \cdot e^{i\theta\Im_\Gamma}, \Re^T_\Gamma \cdot e^{i\theta\Im^T_\Gamma}, \Re^I_\Gamma \cdot e^{i\theta\Im^I_\Gamma}, \Re^F_\Gamma \cdot e^{i\theta\Im^F_\Gamma}$ is a ComCNS in S_1, then $\Re_\Upsilon$ is a ComCNS in S_2, described as

$$\widehat{\Re^T_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\psi) = \begin{cases} \sup \widehat{\Re^T_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^T_\Gamma}}(\jmath) & if \ \jmath \in F^{-1}\psi \\ 0 & otherwise \end{cases}$$

$$\widehat{\Re^I_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\psi) = \begin{cases} \sup \widehat{\Re^I_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^I_\Gamma}}(\jmath) & if \ \jmath \in F^{-1}\psi \\ 0 & otherwise \end{cases}$$

$$\widehat{\Re^F_\Upsilon}(\psi) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\psi) = \begin{cases} \inf \widehat{\Re^F_\Gamma}(\jmath) \cdot e^{i\theta\widehat{\Im^F_\Gamma}}(\jmath) & if \ \jmath \in F^{-1}\psi \\ 1 & otherwise \end{cases}$$

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$$\Re^T_\Upsilon(\psi)\cdot e^{i\theta\Im^T_\Gamma}(\psi)=\begin{cases}\sup\Re^T_\Gamma(\jmath)\cdot e^{i\theta\Im^T_\Gamma}(\jmath) & if\ \ \jmath\in F^{-1}\psi\\ 0 & otherwise\end{cases}$$

$$\Re^I_\Upsilon(\psi)\cdot e^{i\theta\Im^I_\Gamma}(\psi)=\begin{cases}\sup\Re^I_\Gamma(\jmath)\cdot e^{i\theta\Im^I_\Gamma}(\jmath) & if\ \ \jmath\in F^{-1}\psi\\ 0 & otherwise\end{cases}$$

$$\Re^F_\Upsilon(\psi)\cdot e^{i\theta\Im^F_\Gamma}(\psi)=\begin{cases}\inf\Re^F_\Gamma(\jmath)\cdot e^{i\theta\Im^F_\Gamma}(\jmath) & if\ \ \jmath\in F^{-1}\psi\\ 1 & otherwise\end{cases}$$

for each $\jmath\in S_1$ and $\psi\in S_2$ is said to be the image of R_Γ under F.
Similarly,If $\Re_\Upsilon\cdot e^{i\theta\Im_\Upsilon}=[\widehat{\Re^T_\Upsilon}\cdot e^{i\theta\widehat{\Im^T_\Gamma}},\widehat{\Re^I_\Upsilon}\cdot e^{i\theta\widehat{\Im^I_\Upsilon}},\widehat{\Re^F_\Upsilon}\cdot e^{i\theta\widehat{\Im^F_\Upsilon}}]$, $\Re_\Upsilon\cdot e^{i\theta\Im_\Upsilon}$, $\Re^T_\Upsilon\cdot e^{i\theta\Im^T_\Gamma}$, $\Re^I_\Upsilon\cdot e^{i\theta\Im^I_\Upsilon}$, $\Re^F_\Upsilon\cdot e^{i\theta\Im^F_\Upsilon}$ is a ComCNS in S_2,then ComCNS $\Re_\Gamma=F\circ\Re_\Upsilon$ in S_1 ie,the ComCNS defined by $\Re_\Gamma(\jmath)\cdot e^{i\theta\Im_\Gamma(\jmath)}$, $\Re_\Upsilon(F(\jmath))\cdot e^{i\theta\Im_\Gamma(F(\jmath))}$, $\Re_\Gamma=F\circ\Re_\Upsilon$ in S_1 [ie,the ComCNS defined by $\Re_\Gamma(\jmath)\cdot e^{i\theta\Im_\Gamma(\jmath)}=\Re_\Upsilon(F(\jmath))\cdot e^{i\theta\Im_\Gamma(F(\jmath))}$ is described the preimage of $\Re_\Upsilon$ under F.

Theorem 3.13. *A ComCNSBS is the homomorphic image of every ComCNSBS.*

Proof. Let $F:S_1\to S_2$ be any homomorphism. Now, $F((\jmath\Cup_1\psi))=F(\jmath)\Cap_1 F(\psi)$, $F((\jmath\Cup_2\psi))=F(\jmath)\Cap_2 F(\psi)$ and $F((\jmath\Cup_3\psi))=F(\jmath)\Cap_3 F(\psi)$ for all $\jmath,\psi\in S_1$. Let $\widehat{\Upsilon}=F(\Gamma)$,$\Gamma$ is any ComCNSBS of S_1. Let $F(\jmath),F(\psi)\in S_2$. Let $\jmath\in uF^{-1}(F(\jmath))$ and $\psi\in F^{-1}(F(\psi))$ be such that $\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}=\sup\limits_{\jmath\in F^{-1}(F(\jmath))}\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}$ and $\widehat{\Re^T_\Gamma}(\psi)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi)}=\sup\limits_{\jmath\in F^{-1}(F(\psi))}\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)}$. Now,

