Introduction to Neutrosophic Hypernear-rings

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Abstract

This paper is concerned with the introduction of neutrosophic hypernear-rings. The concept of neutrosophic $A-$hypergroup of a hypernear-ring A, neutrosophic $A(I)-$ hypergroup of a neutrosophic hypernear-ring $A(I)$ and their respective neutrosophic substructures are defined. We investigate and present some interesting results arising from the study of hypernear-rings in neutrosophic environment. It is shown that a constant neutrosophic hypernear-ring in general is not a constant hypernear-ring. In addition, we consider the neutrosophic ideals, neutrosophic homomorphism and neutrosophic quotient hypernear-rings of neutrosophic hypernear-rings.
Keywords: neutrosophy, neutrosophic hypernear-ring, neutrosophic $A-$hypergroup, neutrosophic $A(I)-$hypergroup, neutrosophic $A-$subhypergroup, neutrosophic $A(I)-$subhypergroup, neutrosophic hyperideals, neutrosophic hypernear-ring homomorphism, neutrosophic $A-$hypergroup homomorphism.

1 Introduction

Algebraic Hyperstructures are a natural extension/generalization of classical algebraic structures. This theory was introduced in 1934 by Marty. Since then, the theory and its applications to various aspect of sciences have been extensively studied by Corsini [8,910], Mittas [19,20], Stratigopoulos [23] and many other authors. For instance, Dasic in [11] studied the notion of hypernear-ring. He defined hypernear-rings, as the natural generalization of near-rings, endowed with quasicanonical hypergroups $(R, +)$ with multiplication being distributive with respect to the hyperaddition on the left side, and such that $(R, \cdot)$ is a semigroup with bilaterally absorbing element. In [13], Gontineac called the hypernear-ring presented by Dasic as a zero symmetric hypernear-ring and he studied the concept of hypernear-ring in a more general case. Kyung *et al.* presented in[18] the notion of hyper R-sugroups of a hypernear-ring and they investigated some properties of hypernear-rings with respect to the hyper R-subgroups. For more comprehensive details on hyperstructures, the reader should see [121821].

A well established branch of neutrosophic theory is the theory of neutrosophic algebraic structures. This aspect of neutrosophic theory was introduced in [24] by Kandasamy and Smarandache. They combined the elements of a given algebraic structure $(X, \star)$ with the indeterminate element I, and, the new structure $(X(I), \star)$ generated by X and I is called a neutrosophic algebraic structure. For more details about neutrosophic algebraic structures (see[6,14,22,25,26]). Recently, Agboola and Davvaz in [1,23,4] introduced the connections between neutrosophic set and the theory of algebraic hyperstructure. They studied neutrosophic BCI/BCK-Algebras, neutrosophic hypergroups, neutrosophic canonical hypergroups, and neutrosophic hyperrings. In this paper, we will investigate and present some interesting results arising from the study of hypernear-rings in a neutrosophic environment. This paper will add to the growing list of papers connecting algebraic hyperstructures and neutrosophic sets. More of such connections can be found in many recent publications some of which are [5,7,1516,17].

2 Preliminaries

In this section, we will present some definitions and results that will be used later in the paper.

Definition 2.1. [18] A hypernear-ring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1. $(R, +)$ is a quasi canonical hypergroup (not necessarily commutative), i.e., in $(R, +)$ the following hold:

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(a) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;

(b) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;

(c) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the opposite of x);

(d) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If $x \in R$ and A, B are subsets of R, then by $A + B$, $A + x$ and $x + B$ we mean

$$A + B = \bigcup_{a \in A, b \in B} a + b, A + x = A + \{x\}, x + B = \{x\} + B.$$

2. With respect to the multiplication, $(R, \cdot)$ is a semigroup having absorbing element 0, i.e., $x \cdot 0 = 0$ for all $x \in R$. But, in general, $0x \neq 0$ for some $x \in R$.

3. The multiplication is distributive with respect to the hyperoperation $''+''$ on the left side, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

A hypernear-ring R is called zero symmetric if $0x = x0 = 0$ for all $x \in R$.
Note that for all $x, y \in R$, we have $-(-x) = x,\ 0 = -0,\ -(x + y) = -y - x$ and $x(-y) = -xy$.

Definition 2.2. [18] A two sided hyper R-subgroup of a hypernear-ring R is a subset H of R such that

1. $(H, +)$ is a subhypergroup of $(R, +)$,
   (i) $a, b \in H$ implies $a + b \subseteq H$,
   (ii) $a \in H$ implies $-a \in H$,
2. $RH \subseteq H$,
3. $HR \subseteq H$.

If H satisfies (1) and (2), then it is called a left hyper R-subgroup of R. If H satisfies (1) and (3), then it is called a right hyper R-subgroup of R.

Definition 2.3. [6] Let $(N, +, \cdot)$ be any right nearring. The triple $(N(I), +, \cdot)$ is called a right neutrosophic nearring. For all $x = (a, bI),\ y = (c, dI) \in N(I)$ with $a, b, c, d \in N$, we define:

1. $x + y = (a, bI) + (c, dI) = (a + c, (b + d)I)$.
2. $-x = -(a, bI) = (-a, -bI)$.
3. $x \cdot y = (a, bI).(c, dI) = (ac, (ad + bc + bd)I)$.

The zero element in $(N, +)$ is represented by $(0, 0)$ in $(N(I), +)$. Any element $x \in N$ is represented by $(x, 0)$ in $N(I)$. I in $N(I)$ is sometimes represented by $(0, I)$ in $N(I)$.

Definition 2.4. [6] Let $(N(I), +, \cdot)$ be a right neutrosophic nearring.

1. $N(I)$ is called abelian, if $(a, bI) + (c, dI) = (c, dI) + (a, bI)\ \forall\ (a, bI), (c, dI) \in N(I)$.
2. $N(I)$ is called commutative, if $(a, bI) \cdot (c, dI) = (c, dI) \cdot (a, bI)\ \forall\ (a, bI), (c, dI) \in N(I)$.
3. $N(I)$ is said to be distributive, if $N(I) = N_d(I)$, where
$$N_d(I) = \{d \in N(I) : d(m + n) = dm + dn,\ \forall\, m, n \in N(I)\}.$$
4. $N(I)$ is said to be zero-symmetric, if $N(I) = N_0(I)$, where
$$N_0(I) = \{n \in N(I) : n0 = 0\}.$$

The following should be noted:

(i) $N(I)$ is abelian only if $(N, +)$ is abelian.

(ii) $N(I)$ is commutative only if $(N, \cdot)$ is commutative.

(iii) $N(I)$ is distributive only if N is distributive.

(iv) $N(I)$ is zero-symmetric only if N is zero-symmetric

3 Development of neutrosophic hypernear-rings

In this section, we develop the concept of neutrosophic hypernear-rings and present some of their basic properties.

