$N\psi_{\alpha}^{\#0}$ and $N\psi_{\alpha}^{\#1}$-spaces in Neutrosophic Topological Spaces

[1]P. Basker, [2]Broumi Said
[1]Associate Professor, Department of Mathematics, Chandigarh University, Punjab-140413, India.
[2]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, Casablanca, Morocco.
mcpdbasker@gmail.com[1], broumisaid78@gmail.com[2]

Abstract

In this paper, we have introduced the concept of $nk_{\#}^{N\alpha}(n_1)$ by using $CL_N\alpha(\{n_1\})$ where $n_1 \in N$ via $N\alpha$-open sets. Also we have introduced the spaces called $N\psi_{\alpha}^{\#0}$-space and $N\psi_{\alpha}^{\#1}$-space.
Keywords: $nk_{\#}^{N\alpha}(n_1)$, $CL_N\alpha(\{n_1\})$, $N\alpha$-open sets, $N\psi_{\alpha}^{\#0}$-space and $N\psi_{\alpha}^{\#1}$-space

1 Introduction

Fuzzy set (FS) was introduced by Zadeh [13] and after C. L. Chang [4] introduced Fuzzy Topological Spaces (FTS) and Many authors converted general topological spaces in the context of Fuzzy Topological Spaces. The concept of Intuitionistic Fuzzy set (IFS) was introduced by Atanassov [2], some more Intuitionistic Fuzzy set new results are derived by Smarandache, Florentin, Said Broumi, Mamoni Dhar and Pinaki Majumdar [10] which is the generalization of Fuzzy set. Later Coker [5] by using the notion of Intuitionistic Fuzzy Set and derived the concept of Intuitionistic Fuzzy Topological Spaces ($IFTS$), The concept of Neutroscopic Set (NS) was first given by F. Smarandache ([8], [9], [10]). By using the notion of Neutrosophic set many fields have been developed, Analysis of Neutrosophic multiple regression, Intuitionistic Neutrosophic Soft Set, Neutrosophic δ-Open and closed Maps, New Neutrosophic Sets via Neutrosophic Topological Spaces([3], [6], [11], [12]), A. A. Salama and S. A. Alblowi [7] presented the concept of Neutroscopic Topological Space (NTS). Neutrosophic α-open set in Neutroscopic Topological Space was introduced by I. Arokiarani et al.[1].

Definition 1.1 (1)**.** A neutrosophic set A in a neutrosophic topological space is called a neutrosophic α-open set (briefly $N\alpha OS$), if $A \subseteq Nint(Ncl(Nint(A)))$. The complement of neutrosophic α-open set is called neutrosophic α-closed set.

2 On $nk_{\#}^{N\alpha}(A)$ via $N\alpha$-open sets

Definition 2.1. (a) Let N be a NTS and $n \in N$. A subset N_1 of N is called as $N\alpha$-nbhd of n $\exists$ $N\alpha$-open set N_2 such that $n \in N_2 \subset N_1$.
The collection of all $N\alpha$-nbhd of $n \in N$ is called $N\alpha$-neighbourhood system at n and shall be denoted by $NBH_{N\alpha}(n)$.

(b) Let N be a NTS and N_1 be a subset of N, A subset N_2 of N is said to be $N\alpha$-nbhd of N_1 $\exists$ $N\alpha$-open set M such that $N_1 \in M \subseteq N_2$.

(c) Let N_1 be a subset of N. A point $n_1 \in N_1$ is said to be $N\alpha$-interior point of N_1, if N_1 is an $NBH_{N\alpha}(n_1)$. The set of all $N\alpha$-interior points of N_1 is called an $N\alpha$-interior of N_1 and is denoted by $NBH_{N\alpha}(n_1)$.

(d) $N\alpha$-interior of N_1 is the $\bigcup$ of all $N\alpha$-open sets $\subset N_1$ and it is denoted by $INT_N\alpha(N_1)$. $INT_N\alpha(N_1) = \bigcup\{M : M \text{ is a } N\alpha\text{-open set and } M \subseteq N_1\}$.

(e) $N\alpha$-closure of N_1 is the $\bigcap$ of all $N\alpha$-closed sets $\supset N_1$ and it is denoted by $CL_N\alpha(N_1)$. $CL_N\alpha(N_1) = \bigcap\{M : M \text{ is a } N\alpha\text{-closed set and } N_1 \subseteq M\}$.

(f) *bigcap* of all $N\alpha$-open subsets of $(N,\ \tau_N)$ containing N_1 is called the $N\alpha$-kernel of N_1 (briefly, $nk_{\#}^{N\alpha}(N_1)$). $nk_{\#}^{N\alpha}(N_1) = \cap\{M \in N\alpha(N,\ \tau_N) : N_1 \subseteq M\}$.