$$\begin{aligned}\widehat{\Re^T_\Upsilon}((F(\jmath)\Cap_1 F(\psi)))\cdot e^{i\theta\widehat{\Im^T_\Gamma}((F(\jmath)\Cap_1F(\psi)))}&=\sup_{(\jmath')\in F^{-1}(F(\jmath)\Cap_1F(\psi))}\widehat{\Re^T_\Gamma}(\jmath')\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath')}\\&=\sup_{(\jmath')\in F^{-1}(F((\jmath\Cup_1\psi))}\widehat{\Re^T_\Gamma}(\jmath')\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath')}\\&=\widehat{\Re^T_\Gamma}((\jmath\Cup_1\psi))\cdot e^{i\theta\widehat{\Im^T_\Gamma}((\jmath\Cup_1\psi))}\\&\geq\min\{\widehat{\Re^T_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\jmath)},\widehat{\Re^T_\Gamma}(\psi)\cdot e^{i\theta\widehat{\Im^T_\Gamma}(\psi)}\}\\&=\min\{\widehat{\Re^T_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\jmath)},\widehat{\Re^T_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\psi)}\}.\end{aligned}$$

Thus, $\widehat{\Re^T_\Upsilon}((F(\jmath)\Cap_1 F(\psi)))\cdot e^{i\theta\widehat{\Im^T_\Upsilon}((F(\jmath)\Cap_1F(\psi)))}\geq\min\{\widehat{\Re^T_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\jmath)},\widehat{\Re^T_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\psi)}\}$.
Similarly,$\widehat{\Re^T_\Upsilon}((F(\jmath)\Cap_2 F(\psi)))\cdot e^{i\theta\widehat{\Im^T_\Upsilon}((F(\jmath)\Cap_2F(\psi)))}\geq\min\{\widehat{\Re^T_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\jmath)},\widehat{\Re^T_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\psi)}\}$ and $\widehat{\Re^T_\Upsilon}((F(\jmath)\Cap_3 F(\psi)))\cdot e^{i\theta\widehat{\Im^T_\Upsilon}((F(\jmath)\Cap_3F(\psi)))}\geq\min\{\widehat{\Re^T_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\jmath)},\widehat{\Re^T_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\psi)}\}$.
Let $\jmath\in F^{-1}(F(\jmath))$ and $\psi\in F^{-1}(F(\psi))$ be such that $\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}(\jmath)}=\sup\limits_{\jmath\in F^{-1}(F(\jmath))}\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}(\jmath)}$ and $\widehat{\Re^I_\Gamma}(\psi)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}(\psi)}=\sup\limits_{\jmath\in F^{-1}(F(\psi))}\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^T_\Upsilon}F(\jmath)}$. Now,

$$\begin{aligned}\widehat{\Re^I_\Upsilon}((F(\jmath)\Cap_1 F(\psi)))\cdot e^{i\theta\widehat{\Im^I_\Upsilon}((F(\jmath)\Cap_1F(\psi)))}&=\sup_{(\jmath')\in F^{-1}(F(\jmath)\Cap_1F(\psi))}\widehat{\Re^I_\Gamma}(\jmath')\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath')}\\&=\sup_{(\jmath')\in F^{-1}(F((\jmath\Cup_1\psi))}\widehat{\Re^I_\Gamma}(\jmath')\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath')}\\&=\widehat{\Re^I_\Gamma}((\jmath\Cup_1\psi))\cdot e^{i\theta\widehat{\Im^I_\Gamma}((\jmath\Cup_1\psi))}\\&\geq\frac{\widehat{\Re^I_\Gamma}(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\jmath)}+\widehat{\Re^I_\Gamma}(\psi)\cdot e^{i\theta\widehat{\Im^I_\Gamma}(\psi)}}{2}\\&=\frac{\widehat{\Re^I_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\jmath)}+\widehat{\Re^I_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\psi)}}{2}.\end{aligned}$$

Thus, $\widehat{\Re^I_\Upsilon}((F(\jmath)\Cap_1 F(\psi)))\cdot e^{i\theta\widehat{\Im^I_\Upsilon}((F(\jmath)\Cap_1F(\psi)))}\geq\frac{\widehat{\Re^I_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\jmath)}+\widehat{\Re^I_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\psi)}}{2}$.
Similarly,$\widehat{\Re^I_\Upsilon}((F(\jmath)\Cap_2 F(\psi)))\cdot e^{i\theta\widehat{\Im^I_\Upsilon}((F(\jmath)\Cap_2F(\psi)))}\geq\frac{\widehat{\Re^I_\Upsilon}F(\jmath)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\jmath)}+\widehat{\Re^I_\Upsilon}F(\psi)\cdot e^{i\theta\widehat{\Im^I_\Upsilon}F(\psi)}}{2}$ and