Definition 3.1. Let $(R, +, \cdot)$ be any hypernear-ring. The triple $(R(I), +', \odot)$ is a neutrosophic hypernear-ring generated by R and I, where $+'$ and $\odot$ are hyperoperations.
For all $r_1 = (u, vI), r_2 = (s, tI) \in R(I)$ with $u, v, s, t \in R$, we define :

1. $r_1 +' r_2 = (u, vI) +' (s, tI) = \{(p, qI) : p \in u + s, q \in v + t\}$ for all $r_1, r_2 \in R$,
2. $r_1 \odot r_2 = (u, vI) \odot (s, tI) = (u \odot s, (u \odot t + v \odot s + v \odot t)I)$, the $''\odot''$ and $''+''$ on the right are respectively the ordinary multiplication and hyperaddition in R,
3. $-r_1 = -(u, vI) = (-u, -vI)$.

We represent element $x \in R$ by $(x, 0I) \in R(I)$, and $0 \in (R, +)$ by $(0, 0I) \in (R(I), +')$. $I \in R(I)$ may also be written as $(0, I)$.

Lemma 3.2. *Let $(R(I), +', \odot)$ be any neutrosophic hypernear-ring. Let $r_1 = (u, vI), r_2 = (s, tI) \in R(I)$ with $u, v, s, t \in R$. For all $r_1, r_2 \in R(I)$ we have*

1. $-(-r_1) = -(-a, -bI) = (-(-a), -(-b)I) = (a, bI)$,
2. $-(r_1 +' r_2) = -r_1 - r_2$,
3. $-(0, 0I) = (0, 0I)$,
4. $r_1 \odot (-r_2) = -(r_1 \odot r_2)$.

Proof. The proof is similar to the proof in classical case. □

Definition 3.3. Let $(R(I), +', \odot)$ be a neutrosophic hypernear-ring. An element $(x, yI) \in R(I)$ is said to be idempotent if $(x, yI)^2 = (x, yI)$.

Definition 3.4. Let $(R(I), +', \odot)$ be a neutrosophic hypernear-ring.

1. $R(I)$ is called zero-symmetric neutrosophic hypernear-ring, if $R(I) = R_{(0,0I)}(I)$, where
$$R_{(0,0I)}(I) = \{(x, yI) \in R(I) \mid (0, 0I) \odot (x, yI) = (0, 0I)\}.$$
2. $R(I)$ is called a constant neutrosophic hypernear-ring, if $R(I) = R_c(I)$ where
$$R_c(I) = \{(x, yI) \in R(I) \mid (x, yI) \odot (p, qI) = (p, qI),\ \forall\, (p, qI) \in R(I)\}.$$

Proposition 3.5. *Every neutrosophic hypernear-ring is a hypernear-ring.*

Proof. Let $(R(I), +', \odot)$ be a neutrosophic hypernear-ring.

1. We shall show that $(R(I), +')$ is a quasi canonical hypergroup.
   (a) Let $(a, bI), (c, dI), (e, fI) \in R(I)$. Then
$$\begin{aligned}((a, bI) +' (c, dI)) +' (e, fI) &= \{(x, yI) : x \in a + c, y \in b + d\} +' (e, fI)\\ &= \{(p, qI) : p \in x + e, q \in y + f\}\\ &= \{(p, qI) : p \in (a + c) + e,\ q \in (b + d) + f\}\\ &= \{(p, qI) : p \in a + (c + e),\ q \in b + (d + f)\}\\ &= \{(p, qI) : p \in a + u,\ q \in b + v\}\\ &= (a, bI) +' \{(u, vI) : u \in c + e,\ v \in d + f\}\\ &= (a, bI) +' ((c, dI) +' (e, fI)).\end{aligned}$$
   (b) Let $(0, 0I) \in R(I)$, then for all $(a, bI) \in R(I)$ we have
$$\begin{aligned}(a, bI) +' (0, 0I) &= \{(x, yI) : x \in a + 0, y \in b + 0\}\\ &= \{(x, y) : x \in a, y \in b\}\\ &= \{(a, bI)\}.\end{aligned}$$
   Following similar approach we can show that $(0, 0I) + (a, bI) = \{(a, bI)\}$. Hence there exists a neutral element in $R(I)$.

(c) Let $(a,bI), -(a,bI) \in R(I)$, then

$$\begin{aligned}(a,bI) +' (-(a,bI)) &= (a,bI) +' (-a,-bI)\\ &= \{(x,yI) : x \in a + (-a), y \in b + (-b)\}\\ &= \{(x,yI) : x \in (-a) + a, y \in (-b) + b\}\\ &= \{(x,yI) : x \in \{0\}, y \in \{0\}\}\\ &\Longrightarrow (0,0I) \in (a,bI) +' (-(a,bI)).\end{aligned}$$

Hence $-(a,bI)$ is the unique inverse of any $(a,bI) \in R(I)$.

(d) Suppose that $(x,yI) \in (a,bI) +' (c,dI)$ then

$$\begin{aligned}(x,yI) &\in \{(p,qI) : p \in a + c, q \in b + d\}\\ &= \{(p,qI) : c \in -a + p, d \in -b + q\}\\ &= \{(c,dI) : c \in -a + p, d \in -b + q\}\\ &\Longrightarrow (c,dI) \in -(a,bI) +' (x,yI).\end{aligned}$$

Following the approach above we can also establish that $(a,bI) \in (x,yI) - (c,dI)$.

2. Let $(a,bI), (c,dI), (e,fI) \in R(I)$ then

(a) $(a,bI) \odot (c,dI) = (p,qI) \in R(I), \quad p = ac$ and $q \in (ad + bc + bd)$.

(b) Let $(a,bI), (c,dI), (e,fI) \in R(I)$ then

$$\begin{aligned}((a,bI) \odot (c,dI)) \odot (e,fI) &= (ac,(ad + bc + bd)I) \odot (e,fI)\\ &= ((ac)e, ((ac)f + (ad)e + (bc)e + (bd)e + (ad)f + (bc)f + (bd)f)I)\\ &= (a(ce), (a(cf) + a(de) + b(ce) + b(de) + a(df) + b(cf) + b(df))I)\\ &= (a(ce), (a(cf) + a(de) + a(df) + b(ce) + b(cf) + b(de) + b(df))I)\\ &= (a,bI) \odot ((c,dI) \odot (e,fI)).\end{aligned}$$

And $(a,bI) \odot (0,0I) = (a0,(a0 + b0 + b0)I) = (0,0I)$.

3. Now, it remains to show the distributive of $(\odot)$ with respect to $(+')$ on the left side.
Let $(a,bI), (c,dI), (e,dI) \in R(I)$ then

$$\begin{aligned}(a,bI) \odot ((c,dI) +' (e,fI)) &= (a,bI) \odot \{(x,yI) : x \in c + e,\ y \in d + f\}\\ &= \{(a,bI) \odot (x,yI) : x \in c + e,\ y \in d + f\}\\ &= (ax, (ax + ay + bx + by)I)\\ &= (a(c + e), (a(c + e) + a(d + f) + b(c + e) + b(d + f))I)\\ &= (ac + ae,\ q \in (ac + ae + ad + af + bc + be + bd + bf)I)\\ &= (ac, (ac + ad + +bc + bd)I) +' (ae,\ q_2 \in (ae + af + be + bf)I)\\ &= ((a,bI) \odot (c,dI)) +' ((a,bI) \odot (e,fI)).\end{aligned}$$

□

Proposition 3.6. *Let $\{R_i(I)\}_i^n$ be a family of neutrosophic hypernear-rings. Then $(\prod_{i=1}^n R_i(I), +', \odot)$ is a neutrosophic hypernear-ring.*

Proof. 1. Let $(a_i,b_iI), (c_i,d_iI) \in \prod_{i=1}^n R_i(I),$ with $a_i,b_i,c_i,d_i \in R_i$ for $i = 1 \cdots n$.
Following the approach in 1 of Proposition 3.5 we have that $(\prod_{i=1}^n R_i(I), +')$ is a neutrosophic quasi canonical hypernear-ring.