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$\widehat{\Re^{I}_{\Upsilon}}((F(\jmath) \Cap_3 F(\psi))) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}((F(\jmath)\Cap_3 F(\psi)))} \geq \frac{\widehat{\Re^{I}_{\Upsilon}}F(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}F(\jmath)}+\widehat{\Re^{I}_{\Upsilon}}F(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}F(\psi)}}{2}$.
Let $F(\jmath), F(\psi) \in S_2$. Let $\jmath \in F^{-1}(F(\jmath))$ and $\psi \in F^{-1}(F(\psi))$ be such that $\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)} = \inf_{\jmath\in F^{-1}(F(\jmath))} \widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}$ and $\widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)} = \inf_{\jmath\in F^{-1}(F(\psi))} \widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}$. Now,

$$\begin{aligned}
\widehat{\Re^{F}_{\Upsilon}}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((F(\jmath)\Cap_1 F(\psi)))} &= \inf_{(\jmath')\in F^{-1}(F(\jmath)\Cap_1 F(\psi))} \widehat{\Re^{F}_{\Gamma}}(\jmath') \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath')} \\
&= \inf_{(\jmath')\in F^{-1}(F((\jmath\Cup_1\psi))} \widehat{\Re^{F}_{\Gamma}}(\jmath') \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath')} \\
&= \widehat{\Re^{F}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\Cup_1\psi))} \\
&\leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\} \\
&= \max\{\widehat{\Re^{F}_{\Upsilon}}F(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\jmath)}, \widehat{\Re^{F}_{\Upsilon}}F(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\psi)}\}.
\end{aligned}$$

Thus, $\widehat{\Re^{F}_{\Upsilon}}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((F(\jmath)\Cap_1 F(\psi)))} \leq \max\{\widehat{\Re^{F}_{\Upsilon}}F(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\jmath)}, \widehat{\Re^{F}_{\Upsilon}}F(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\psi)}\}$.
Similarly,$\widehat{\Re^{F}_{\Upsilon}}((F(\jmath) \Cap_2 F(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((F(\jmath)\Cap_2 F(\psi)))} \leq \max\{\widehat{\Re^{F}_{\Upsilon}}F(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\jmath)}, \widehat{\Re^{F}_{\Upsilon}}F(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\psi)}\}$ and $\widehat{\Re^{F}_{\Upsilon}}((F(\jmath) \Cap_3 F(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((F(\jmath)\Cap_1 F(\psi)))} \leq \max\{\widehat{\Re^{F}_{\Upsilon}}F(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\psi)}, \widehat{\Re^{F}_{\Upsilon}}F(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}F(\psi)}\}$.

Let $F : S_1 \to S_2$ be any homomorphism. Now, $F((\jmath\Cup_1\psi)) = F(\jmath)\Cap_1 F(\psi), F((\jmath\Cup_2\psi)) = F(\jmath)\Cap_2 F(\psi)$ and $F((\jmath \Cup_3 \psi)) = F(\jmath) \Cap_3 F(\psi)$ for all $\jmath, \psi \in S_1$. Let $\Upsilon = F(\Gamma)$,Γ is any ComCNSBS of S_1. Let $F(\jmath), F(\psi) \in S_2$. Let $\jmath \in F^{-1}(F(\jmath))$ and $\psi \in F^{-1}(F(\psi))$ be such that $\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)} = \sup_{\jmath\in F^{-1}(F(\jmath))} \Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}$ and $\Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)} = \sup_{\jmath\in F^{-1}(F(\psi))} \Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}$. Now,

$$\begin{aligned}
\Re^{T}_{\Upsilon}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\Im^{T}_{\Gamma}((F(\jmath)\Cap_1 F(\psi)))} &= \sup_{(\jmath')\in F^{-1}(F(\jmath)\Cap_1 F(\psi))} \Re^{T}_{\Gamma}(\jmath') \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath')} \\
&= \sup_{(\jmath')\in F^{-1}(F((\jmath\Cup_1\psi))} \Re^{T}_{\Gamma}(\jmath') \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath')} \\
&= \Re^{T}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\Cup_1\psi))} \\
&\geq \min\{\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\} \\
&= \min\{\Re^{T}_{\Upsilon}F(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\jmath)}, \Re^{T}_{\Upsilon}F(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\psi)}\}.
\end{aligned}$$