2. Let $(a_i,b_iI), (c_i,d_iI) \in \prod_{i=1}^n R_i(I),$ with $a_i,b_i,c_i,d_i \in R_i$ for $i = 1 \cdots n$.
Following the approach in 2 of Proposition 3.5 we have that $(\prod_{i=1}^n R_i(I), \odot)$ is a neutrosophic semihypergroup.

3. To show that $(\odot)$ is distributive with respect to $(+')$ on the left side, we follow the approach in 3 of Proposition 3.5
Hence we have that $(\prod_{i=1}^n R_i(I), +', \odot)$ is a neutrosophic hypernear-ring.

□

Proposition 3.7. *Let $M(I)$ be neutrosophic hypernear-rings and N be a hypernear-ring. Then $M(I) \times N$ is a neutrosophic hypernear-ring.*

Proof. The proof follows from Proposition 3.6 . □

Example 3.8. Let $(R(I),+)$ be a neutrosophic hypergroup and let $M^{R(I)}_{(0,0I)}$ be defined by

$$M^{R(I)}_{(0,0I)} = \{f : R(I) \longrightarrow R(I)\},$$

such that $f((0,0I)) = (0,0I)$. For all $f,g \in M^{R(I)}_{(0,0I)}$ we define the hyperoperation $f +' g$ of mappings as follows:

$$(f +' g)((x,yI)) = \{h \in M^{R(I)}_{(0,0I)} \mid \forall\,(x,yI) \in R(I), h((x,yI)) \in f((x,yI)) + g((x,yI))\}.$$

$$f \circ g((x,yI)) = f(g((x,yI))).$$

Here the $''+''$ on the right is the hyperoperation in $(R,+)$ and $\circ$ is a composition of functions.
Then $(M^{R(I)}_{(0,0I)}, +', \circ)$ is a zero-symmetric neutrosophic hypernear-ring.

Let $f,g \in M^{R(I)}_{(0,0I)}$ and $(x,yI) \in R(I)$. Since $f(x,yI) +' g(x,yI) \neq \emptyset$, then there exists $h \in M^{R(I)}_{(0,0I)}$ such that $h((x,yI)) \in f((x,yI)) +' g((x,yI))$. Obviously, $h((0,0I)) \in f((0,0I)) +' g((0,0I)) = \{(0,0I)\}$, i.e., $h((0,0I)) = (0,0I)$.
Now we shall show that $(M^{R(I)}_{(0,0I)}, +')$ is a neutrosophic quasi canonical hypergroup .

1. Let $f,g,h \in M^{R(I)}_{(0,0I)}$ and $(x,yI) \in R(I)$ then

$$\begin{aligned}
(f +' g) +' h &= \{p \mid \forall\,(x,yI) \in R(I), p((x,yI)) \in f((x,yI)) + g((x,yI))\} +' h \\
&= \{q \mid \forall\,(x,yI) \in R(I), q((x,yI)) \in p((x,yI)) + h((x,yI))\} \\
&= \{u \mid \forall\,(x,yI) \in R(I), u((x,yI)) \in (f((x,yI)) + g((x,yI))) + h((x,yI))\} \\
&= \{u \mid \forall\,(x,yI) \in R(I), u((x,yI)) \in f((x,yI)) + g((x,yI)) + h((x,yI))\} \\
&= \{u \mid \forall\,(x,yI) \in R(I), u((x,yI)) \in f((x,yI)) + (g((x,yI)) + h((x,yI)))\} \\
&= \{s \mid \forall\,(x,yI) \in R(I), s((x,yI)) \in f((x,yI)) + t((x,yI))\} \\
&= f +' \{t \mid \forall (x,yI) \in R(I), t((x,yI)) \in g((x,yI)) + h((x,yI))\} \\
&= f +' (g +' h).
\end{aligned}$$

2. Let $\tau \in M^{R(I)}_{(0,0I)}$ be defined by $\tau((x,yI)) = (0,0I)$, then for all $f \in M^{R(I)}_{(0,0I)}$ we have

$$\begin{aligned}
(f +' \tau)((x,yI)) &= \{g \mid \forall\,(x,yI) \in R(I), g \in f((x,yI)) + \tau((x,yI))\} \\
&= \{g \mid \forall\,(x,yI) \in R(I), g \in f((x,yI)) + (0,0I)\} \\
&= \{f((x,yI))\}.
\end{aligned}$$

Similarly, it can be shown that $(\tau +' f)((x,yI)) = \{f((x,yI))\}$. Hence, there exists a neutral function $\tau \in M^{R(I)}_{(0,0I)}$.

3. Let $f, -f \in M^{R(I)}_{(0,0I)}$ with $(-f((x,yI))) = -f((x,yI))$ then

$$\begin{aligned}
(f +' (-f))((x,yI)) &= \{g \mid \forall\,(x,yI) \in R(I),\ g \in f((x,yI)) + (-f((x,yI)))\} \\
&= \{g \mid \forall\,(x,yI) \in R(I),\ g \in f((x,yI)) - f((x,yI))\} \\
&= \{g \mid \forall\,(x,yI) \in R(I),\ g \in \{\tau((x,yI))\}
\end{aligned}$$

$\therefore\ \tau((x,yI) \in f((x,yI)) - f((x,yI)) \Longrightarrow (0,0I) \in f((x,yI)) - f((x,yI))$.
Hence $-f$ is the unique inverse of any $f\ \in M^{R(I)}_{(0,0I)}$.

4. Suppose that $h \in f +' g$. Then

$$\begin{aligned}
h &\in \{p \mid \forall\,(x,yI) \in R(I)\ p((x,yI)) \in f((x,yI)) + g((x,yI))\} \\
&= \{p \mid \forall\,(x,yI) \in R(I)\ g((x,yI)) \in -f((x,yI)) + p((x,yI))\} \\
&= \{g \mid \forall\,(x,yI) \in R(I)\ g((x,yI)) \in -f((x,yI)) + p((x,yI))\}.
\end{aligned}$$

Then we have that $g \in (-f) +' h$. Similarly, it can be shown that $f \in h +' (-g)$. Hence $h \in f +' g$ implies that $f \in h +' (-g)$ and $g \in (-f) +' h$.
Accordingly, $(M^{R(I)}_{(0,0I)}, +')$ is a neutrosophic quasicanonical hypergroup.

It can easily be established that $(M^{R(I)}, \circ)$ is a semihypergroup having τ as a bilaterally absorbing element such that $(f \circ \tau)(x, yI) = f(\tau(x, yI)) = f((0, 0I)) = (0, 0I) = \tau((x, yI))$ i.e., $f \circ \tau = f\tau = \tau$.
So, it remains to prove that the operation $\circ$ is distributive with respect to the hyperoperation on the left side.
Let $f, g, h \in M^{R(I)}_{(0,0I)}$ then

$$\begin{aligned} f \circ (g +' h) &= f \circ \{t \mid \forall\, (x, yI) \in R(I), t((x, yI)) \in g((x, yI)) + h((x, yI))\} \\ &= \{p \mid \forall\, (x, yI) \in R(I), p((x, yI)) \in ft((x, yI))\} \\ &= \{p \mid \forall\, (x, yI) \in R(I), p((x, yI)) \in fg((x, yI)) + fh((x, yI))\} \\ &= f \circ g((x, yI)) +' f \circ h((x, yI)). \end{aligned}$$

Then it follows that $(M^{R(I)}_{(0,0I)}, \circ, +')$ is a zero-symmetric neutrosophic hypernear-ring.

Example 3.9. Let $R(I) = \{x_0 = (0, 0I), x_1 = (a, 0I), x_2 = (b, 0I), x_3 = (0, aI), x_4 = (0, bI),$ $y_1 = (a, aI), y_2 = (a, bI), y_3 = (b, aI), y_4 = (b, bI)\}$ be a neutrosophic set. Let $N_x = \{x_0, x_1, x_2, x_3, x_4\}$ and $N_y = \{y_1, y_2, y_3, y_4\}$. Define hyperoperations $+'$ and $''\odot''$ on $R(I)$ as in the table below.

Table 1: (i) Cayley table for the hyper operation $+'$ and (ii) Cayley table for the hyper operation " $\odot$ "

(i)

$+'$	x_0	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
x_0	$\{x_0\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{y_1\}$	$\{y_2\}$	$\{y_3\}$	$\{y_4\}$
x_1	$\{x_1\}$	$\{x_0, x_1, x_2\}$	$\{x_1, x_2\}$	$\{y_1\}$	$\{y_2\}$	$\{x_3, y_1, y_3\}$	$\{x_4, y_2, y_4\}$	$\{y_1, y_3\}$	$\{y_2\ y_4\}$
x_2	$\{x_2\}$	$\{x_1, x_2\}$	$\{x_0, x_1, x_2\}$	$\{y_3\}$	$\{y_4\}$	$\{y_1, y_3\}$	$\{y_2, y_4\}$	$\{x_3, y_1, y_3\}$	$\{x_4\ y_2\ y_4\}$
x_3	$\{x_3\}$	$\{y_1\}$	$\{y_3\}$	$\{x_0, x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_1, y_1, y_2\}$	$\{y_1, y_2\}$	$\{x_2, y_3, y_4\}$	$\{y_3\ y_4\}$
x_4	$\{x_4\}$	$\{y_2\}$	$\{y_4\}$	$\{x_3, x_4\}$	$\{x_0, x_3, x_4\}$	$\{y_1, y_2\}$	$\{x_1, y_1, y_2\}$	$\{y_3, y_4\}$	$\{x_2, y_3, y_4\}$
y_1	$\{y_1\}$	$\{x_3, y_1, y_3\}$	$\{y_1, y_3\}$	$\{x_1, y_1, y_2\}$	$\{y_1, y_2\}$	$R(I)$	$\{x_3, x_4, N_y\}$	$\{x_1, x_2, N_y\}$	$\{N_y\}$
y_2	$\{y_2\}$	$\{x_4, y_2, y_4\}$	$\{y_2, y_4\}$	$\{y_1, y_2\}$	$\{x_1, y_1, y_2\}$	$\{x_3, x_4, N_y\}$	$R(I)$	$\{N_y\}$	$\{x_1, x_2, N_y\}$
y_3	$\{y_3\}$	$\{y_1, y_3\}$	$\{x_3, y_1, y_3\}$	$\{x_2\ y_3, y_4\}$	$\{y_3, y_4\}$	$\{x_1, x_2, N_y\}$	$\{N_y\}$	$R(I)$	$\{x_3, x_4, N_y\}$
y_4	$\{y_4\}$	$\{y_2, y_4\}$	$\{x_4, y_2, y_4\}$	$\{y_3, y_4\}$	$\{x_2, y_3, y_4\}$	$\{N_y\}$	$\{x_1, x_2\ N_y\}$	$\{x_3, x_4, N_y\}$	$R(I)$

(ii)

$\odot$	x_0	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0
x_1	x_0	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
x_2	x_0	x_2	x_1	x_4	x_3	y_4	y_3	y_2	y_1
x_3	x_0	x_3	x_4	x_3	x_4	$\{x_0, x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_0, x_3, x_4\}$
x_4	x_0	x_4	x_3	x_4	x_3	$\{x_0, x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_0, x_3, x_4\}$
y_1	x_0	y_1	y_4	$\{x_0, x_3, x_4\}$	$\{x_0, x_3, x_4\}$	$\{x_1, y_1, y_2\}$	$\{x_1, y_1, y_2\}$	$\{x_2, y_3, y_4\}$	$\{x_2, y_3, y_4\}$
y_2	x_0	y_2	y_3	$\{x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_1, y_1, y_2\}$	$\{x_1, y_1, y_2\}$	$\{x_2, y_3, y_4\}$	$\{x_2, y_3, y_4\}$
y_3	x_0	y_3	y_2	$\{x_3, x_4\}$	$\{x_3, x_4\}$	$\{x_2, y_3, y_4\}$	$\{x_2, y_3, y_4\}$	$\{x_1, y_1, y_2\}$	$\{x_1, y_1, y_2\}$
y_4	x_0	y_4	y_1	$\{x_0, x_3, x_4\}$	$\{x_0, x_3, x_4\}$	$\{x_2, y_3, y_4\}$	$\{x_2, y_3, y_4\}$	$\{x_1, y_1, y_2\}$	$\{x_1, y_1, y_2\}$

Then $(R(I), +', \odot)$ is neutrosophic hypernear-ring .

Proposition 3.10. *Let $R(I)$ be a neutrosophic hypernear-ring. $R(I)$ is zero-symmetric only if R is zero-symmetric hypernear-ring.*

Proof. Let $R(I)$ be a neutrosophic hypernear-ring and let R be a zero symmetric hypernear-ring.
Then for all $(x, yI) \in R(I)$ and for $(0, 0I) \in R(I)$ we have

$$\begin{aligned}(0, 0I) \odot (x, yI) &= (0x, (0y + 0x + 0y)I) \\ &= (0, 0I).\end{aligned}$$

Hence, $R(I)$ is zero-symmetric neutrosophic hypernear-ring. □

Proposition 3.11. *Let $R(I)$ be a neutrosophic hypernear-ring and let R be a constant hypernear-ring. Then generally, $R(I)$ is not a constant hypernear-ring.*

Proof. Let $R(I)$ be a neutrosophic hypernear-ring and let R be a constant hypernear-ring. Then, for all $(a, bI), (x, yI) \in R(I)$ we have,

$$\begin{aligned}(a, bI) \odot (x, yI) &= (ax,\ (ay + bx + by)I) \\ &= (x,\ (y + x + y)I) \qquad \because \text{ R is constant.} \\ &\neq (x, yI).\end{aligned}$$

□

Remark 3.12. $R(I) = \{(x, yI) : x, y \in R\}$ will be a constant hypernear-ring if x is the zero element in the constant hypernear-ring R. And, for each $z \in R$, $z + z = z$, and $0z = z$.