Thus,$\Re^{T}_{\Upsilon}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}((F(\jmath)\Cap_1 F(\psi)))} \geq \min\{\Re^{T}_{\Upsilon}F(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\jmath)}, \Re^{T}_{\Upsilon}F(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\psi)}\}$.
Similarly,$\Re^{T}_{\Upsilon}((F(\jmath) \Cap_2 F(\psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}((F(\jmath)\Cap_2 F(\psi)))} \geq \min\{\Re^{T}_{\Upsilon}F(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\jmath)}, \Re^{T}_{\Upsilon}F(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\psi)}\}$ and $\Re^{T}_{\Upsilon}((F(\jmath) \Cap_3 F(\psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}((F(\jmath)\Cap_3 F(\psi)))} \geq \min\{\Re^{T}_{\Upsilon}F(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\jmath)}, \Re^{T}_{\Upsilon}F(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\psi)}\}$.
Let $\jmath \in F^{-1}(F(\jmath))$ and $\psi \in F^{-1}(F(\psi))$ be such that $\Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\jmath)} = \sup_{\jmath\in F^{-1}(F(\jmath))} \Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\jmath)}$ and $\Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\psi)} = \sup_{\jmath\in F^{-1}(F(\psi))} \Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Upsilon}F(\jmath)}$. Now,

$$\begin{aligned}
\Re^{I}_{\Upsilon}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((F(\jmath)\Cap_1 F(\psi)))} &= \sup_{(\jmath')\in F^{-1}(F(\jmath)\Cap_1 F(\psi))} \Re^{I}_{\Gamma}(\jmath') \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath')} \\
&= \sup_{(\jmath')\in F^{-1}(F((\jmath\Cup_1\psi))} \Re^{I}_{\Gamma}(\jmath') \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath')} \\
&= \Re^{I}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\Cup_1\psi))} \\
&\geq \frac{\Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} + \Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2} \\
&= \frac{\Re^{I}_{\Upsilon}F(\jmath) \cdot e^{i\theta\Im^{I}_{\Upsilon}F(\jmath)} + \Re^{I}_{\Upsilon}F(\psi) \cdot e^{i\theta\Im^{I}_{\Upsilon}F(\psi)}}{2}.
\end{aligned}$$

Thus,$\Re^{I}_{\Upsilon}((F(\jmath) \Cap_1 F(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((F(\jmath)\Cap_1 F(\psi)))} \geq \frac{\Re^{I}_{\Upsilon}F(\jmath)\cdot e^{i\theta\Im^{I}_{\Upsilon}F(\jmath)}+\Re^{I}_{\Upsilon}F(\psi)\cdot e^{i\theta\Im^{I}_{\Upsilon}F(\psi)}}{2}$.
Similarly,$\Re^{I}_{\Upsilon}((F(\jmath) \Cap_2 F(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((F(\jmath)\Cap_2 F(\psi)))} \geq \frac{\Re^{I}_{\Upsilon}F(\jmath)\cdot e^{i\theta\Im^{I}_{\Upsilon}F(\jmath)}+\Re^{I}_{\Upsilon}F(\psi)\cdot e^{i\theta\Im^{I}_{\Upsilon}F(\psi)}}{2}$ and

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$\Re^{I}_{\Upsilon}((\digamma(\jmath) \Cap_3 \digamma(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((\digamma(\jmath)\Cap_3\digamma(\psi)))} \geq \frac{\Re^{I}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{I}_{\Upsilon}\digamma(\jmath)}+\Re^{I}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{I}_{\Upsilon}\digamma(\psi)}}{2}$.
Let $\digamma(\jmath), \digamma(\psi) \in S_2$. Let $\jmath \in \digamma^{-1}(\digamma(\jmath))$ and $\psi \in \digamma^{-1}(\digamma(\psi))$ be such that $\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)} = \inf\limits_{\jmath\in \digamma^{-1}(\digamma(\jmath))} \Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}$ and $\Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)} = \inf\limits_{\jmath\in \digamma^{-1}(\digamma(\psi))} \Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}$. Now,

$$\begin{aligned}
\Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} &= \inf_{(\jmath')\in \digamma^{-1}(\digamma(\jmath)\Cap_1\digamma(\psi))} \Re^{F}_{\Gamma}(\jmath')\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath')}\\
&= \inf_{(\jmath')\in \digamma^{-1}(\digamma((\jmath\Cup_1\psi))} \Re^{F}_{\Gamma}(\jmath')\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath')}\\
&= \Re^{F}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\Cup_1\psi))}\\
&\leq \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)}\}\\
&= \max\{\Re^{F}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\jmath)}, \Re^{F}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}\}.
\end{aligned}$$

Thus, $\Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\Re^{F}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\jmath)}, \Re^{F}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}\}$.
Similarly,$\Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_2 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_2\digamma(\psi)))} \leq \max\{\Re^{F}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\jmath)}, \Re^{F}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}\}$ and $\Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_3 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\Re^{F}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}, \Re^{F}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}\}$. Thus, Υ is a ComCNSBS of S_2.