To see this, pick any $(0, bI), (0, aI) \in R(I)$. Since $z + z = z$ and $0z = z$ for all $z \in R$ we have

$$\begin{aligned}(0, bI) \odot (0, aI) &= (0, (0a + b0 + ba)I) \\ &= (0, (a + 0 + a)I) \\ &= (0, (a + a)I) \\ &= (0, aI).\end{aligned}$$

Proposition 3.13. *Every element in a constant neutrosophic hypernear-ring is idempotent.*

Proof. The proof follows easily from definition of a constant neutrosophic hypernear-ring. □

Definition 3.14. Let $R(I)$ be a neutrososphic hypernear-ring and let $N(I)$ be a nonempty subset of $R(I)$. $N(I)$ is called a neutrosophic subhypernear-ring if $N(I)$ is a neutrosophic hypernear-ring and $N(I)$ contains a proper subset which is a subhypernear-ring of R.

Proposition 3.15. *Let $R(I)$ be a neutrosophic hypernear-ring. The neutrosophic subset*

$$M(I)_{(0,0I)} = \{(x, yI) \in R(I) : (0, 0I) \odot (x, yI) = (0, 0I)\}$$

of $R(I)$ is a zero-symmetric neutrosophic subhypernear-ring of $R(I)$.

Proof. Let $(a, bI), (c, dI) \in M(I)_{(0,0I)}$, then $(0, 0I) \odot (a, bI) = (0, 0I)$ and $(0, 0I) \odot (c, dI) = (0, 0I)$.

1. Since every element in $M(I)_{(0,0I)}$ is of the form (a, bI), with $a, b \in R$. $M(I)_{(0,0I)}$ can be written as $(M_0, M_0(I))$. Here M_0 is a zero-symmetry subhypernear-ring of R. Therefore, we can conclude that $M(I)_{(0,0I)}$ contains a proper subset which is a zero-symmetric subhypernear-ring of R.

2. We shall show that $(M(I)_{(0,0I)}, +')$ is a zero-symmetric neutrosophic subhypergroup.
$$\begin{aligned} (0,0I)\odot[(a,bI)+'(c,dI)] &= (0,0I)\odot\{(p,qI) : p\in a+c,\ q\in b+d\} \\ &= (0,0I)\odot(p,qI) \\ &= (0p,\ (0q+0p+0q)I) \\ &= (0,0I). \end{aligned}$$
This shows that $(a, bI) + (c, dI) \subseteq M(I)_{(0,0I)}$.
And, for all $(a, bI) \in M(I)_{(0,0I)}$,
$$\begin{aligned} (0,0I)\odot(-(a,bI)) &= -((0,0I)\odot(a,bI)) \quad \text{by Lemma 3.2} \\ &= -(0a,(0b+0a+0b)I) \\ &= -(0,0I) = (0,0I) \quad \text{by Lemma 3.2 .} \end{aligned}$$
This shows that $-(a, bI) \in M(I)_{(0,0I)}$.
Thus, $(M(I)_{(0,0I)}, +')$ is a zero-symmetric neutrosophic subhypergroup.

3. We shall show that $(M(I)_{(0,0I)}, \odot)$ is a zero-symmetric neutrosophic subsemihypergroup.
$$\begin{aligned} (0,0I)\odot[(a,bI)\odot(c,dI)] &= (0,0I)\cdot[(ac,\ (ad+bc+bd)I)] \\ &= (0(ac),\ (0(ad)+0(bc)+0(bd)+0(ac)+0(ad)+0(bc)+0(bd))I) \\ &= ((0a)c,\ ((0a)d+(0b)c+(0b)d+(0a)c+(0a)d+(0b)c+(0b)d)I) \\ &= (0c,\ (0d+0c+0d+0c+0d+0c+0d)I) \\ &= (0,0I). \end{aligned}$$
This shows that $(a, bI) \odot (c, dI) \in M(I)_{(0,0I)}$.
Thus $(M(I)_{(0,0I)}, \odot)$ is a zero-symmetric neutrosophic subsemihypergroup.

Hence, we can conclude that $(M(I)_{(0,0I)}, +', \odot)$ is a zero-symmetric neutrosophic subhypernear-ring of $R(I)$. □

Definition 3.16. Let $(A, +, \cdot)$ be any hypernear-ring and let $(M(I), +')$ be neutrosophic hypergroup. Suppose that
$$\psi : A \times M(I) \longrightarrow M(I)$$
is an action of A on $M(I)$ defined by $a \cdot (x, yI) = (ax, ayI)$, for all $a \in A$ and $x, y \in M$.
$M(I)$ is called a neutrosophic A-hypergroup, for all $a, b \in A$ and $(x, yI) \in M(I)$, the following conditions hold:

1. $(a + b)(x, yI) = a(x, yI) +' b(x, yI)$.
2. $(a \cdot b) \cdot (x, yI) = a(b(x, yI))$.
3. $a \cdot I = aI$.
4. $0 \cdot (x, yI) = (0, 0I)$ and $a \cdot (0, 0I) = (0, 0I)$ for all $(x, yI) \in M(I)$ and $a \in A$.

If we replace A with a neutrosophic hypernear-ring $A(I), then$ $M(I)$ becomes a neutrosophic $A(I)-$hypergroup.

Proposition 3.17. *Every neutrosophic A-hypergroup is an A-hypergroup.*

Proof. Suppose that $M(I)$ is a neutrosophic A-hypergroup. Then $(M(I), +')$ is a hypergroup. The required result follows. □

Definition 3.18. Let $S(I)$ be a nonempty subset of a neutrosophic A-hypergroup $M(I)$. $S(I)$ is said to be a two-sided neutrosophic A-subhypergroup of $M(I)$ if

1. $(S(I), +')$ is a neutrosophic subhypergroup of $(M(I), +')$,
2. $AS(I) \subseteq S(I)$ and
3. $S(I)A \subseteq S(I)$.

$S(I)$ is said to be left neutrosophic A-subhypergroup if 1 and 2 are met. And (I) is called right neutrosophic A-subhypergroup if properties 1 and 3 are satisfied.

Example 3.19. Let $M(I)$ be a neutrosophic A-hypergroup and $(x, yI) \in M(I)$, then the set

$$A(x, yI) = \{a(x, yI) : a \in A\}$$

is a left neutrosophic A-subhypergroup of $M(I)$.

To see this, let $u, v \in A(x, yI)$. Then there exist $a_1, a_2 \in A$ such that $u = a_1(x, yI)$ and $v = a_2(x, yI)$ so that

$$\begin{aligned} u +' v &= a_1(x, yI) +' a_2(x, yI) = (a_1x, a_1yI) +' (a_2x, a_2yI) \\ &= \{(p, qI) : p \in a_1x + a_2x, \ \ q \in a_1y + a_2y\} \\ &= \{(p, qI) : p \in (a_1 + a_2)x, \ \ q \in (a_1 + a_2)y\} \\ &= ((a_1 + a_2)x, \ (a_1 + a_2)yI) \\ &= (a_1 + a_2)(x, yI) \\ &\subseteq A(x, yI) \qquad\qquad \because a_1 + a_2 \subseteq A. \end{aligned}$$

Also, for any $u \in A(x, yI)$, there exists $a_1 \in A$ such that $u = a_1(x, yI)$.
Then $-u = -(a_1(x, yI)) = -a_1(x, yI)$, this implies that $-u \in A(x, yI)$, since $-a_1 \in A$.
Since, $A(x, yI)$ can be written as $(Ax, Ay(I))$, $A(x, yI)$ contains a proper subset which is a subhypergroup. Hence, $A(x, yI)$ is a neutrosophic subhypergroup.
Now, it remains to show that $AA(x, yI) \subseteq A(x, yI)$. Let $a \in A$ and $u = a_1(x, yI) \in A(x, yI)$.
We have that

$$au = a(a_1(x, yI)) = (aa_1)(x, yI) \in A(x, yI), \quad \text{since } \ aa_1 \in A.$$

Hence, the set $A(x, yI) = \{a(x, yI) : a \in A\}$ is a left neutrosophic A-subhypergroup of $M(I)$.

Proposition 3.20. *Let $B(I)$ and $D(I)$ be two left neutrosophic A-subhypergroups of a neutrosophic A-hypergroup $M(I)$ and let $B_i(I)_{i\in\Lambda}$ be a family of left neutrosophic A-subhypergroups of an A-hypergroup $M(I)$. Then,*

1. *$B(I) + D(I)$ is a left neutrosophic A-subhypergroup of $M(I)$.*
2. *$\bigcap_{i\in\Lambda} B_i(I)$ is also a left neutrosophic A-subhypergroups of $M(I)$.*

Proof. 1. We can easily show that $B(I) + D(I)$ is a neutrosophic subhypergroup of $M(I)$.
Now it remain to show that $A(B(I) + D(I)) \subseteq B(I) + D(I)$.
Since $B(I)$ and $D(I)$ are left neutrosophic A-subhypergroup of $M(I)$, $AB(I) \subseteq B(I)$ and $AD(I) \subseteq D(I)$.
So, for $a \in A$, $(b_1, b_2I) \in B(I)$ and $(d_1, d_2I) \in D(I)$ we have

$$\begin{aligned} A(B(I) + D(I)) &= a((b_1, b_2I) + (d_1, d_2I)) \\ &= a\{(p, qI) : p \in b_1 + d_1, \ q \in b_2 + d_2\} \\ &= \{(ap, aqI) : ap \in ab_1 + ad_1, \ aq \in ab_2 + ad_2\} \\ &= \{(u, vI) : u \in ab_1 + ad_1, \ v \in ab_2 + ad_2\} \\ &= (u, vI) \in AB(I) + AD(I) \\ &\subseteq B(I) + D(I). \end{aligned}$$

Hence, $B(I) + D(I)$ is a left neutrosophic A-subhypergroup.

2. Let $b_1 = (x, yI), b_2 = (u, vI) \in \bigcap_{i\in\Lambda} B_i(I)$ and $a \in A$. Then for all $i \in \Lambda$, $b_1, b_2 \in B_i(I)$.
Since each $B_i(I)$ is a left neutrosophic A-subhyperspace, for all $i \in \Lambda$, $b_1 - b_2 = b_1 + (-b_2) \subseteq B_i(I)$, $B_i(I)$ cantains a proper subset which is a subhypergroup and $Ab_1 \subseteq B_i(I)$.
Thus,

$$b_1 - b_2 = b_1 + (-b_2) \subseteq \bigcap_{i\in\Lambda} B_i(I),$$

$$\bigcap_{i\in\Lambda} B_i(I) \text{ contains } \bigcap_{i\in\Lambda} B_i \text{ which is a subhyperspace,}$$

$$\text{and } Ab_1 \subseteq \bigcap_{i=1} B_i(I).$$

□

Proposition 3.21. *Let $M(I)$ be a constant neutrosophic $M(I)$-hypergroup. Then*

1. *any neutrosophic subhypergroup of $M(I)$ is a left neutrosophic $M(I)$-subhypergroup of $M(I)$.*
2. *$M(I)$ is the only right neutrosophic $M(I)$-subhypergroup of $M(I)$.*

Proof. 1. We know from Remark 3.12 that

$$M(I) = \{(0, mI) : m \in M\}.$$

Now, let $N(I)$ be any neutrosophic subhypergroup of $M(I)$.
Let $(0, xI) \in N(I)$ and $(0, mI) \in M(I)$ be arbitrary. Then

$$(0, mI)\cdot(0, xI) = (0\cdot 0,\ 0\cdot x + m\cdot 0 + mxI) = (0,\ (x + 0 + x)I) = (0, xI) \in N(I).$$

$$\therefore M(I)N(I) \subseteq N(I).$$

Hence, $N(I)$ a left neutrosophic $M(I)-$subhypergroup of $M(I)$.

2. First, $(M(I), +')$ is a neutrosophic subhypergroup of $(M(I), +')$.
It remains to show that $M(I)$ is the only neutrosophic subhypergroup of $(M(I), +')$ satisfying axiom 3 of Definition 3.18 , i.e., $M(I)M(I) \subseteq M(I)$.
To see this, suppose we can find another neutrosophic subhypergroup $U(I)$ of $M(I)$ such that $U(I)M(I) \subseteq U(I)$. Then we have $(0, 0I)M(I) = M(I) \subseteq U(I)$, since $M(I)$ is constant.
So we must have that $M(I) = U(I)$. The proof is complete.

□

Definition 3.22. A neutrosophic subhypergroup $A(I)$ of a neutrosophic hypergroup $(R(I), +')$ is said to be normal if for all $(x, yI) \in R(I)$, we have $(x, yI) + A(I) - (x, yI) \subseteq A(I)$.

Definition 3.23. Let $(R(I), +', \cdot)$ be a neutrosophic hypernear-ring and let $A(I)$ be a normal neutrosophic subhypergroup of $(R(I), +)$.

1. $A(I)$ is called a left neutrosophic hyperideal of $R(I)$, if for all $(a, bI) \in A(I), (x, yI) \in R(I),$ we have $(x, yI)(a, bI) \in A(I)$.
2. $A(I)$ is called a right neutrosophic hyperideal of $R(I)$, if for all $(x, yI), (u, vI) \in R(I)$, we have $((x, yI) +' A(I))(u, vI) - (x, yI)(u, vI) \subseteq A(I)$.
3. $A(I)$ is called a neutrosophic hyperideal of $R(I)$, if it is both a left and right neutrosophic hyperideal of $R(I)$.