Theorem 3.14. *A ComCNSBS is the homomorphic preimage of every ComCNSBS.*

Proof. Let $\digamma : S_1 \to S_2$ be a homomorphism. Now,$\digamma((\jmath \Cup_1 \psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath \Cup_2 \psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\widehat{\Upsilon} = \digamma(\Gamma)$,$\widehat{\Upsilon}$ is a ComCNSBS of S_2. Let $\jmath, \psi \in S_1$. Now,$\widehat{\Re^{T}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\Cup_1\psi))} = \widehat{\Re^{T}_{\Upsilon}}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\digamma((\jmath\Cup_1\psi)))} = \widehat{\Re^{T}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \min\{\widehat{\Re^{T}_{\Upsilon}}\digamma(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}\digamma(\jmath)}, \widehat{\Re^{T}_{\Upsilon}}\digamma(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}\digamma(\psi)}\} = \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)}, \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)}\}$. Thus,$\widehat{\Re^{T}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)}, \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)}\}$.
Now,$\widehat{\Re^{I}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\Cup_1\psi))} = \widehat{\Re^{I}_{\Upsilon}}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}((\jmath\Cup_1\psi))} = \widehat{\Re^{I}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}((\jmath\Cup_1\psi))} \geq \frac{\widehat{\Re^{I}_{\Upsilon}}\digamma(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}\digamma(\jmath)}+\widehat{\Re^{I}_{\Upsilon}}\digamma(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}\digamma(\psi)}}{2} = \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)}+\widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}$. Thus,$\widehat{\Re^{I}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)}+\widehat{\Re^{I}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2}$. Now,$\widehat{\Re^{F}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\Cup_1\psi))} = \widehat{\Re^{F}_{\Upsilon}}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}(\digamma((\jmath\Cup_1\psi)))} = \widehat{\Re^{F}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\widehat{\Re^{F}_{\Upsilon}}\digamma(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}\digamma(\jmath)}, \widehat{\Re^{F}_{\Upsilon}}\digamma(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}\digamma(\psi)}\} = \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}$. Thus,$\widehat{\Re^{F}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\Cup_1\psi))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\}$.

Let $\digamma : S_1 \to S_2$ be a homomorphism. Now,$\digamma((\jmath \Cup_1 \psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath \Cup_2 \psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\Upsilon = \digamma(\Gamma)$,$\Upsilon$ is a ComCNSBS of S_2. Let $\jmath, \psi \in S_1$. Now,$\Re^{T}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\Cup_1\psi))} = \Re^{T}_{\Upsilon}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\digamma((\jmath\Cup_1\psi)))} = \Re^{T}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \min\{\Re^{T}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{T}_{\Upsilon}\digamma(\jmath)}, \Re^{T}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{T}_{\Upsilon}\digamma(\psi)}\} = \min\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\}$. Thus,$\Re^{T}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\Cup_1\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\}$.
Now,$\Re^{I}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\Cup_1\psi))} = \Re^{I}_{\Upsilon}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((\jmath\Cup_1\psi))} = \Re^{I}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((\jmath\Cup_1\psi))} \geq \frac{\Re^{I}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{I}_{\Upsilon}\digamma(\jmath)}+\Re^{I}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{I}_{\Upsilon}\digamma(\psi)}}{2} = \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)}+\Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2}$. Thus,$\Re^{I}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\Cup_1\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)}+\Re^{I}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2}$. Now,$\Re^{F}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\Cup_1\psi))} = \Re^{F}_{\Upsilon}(\digamma((\jmath \Cup_1 \psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}(\digamma((\jmath\Cup_1\psi)))} = \Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\Re^{F}_{\Upsilon}\digamma(\jmath)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\jmath)}, \Re^{F}_{\Upsilon}\digamma(\psi)\cdot e^{i\theta\Im^{F}_{\Upsilon}\digamma(\psi)}\} = \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)}\}$. Thus,$\Re^{F}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\Cup_1\psi))} \leq \max\{\Re^{F}_{\Gamma}(\jmath)\cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}, \Re^{F}_{\Gamma}(\psi)\cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)}\}$.

Theorem 3.15. *If $\digamma : S_1 \to S_2$ is a homomorphism, then $\digamma(\widehat{\Gamma_{(\wp,\varkappa)}})$ is a SBS of ComCNSBS $\widehat{\Upsilon}$ of S_2.*

Proof. The mapping $\digamma : S_1 \to S_2$ be a homomorphism. Now,$\digamma((\jmath \Cup_1 \psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath \Cup_2 \psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\widehat{\Upsilon} = \digamma(\Gamma)$,$\Gamma$ is a ComCNSBS of S_1. By Theorem 3.13, $\widehat{\Upsilon}$ is a ComCNSBS of S_2. Let $\widehat{\Gamma_{(\wp,\varkappa)}}$ be any SBS of Γ. Suppose that $\jmath, \psi \in \widehat{\Gamma_{(\wp,\varkappa)}}$. Then $\jmath \Cup_1 \psi, \jmath \Cup_2 \psi$ and $\jmath \Cup_3 \psi \in \widehat{\Gamma_{(\wp,\varkappa)}}$. Now,$\widehat{\Re^{T}_{\Upsilon}}(\digamma(\jmath))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\digamma(\jmath))} = \widehat{\Re^{T}_{\Gamma}}(\jmath)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)} \geq \wp, \widehat{\Re^{T}_{\Upsilon}}(\digamma(\psi))\cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\digamma(\psi))} = \widehat{\Re^{T}_{\Gamma}}(\psi)\cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)} \geq \wp$. Thus,$\widehat{\Re^{T}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \widehat{\Re^{T}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \wp$.

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Now,$\widehat{\Re^{I}_{\Upsilon}}(\digamma(\jmath)) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}(\digamma(\jmath))} = \widehat{\Re^{I}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} \geq \wp, \widehat{\Re^{I}_{\Upsilon}}(\digamma(\psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}(\digamma(\psi))} = \widehat{\Re^{I}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)} \geq \wp$. Thus,$\widehat{\Re^{I}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \widehat{\Re^{I}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \wp$. Now,$\widehat{\Re^{F}_{\Upsilon}}(\digamma(\jmath)) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}(\digamma(\jmath))} = \widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)} \leq \varkappa, \widehat{\Re^{F}_{\Upsilon}}(\digamma(\psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}(\digamma(\psi))} = \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)} \leq \varkappa$. Thus,$\widehat{\Re^{F}_{\Upsilon}}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \widehat{\Re^{F}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\Cup_1\psi))} \leq \varkappa$,for all $\digamma(\jmath), \digamma(\psi) \in S_2$. Similarly other operations,$\digamma(\widehat{\Gamma_{(\wp,\varkappa)}})$ is a SBS of ComCNSBS $\widehat{\Upsilon}$ of S_2.

Let $\digamma : S_1 \to S_2$ be a homomorphism. Now,$\digamma((\jmath \Cup_1 \psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath \Cup_2 \psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\Upsilon = \digamma(\Gamma)$,$\Gamma$ is a ComCNSBS of S_1. By Theorem 3.13, Υ is a ComCNSBS of S_2. Let $\Gamma_{(\wp,\varkappa)}$ be any SBS of Γ. Suppose that $\jmath, \psi \in \Gamma_{(\wp,\varkappa)}$. Then $\jmath \Cup_1 \psi, \jmath \Cup_2 \psi$ and $\jmath \Cup_3 \psi \in \Gamma_{(\wp,\varkappa)}$. Now,$\Re^{T}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\digamma(\jmath))} = \Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)} \geq \wp, \Re^{T}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\digamma(\psi))} = \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)} \geq \wp$. Thus,$\Re^{T}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{T}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \Re^{T}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\Cup_1\psi))} \geq \wp$. Now,$\Re^{I}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{I}_{\Upsilon}(\digamma(\jmath))} = \Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} \geq \wp, \Re^{I}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{I}_{\Upsilon}(\digamma(\psi))} = \Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)} \geq \wp$. Thus,$\Re^{I}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{I}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \geq \Re^{I}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\Cup_1\psi))} \geq \wp$. Now,$\Re^{F}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{F}_{\Upsilon}(\digamma(\jmath))} = \Re^{F}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)} \leq \varkappa, \Re^{F}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{F}_{\Upsilon}(\digamma(\psi))} = \Re^{F}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)} \leq \varkappa$. Thus,$\Re^{F}_{\Upsilon}((\digamma(\jmath) \Cap_1 \digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \Re^{F}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\Cup_1\psi))} \leq \varkappa$,for all $\digamma(\jmath), \digamma(\psi) \in S_2$. Similarly other operations,$\digamma(\Gamma_{(\wp,\varkappa)})$ is a SBS of ComCNSBS Υ of S_2.

Theorem 3.16. *If $\digamma : S_1 \to S_2$ is any homomorphism,then $\widehat{\Gamma_{(\wp,\varkappa)}}$ is a SBS of ComCNSBS Γ of S_1.*