Definition 3.24. Let $(M(I), +')$ be a neutrosophic A-hypergroup over a hypernear-ring A. A normal neutrosophic subhypergroup $B(I)$ of $M(I)$ is called a neutrosophic hyperideal of $M(I)$ if for all $(b_1, b_2I) \in B(I)$, $(x, yI) \in M(I)$ and $a \in A$ we have

$$a\cdot((x, yI) +' (b_1, b_2I)) - a\cdot(x, yI) \subseteq B(I).$$

Proposition 3.25. *Let $M(I)$ be a neutrosophic hypernear-ring. If $U(I)$ and $V(I)$ are any two neutrosophic hyperideals of $M(I)$ and $U_i(I)_{i\in\Lambda}$ is a family of neutrosophic hyperideals of $M(I)$, then*

1. *$U(I) + V(I) = \{a | a \in u + v$ for some $u \in U(I),\ v \in V(I)\}$ is a neutrosophic hyperideal of $M(I)$.*
2. *$U(I)V(I) = \{a | a \in \sum_{i=1}^{n} u_i v_i$ for some $u_i \in U(I),\ v_i \in V(I)\}$ is a neutrosophic hyperideal of $M(I)$.*
3. *$\bigcap_{i\in\Lambda} U_i(I)$ is a neutrosophic hyperideal of $M(I)$.*

Proof. The proof is the same as the proof in classical case. □

Proposition 3.26. *Let $(M(I), +)$ be a neutrosophic A-hypergroup over a hypernear-ring A. Let $B(I)$ be a neutrosophic hyperideal of $M(I)$ and $D(I)$ be a left neutrosophic A-subhypergroup of $M(I)$. Then $D(I) + B(I)$ is a left neutrosophic A-subhypergroup of $M(I)$.*

Proof. We want to show that $A(D(I) + B(I)) \subseteq D(I) + B(I)$.
Since $B(I)$ is a neutrosophic hyperideal of $M(I)$ and $D(I)$ is a left neutrosophic A-subhypergroup of $M(I)$, for $(b_1, b_2I) \in B(I)$, $(x, yI) \in D(I)$ and $a \in A$ we have

$$a \cdot ((x, yI) + (b_1, b_2I)) - a \cdot (x, yI) \subseteq B(I).$$

So, for $(b_3, b_4I) \in B(I)$,

$$\begin{aligned} a \cdot ((x, yI) + (b_1, b_2I)) &= (b_3, b_4I) + a \cdot (x, yI) \\ &= (ax, (ay)I) + (b_3, b_4I) \\ &\subseteq D(I) + B(I). \end{aligned}$$

Hence, $D(I) + B(I)$ is a left neutrosophic A-subhypergroup of $M(I)$. □

Proposition 3.27. *Let $M(I)$ be a neutrosophic $A(I)-$ hypergroup over a neutrosophic hypernear-ring $A(I)$. If $B(I)$ and $D(I)$ are any two neutrosophic hyperideals of $M(I)$, then*

$$(B(I) : D(I)) = \{(a_1, a_2I) \in A(I) : (a_1, a_2I)D(I) \subseteq B(I)\}$$

is a neutrosophic hyperideal of $M(I)$.

Proof. Let $(b_1, b_2I) \in (B(I) : D(I))$, $(d_1, d_2I) \in D(I), (a_1, a_2I) \in A(I)$ and $(m_1, m_2I) \in M(I)$ be arbitrary elements. Then $(b_1, b_2I)(d_1, d_2I) = (b_1, b_2I)$ and it implies that $b_1d_1 = b_1,\ b_1d_2+b_2d_1+b_2d_2 = b_2$. We want to show that $(B(I) : D(I))$ is a neutrosophic hyperideal of $M(I)$.
To see this, we only need to show that

$$(a_1, a_2I) \cdot ((m_1, m_2I) + (b_1, b_2I)) - (a_1, a_2I)(m_1, m_2I) \subseteq (B(I) : D(I)).$$

Now,

$$\begin{aligned} & [(a_1, a_2I) \cdot ((m_1, m_2I) + (b_1, b_2I)) - (a_1, a_2I)(m_1, m_2I)](d_1, d_2I) \\ = & [(a_1, a_2I)(m_1 + b_1, (m_2 + b_2)I) - (a_1, a_2I)(m_1, m_2I)](d_1, d_2I) \\ = & [(a_1m_1 + a_1b_1, (a_1m_2 + a_1b_2 + a_2m_1 + a_2b_1 + a_2m_2 + a_2b_2)I) \\ & -(a_1m_1, (a_1m_2 + a_2m_1 + a_2m_2I)](d_1, d_2I) \\ = & [a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)I](d_1, d_2I) \\ = & ((a_1b_1)d_1, ((a_1b_1)d_2 + (a_1b_2)d_1 + (a_1b_2)d_2 + (a_2b_1)d_1 + (a_2b_1)d_2 + (a_2b_2)d_1 + (a_2b_2)d_2)I) \\ = & (a_1(b_1d_1), (a_1(b_1d_2) + a_1(b_2d_1) + a_1(b_2d_2) + a_2(b_1d_1) + a_2(b_1d_2) + a_2(b_2d_1) + a_2(b_2d_2))I) \\ = & (a_1b_1, (a_1(b_1d_2 + b_2d_1 + b_2d_2) + a_2(b_1d_2 + b_2d_1 + b_2d_2) + a_2b_1)I) \\ = & (a_1b_1, (a_1b_2 + a_2b_2 + a_2b_1)I) \\ = & (a_1, a_2I)(b_1, b_2I). \end{aligned}$$

Hence, $(a_1, a_2I) \cdot ((m_1, m_2I) + (b_1, b_2I)) - (a_1, a_2I)(m_1, m_2I) \subseteq (B(I) : D(I))$. □

Definition 3.28. Let $M(I)$ be a neutrosophic hypernear-ring and let $(x, yI) \in M(I)$. The set

$$Ann((x, yI)) = \{(m, nI) \in M(I) : (x, yI)(m, nI) = (0, 0I)\}$$

is called the right annihilator of (x, yI).

Proposition 3.29. *Let $M(I)$ be a zero-symmetric neutrosophic hypernear-ring. For any $(x, yI) \in M(I)$, $Ann((x, yI))$ is a right neutrosophic $M(I)-$subhypergroup of $M(I)$.*

Proof. $Ann((x, yI)) \neq \emptyset$. Since there exists $(0, 0I) \in M(I)$ such that

$$(x, yI)(0, 0I) = (x0, (x0 + y0 + y0)I) = (0, 0I).$$

Let $(a, bI), (c, dI) \in Ann((x, yI))$, then $(x, yI)(a, bI) = (0, 0I)$ and $(x, yI)(c, dI) = (0, 0I)$ from which we have $xa = 0, xb + ya + yb = 0, xc = 0$ and $xd + yc + yd = 0$.
So,

$$\begin{aligned} (x, yI)[(a, bI) + (c, dI)] &= (x, yI)\{(p, qI) : p \in a + c, q \in b + d\} \\ &= (x, yI)(p, qI) \\ &= (xp, (xq + yp + yq)I) \\ &= (0, 0I). \end{aligned}$$

This implies that $(a, bI) + (c, dI) \subseteq Ann((x, yI))$.
Also,

$$\begin{aligned}(x, yI)(-(a, bI)) &= -((x, yI)(a, bI)) \quad \text{From Lemma 3.2}\\ &= -(0, 0I)\\ &= (0, 0I).\end{aligned}$$