Proof. Let $\digamma : S_1 \to S_2$ be any homomorphism. We have $\digamma((\jmath \Cup_1 \psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath \Cup_2 \psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\widehat{\Upsilon} = \digamma(\Gamma)$,$\widehat{\Upsilon}$ is a ComCNSBS of S_2. By Theorem 3.14,Γ is a ComCNSBS of S_1. Let $\digamma(\widehat{\Gamma_{(\wp,\varkappa)}})$ be a SBS of $\widehat{\Upsilon}$. Suppose that $\digamma(\jmath), \digamma(\psi) \in \digamma(\widehat{\Gamma_{(\wp,\varkappa)}})$. Now,$\digamma((\jmath\Cup_1\psi)), \digamma((\jmath\Cup_2\psi))$ and $\digamma((\jmath\Cup_3\psi)) \in \digamma(\widehat{\Gamma_{(\wp,\varkappa)}})$. Now,$\widehat{\Re^{T}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)} = \widehat{\Re^{T}_{\Upsilon}}(\digamma(\jmath)) \cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\digamma(\jmath))} \geq \wp, \widehat{\Re^{T}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)} = \widehat{\Re^{T}_{\Upsilon}}(\digamma(\psi)) \cdot e^{i\theta\widehat{\Im^{T}_{\Upsilon}}(\digamma(\psi))} \geq \wp$. Thus,$\widehat{\Re^{T}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \min\{\widehat{\Re^{T}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\jmath)}, \widehat{\Re^{T}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{T}_{\Gamma}}(\psi)}\} \geq \wp$. Now,$\widehat{\Re^{I}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} = \widehat{\Re^{I}_{\Upsilon}}(\digamma(\jmath)) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}(\digamma(\jmath))} \geq \wp, \widehat{\Re^{I}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)} = \widehat{\Re^{I}_{\Upsilon}}(\digamma(\psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Upsilon}}(\digamma(\psi))} \geq \wp$.
Thus,$\widehat{\Re^{I}_{\Gamma}}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}((\jmath\Cup_1\psi))} \geq \frac{\widehat{\Re^{I}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\jmath)} + \widehat{\Re^{I}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{I}_{\Gamma}}(\psi)}}{2} \geq \wp$. Now,$\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)} = \widehat{\Re^{F}_{\Upsilon}}(\digamma(\jmath)) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}(\digamma(\jmath))} \leq \varkappa, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)} = \widehat{\Re^{F}_{\Upsilon}}(\digamma(\psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}(\digamma(\psi))} \leq \varkappa$.
Thus,$\widehat{\Re^{F}_{\Gamma}}((\jmath\Cup_1\psi)) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}((\jmath\Cup_1\psi))} = \widehat{\Re^{F}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi))) \cdot e^{i\theta\widehat{\Im^{F}_{\Upsilon}}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\widehat{\Re^{F}_{\Gamma}}(\jmath) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\jmath)}, \widehat{\Re^{F}_{\Gamma}}(\psi) \cdot e^{i\theta\widehat{\Im^{F}_{\Gamma}}(\psi)}\} \leq \varkappa$,for all $\jmath, \psi \in S_1$. Similarly other operations,$\widehat{\Gamma_{(\wp,\varkappa)}}$ is a SBS of ComCNSBS Γ of S_1.

Let $\digamma : S_1 \to S_2$ be any homomorphism. We have $\digamma((\jmath\Cup_1\psi)) = \digamma(\jmath) \Cap_1 \digamma(\psi), \digamma((\jmath\Cup_2\psi)) = \digamma(\jmath) \Cap_2 \digamma(\psi)$ and $\digamma((\jmath \Cup_3 \psi)) = \digamma(\jmath) \Cap_3 \digamma(\psi)$ for all $\jmath, \psi \in S_1$. Let $\Upsilon = \digamma(\Gamma)$,$\Upsilon$ is a ComCNSBS of S_2. By Theorem 3.14,Γ is a ComCNSBS of S_1. Let $\digamma(\Gamma_{(\wp,\varkappa)})$ be a SBS of Υ. Suppose that $\digamma(\jmath), \digamma(\psi) \in \digamma(\Gamma_{(\wp,\varkappa)})$. Now,$\digamma((\jmath\Cup_1\psi)), \digamma((\jmath\Cup_2\psi))$ and $\digamma((\jmath\Cup_3\psi)) \in \digamma(\Gamma_{(\wp,\varkappa)})$. Now,$\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)} = \Re^{T}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\digamma(\jmath))} \geq \wp, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)} = \Re^{T}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{T}_{\Upsilon}(\digamma(\psi))} \geq \wp$. Thus,$\Re^{T}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{T}_{\Gamma}((\jmath\Cup_1\psi))} \geq \min\{\Re^{T}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{T}_{\Gamma}(\jmath)}, \Re^{T}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{T}_{\Gamma}(\psi)}\} \geq \wp$. Now,$\Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} = \Re^{I}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{I}_{\Upsilon}(\digamma(\jmath))} \geq \wp, \Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)} = \Re^{I}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{I}_{\Upsilon}(\digamma(\psi))} \geq \wp$. Thus,$\Re^{I}_{\Gamma}((\jmath \Cup_1 \psi)) \cdot e^{i\theta\Im^{I}_{\Gamma}((\jmath\Cup_1\psi))} \geq \frac{\Re^{I}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{I}_{\Gamma}(\jmath)} + \Re^{I}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{I}_{\Gamma}(\psi)}}{2} \geq \wp$. Now,$\Re^{F}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)} = \Re^{F}_{\Upsilon}(\digamma(\jmath)) \cdot e^{i\theta\Im^{F}_{\Upsilon}(\digamma(\jmath))} \leq \varkappa, \Re^{F}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)} = \Re^{F}_{\Upsilon}(\digamma(\psi)) \cdot e^{i\theta\Im^{F}_{\Upsilon}(\digamma(\psi))} \leq \varkappa$. Thus,$\Re^{F}_{\Gamma}((\jmath\Cup_1\psi)) \cdot e^{i\theta\Im^{F}_{\Gamma}((\jmath\Cup_1\psi))} = \Re^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi))) \cdot e^{i\theta\Im^{F}_{\Upsilon}((\digamma(\jmath)\Cap_1\digamma(\psi)))} \leq \max\{\Re^{F}_{\Gamma}(\jmath) \cdot e^{i\theta\Im^{F}_{\Gamma}(\jmath)}, \Re^{F}_{\Gamma}(\psi) \cdot e^{i\theta\Im^{F}_{\Gamma}(\psi)}\} \leq \varkappa$,for all $\jmath, \psi \in S_1$. Similarly other operations,$\Gamma_{(\wp,\varkappa)}$ is a SBS of ComCNSBS Γ of S_1.

4 Conclusion and future direction

A novel kind of ComCNSBS is presented in this work. The concept of three grades is approached in a novel way by the complex cubic neutrosophic SBS , which describes three grades in terms of a complex number. A complicated form of three grades signifies a paradigm change since it converts three grades into a two dimensional parameter. Neutrosophic SBSwith complicated interval values was defined. The concepts of

ComCNNSBS and ComCNSBS level sets were established. Applying the cubic set and anti cubic set to bisemiring is our aim. Additionally, a study of the characteristics of different conversions is done. In addition to data analysis, we are trying to handle images and signals, thus it makes sense to use F-transforms to expand the use of new fuzzy structures. Thus, we ought to consider using cubic SBS, soft set ComCNSBS, and soft set ComCNNSBS in the future.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control, 8, (1965), 338-353.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1), (1986) 87-96.

[3] R. R. Yager, Pythagorean membership grades in multi criteria decision-making, IEEE Trans. Fuzzy Systems, 22, (2014), 958-965.

[4] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani and T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, Journal of Intelligent and Fuzzy Systems, 36, (2019), 2829-284.

[5] B.C. Cuong and V. Kreinovich, Picture fuzzy sets a new concept for computational intelligence problems, in Proceedings of 2013 Third World Congress on Information and Communication Technologies (WICT 2013), IEEE, (2013), 1-6.

[6] M Palanikumar, K Arulmozhi, Novel possibility Pythagorean interval valued fuzzy soft set method for a decision making, TWMS J. App. and Eng. Math., 13(1), (2023), 327–340.

[7] Ali, O. Mashhadani, S. Alhakam, I. M., S. "A New Paradigm for Decision Making under Uncertainty in Signature Forensics Applications based on Neutrosophic Rule Engine," Journal of International Journal of Neutrosophic Science, vol. 24, no. 2, pp. 268-282, 2024.

[8] Hasan, Z. "Deep Learning for Super Resolution and Applications," Journal of Galoitica: Journal of Mathematical Structures and Applications, vol. 8, no. 2, pp. 34-42, 2023.

[9] F. Smarandache, A unifying field in logics Neutrosophy Neutrosophic Probability, Set and Logic, Rehoboth American Research Press (1999).

[10] Daniel Ramot, Ron Milo, Menahem Friedman, and Abraham Kandel, Complex fuzzy set, IEEE Transactions on Fuzzy System, 10(2), 2002.

[11] S.J Golan, Semirings and their Applications, Kluwer Academic Publishers, London, 1999.

[12] Faward Hussian, Raja Muhammad Hashism, Ajab Khan, Muhammad Naeem, Generalization of bisemirings, International Journal of Computer Science and Information Security, 14(9), (2016), 275-289.

[13] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. Intelligent Technologies Bangkok, Thailand, (2000) 307-312.

[14] J. Ahsan, K. Saifullah, and F. Khan, Fuzzy semirings, Fuzzy Sets and systems, 60, (1993), 309-320.

[15] Javed Ahsan, John N. Mordeson, and Muhammad Shabir, Fuzzy Semirings with Applications to Automata Theory, Springer Heidelberg New York Dordrecht, London, 2012.

[16] M.K Sen, S. Ghosh An introduction to bisemirings, Southeast Asian Bulletin of Mathematics, 28(3), (2001), 547-559.

[17] Ozcek, M. "A Review on the Structure of Fuzzy Regular Proper Mappings in Fuzzy Topological Spaces and Their Properties," Journal of Pure Mathematics for Theoretical Computer Science, vol. 3, no. 2, pp.60-71, 2023.

[18] Palanikumar M, Selvi G, Ganeshsree Selvachandran and Tan S.L, New approach to bisemiring theory via the bipolar-valued neutrosophic normal sets, Neutrosophic Sets and Systems, 55, 427-450, 2023.