Lastly, we will show that $Ann((x, yI))M(I) \subseteq Ann((x, yI))$. To this end, let $(u, vI) \in M(I)$ and $(a, bI) \in Ann((x, yI))$ so that

$$\begin{aligned}(x, yI)[(a, bI)(u, vI)] &= (x, yI)(au, (av + bu + bv)I)\\ &= (x(au), (x(av) + x(bu) + x(bv) + y(au) + y(av) + y(bu) + y(bv))I)\\ &= ((xa)u, ((xa)v + (xb)u + (xb)v + (ya)u + (ya)v + (yb)u + (yb)v)I)\\ &= ((xa)u, (((xb + ya + yb)u) + ((xa + xb + ya + yb)v))I)\\ &= (0, 0I).\end{aligned}$$

So, $(a, bI)(u, vI) \in Ann((x, yI))$ from which it follows that $Ann((x, yI))M(I) \subseteq Ann((x, yI))$.
Hence, $Ann((x, yI))$ is a right neutrosophic $M(I)-$ subhypergroup of $M(I)$. □

Definition 3.30. Let $W(I)$ be a neutrosophic hyperideal of a neutrosophic hypernear-ring $(M(I), +', \cdot)$. The quotient $M(I)/W(I)$ is defined by the set $\{[m] = m + W(I) : m \in M(I)\}$.

Proposition 3.31. *Let $M(I)/W(I) = \{[m] = m + W(I) : m \in V(I)\}$.*
For every $[m] = (m_1, m_2I) + W(I), [n] = (n_1, n_2I) + W(I) \in M(I)/W(I)$ we define:

$$[m] \oplus [n] = (m_1, m_2I) + W(I) \oplus (n_1, n_2I) + W(I) = ((m_1 +' n_1), (m_2 +' n_2)I) + W(I)$$

and

$$[m] \odot [n] = [m \cdot n] = (m_1n_1, (m_1n_2 + m_2n_1 + m_2n_2)I) + W(I).$$

$(M(I)/W(I), \oplus, \odot)$ is a neutrosophic hypernear-ring called neutrosophic quotient hypernear-ring.

Proof. The proof is similar to the proof in classical case. □

Definition 3.32. Let $M(I)$ and $N(I)$ be any two neutrosophic hypernear-rings.

$$\alpha : M(I) \longrightarrow N(I)$$

is called a neutrosophic hypernear-ring homomorphism, if the following conditions hold:

1. α is a hypernear-ring homomorphism,
2. $\alpha(I) = I$.

Note: If $M(I)$ and $N(I)$ are any two neutrosophic A-hypergroups. Then α is called a neutrosophic A-hypergroup homomorphism if α is a A-hypergroup homomorphism and $\alpha(I) = I$.

Definition 3.33. Let α be a neutrosophic homomorphism from $M(I)$ into $N(I)$ then

1. $Ker\alpha = \{(x, yI) \in M(I) : \alpha((x, yI)) = (0, 0I)\}$ and
2. $Im\alpha = \{(a, bI) \in N(I) : (a, bI) = \alpha((x, yI)), (x, yI) \in M(I)\}$.

Proposition 3.34. *Let $A(I)$ and $B(I)$ be two neutrosophic A-hypergroup over a zero-symmetric hypernear-ring A. Let $\alpha : A(I) \longrightarrow B(I)$ be a neutrosophic A-hypergroup homomorphism, then*

1. *$Ker\alpha$ is not a neutrosophic hyperideal of $A(I)$.*
2. *$Ker\alpha$ is a two-sided A-subhypergroup of A.*
3. *$Im\alpha$ is a left $A-$neutrosophic subhypegroup of $B(I)$.*

Proof. 1. Since α is a neutrosophic A-hypergroup homomorphism, we know that $\alpha(I) = I$.
Then $(a_1, a_2I) \in A(I)$ with $a_1, a_2 \in A$ will be in the $Ker\alpha$ if and only if $a_2 = 0$. This implies that

$$ker\alpha = \{(a_1, 0I) \in A(I)\}$$

which is just a subhypergroup of A. Hence, $\ker \alpha$ is not a neutrosophic hyperideal of $A(I)$.

2. From 1, we have that $ker\alpha = \{(a_1, 0I) \in A(I)\}$ is a subhypergroup of A. So, it remains to show that $A(Ker\alpha) \subseteq Ker\alpha$ and $(Ker\alpha)A \subseteq Ker\alpha$
   To see this, let $b \in A$ and $(a_1, 0I) \in Ker\alpha$ be arbitrary then
   $$b\alpha((a_1, 0I)) = \alpha((ba_1, (b0)I)) = \alpha((ba_1, 0I)) = (0, 0I) \in Ker\alpha,$$
   and
   $$\alpha((a_1, 0I))b = \alpha((a_1b, (0b)I)) = \alpha((a_1b, 0I)) = (0, 0I) \in Ker\alpha.$$
   It implies that $A(Ker\alpha) \subseteq Ker\alpha$ and $(Ker\alpha)A \subseteq Ker\alpha$. Hence $\ker\alpha$ is a two-sided A-subhypergroup of A.

3. By definition $Im\alpha = \{(a, bI) \in B(I) : (a, bI) = \alpha((x, yI)), (x, yI) \in A(I)\}$. It is clear that $Im\alpha$ is a neutrosophic subhypergroup of $B(I)$. So, it remains to show that $A(Im\alpha) \subseteq (Im\alpha)$. Now, let $(a, bI) \in Im\alpha$ and $a_1 \in A$ be arbitrary, where $(a, bI) = \alpha((x, yI))$ and $(x, yI) \in A(I)$, then
   $$a_1(a, bI) = a_1\alpha((x, yI)) = \alpha((a_1x), (a_1y)I).$$
   Since $A(I)$ is a $A-$ hypergroup, then $((a_1x), (a_1y)I) \in A(I)$. So, we can say that $\alpha(((a_1x), (a_1y)I)) \in Im\alpha$. And it implies that $A(Im\alpha) \subseteq Im\alpha$. Hence, $Im\alpha$ is a left neutrosophic $A-$subhypergroup of $B(I)$.

□

Proposition 3.35. *Let $A(I)$ and $B(I)$ be any two neutrosophic hypernear-ring and let $\alpha : A(I) \longrightarrow B(I)$ be a neutrosophic hypernear-ring homomorphism, then*

1. *$Ker\alpha$ is not a neutrosophic hyperideal of $A(I)$.*
2. *$Ker\alpha$ is a subhypernear-ring of A.*
3. *$Im\alpha$ is neutrosophic subhypernear-ring of $B(I)$.*

Proof. 1. The proof follows similar approach as proof 1 of Proposition 3.34.

2. $Ker\alpha = \{(a_1, 0) \in A(I)\}$. It is easy to show that $ker\alpha$ is a subhypernear-ring of A.
3. The proof is similar to the proof in classical case.

□

4 Conclusion

We investigated and presented some of the interesting results arising from the study of hypernear-rings in the neutrosophic environment. The concept of neutrosophic $A-$hypergroup of a hypernear-ring A, neutrosophic $A(I)-$hypergroup of a neutrosophic hypernear-ring $A(I)$ and their respective neutrosophic substructures were presented. It was shown that a constant neutrosophic hypernear-ring in general is not a constant hypernear-rings. We hope to study more advanced properties of neutrosophic hypernear-ring in our future work.

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