(g) Let $n \in N_1$. Then $N\alpha$-kernel of n is denoted by $nk_{\#}^{N\alpha}(\{n\}) = \cap\{M \in N\alpha(N,\ \tau_N) : n \in M\}$. $CL_N\alpha(N_1) = \bigcap\{M : N_1 \subset M \in N\alpha(N,\ \tau_N)\}$.

Theorem 2.2. *A subset N_1 of NTS (N, τ_N) is $N\alpha$-open if it is a $N\alpha$-neighborhood of each of its points.*

Proof. Let a subset N_1 of a NTS (X, τ) be $N\alpha$-open. Then for every $n \in N_1$, $n \in N_1 \subseteq N_1$, and therefore N_1 is a $N\alpha$-neighborhood of each of its points.

□

Theorem 2.3. *Let N be NTS (N, τ_N). If N_1 is $N\alpha$-closed subset of N and $n \in CL_N\alpha(N_1)$ iff for any $N\alpha$-neighborhood N of n in N, $N \cap N_1 \neq_N$.*

Proof. Let us assume that there is an $N\alpha$-neighborhood M of the point n in N such that $M \cap N_1 = 0_N$. There exist an $N\alpha$-open set P of N such that $n \in P \subseteq N$. Therefore we have $P \cap N_1 = 0_N$ and so $n \in N - P$. Then $CL_N\alpha(N_1) \in N - P$ and therefore $n \notin CL_N\alpha(N_1)$, which is the contradiction to the hypothesis $n \in CL_N\alpha(N_1)$. Therefore $N \cap N_1 \neq 0_N$.

Conversely, $n \notin CL_N\alpha(N_1)$, then there exists a $N\alpha$-closed set P of N such that $N_1 \subseteq P$ and $n \notin P$. Thus $n \in N - P$ and $N - P$ is $N\alpha$-open in N and hence $N - P$ is an $N\alpha$-neighborhood of n in N. But $N_1 \cap (N - P) = 0_N$ which is a contradiction. Hence $n \in CL_N\alpha(N_1)$.

□

Theorem 2.4. *Let N be NTS (N, τ_N). Then for any nonempty subset N_1 of N, $nk_{\#}^{N\alpha}(N_1) = \{n \in N : CL_N\alpha(\{n\}) \cap N_1 \neq 0_N\}$.*

Proof. Let $n \in nk_{\#}^{N\alpha}(N_1)$. Suppose that $CL_N\alpha(\{n\}) \cap N_1 = 0_N$. Then $N_1 \subseteq N - CL_N\alpha(\{n\})$ and $N - CL_N\alpha(\{n\})$ is $N\alpha$-open set containing N_1 but not n, which is a contradiction.

Conversely, let us assume that $n \notin nk_{\#}^{N\alpha}(N_1)$ and $CL_N\alpha(\{n\}) \cap N_1 \neq 0_N$. Then there exist a $N\alpha$-open set D containing N_1 but not n and $o \in CL_N\alpha(\{n\}) \cap N_1$. Hence $N\alpha$-closed set $N - D$ contains n, and $\{n\} \subset N - D$, $o \notin N - D$. This is a contradiction to $o \in CL_N\alpha(\{n\}) \cap A$. Therefore $x \in nk_{\#}^{N\alpha}(N_1)$.

□

Definition 2.5. In a NTS $(N,\ \tau_N)$, a NS N_1 is said to be weakly ultra-$N\alpha$-separated (briefly, $\|wu\|$ $N\alpha$-separated) from a set N_2 if there exists an $N\alpha$-open set M such that $N_1 \subseteq M$ and $M \cap N_2 = 0_N$ or $N_1 \cap CL_N\alpha(N_2) = 0_N$.

By the previous definition and theorem, we have the following $n,\ o \in N$ of a NTS,

$CL_N\alpha(\{n\}) = \{o : \{n\}$ is not $\|wu\|$ $N\alpha$-separated from $\{n\}\}$

$nk_{\#}^{N\alpha}(\{n\}) = \{o : \{o\}$ is not $\|wu\|$ $N\alpha$-separated from $\{o\}\}$

Definition 2.6. For any point n of a space (N, τ_N) is called

(a) $N\alpha$-derived (briefly, $D_{N\alpha}^{\#}(\{n\})$) set of n is defined to be the set.

$D_{N\alpha}^{\#}(\{n\}) = CL_N\alpha(\{n\}) - \{n\} = \{o : o \neq n$ and $\{o\}$ is not $\|wu\|$ $N\alpha$-separated from $\{n\}\}$,

(b) $N\alpha$-shell (briefly, $Sh_{*N\alpha}(\{n\})$) of a singleton set $\{n\}$ is defined to be the set.

$$Sh_{*N\alpha}(\{n\}) = nk_{\#}^{N\alpha}(\{n\}) - \{n\} = \{o : o \neq n \text{ and } \{n\} \text{ is not } \|wu\|\ N\alpha\text{-separated from } \{o\}\}.$$

Definition 2.7. Let N be NTS (N, τ_N). Then we define

(a) $D_{N\alpha}^{(N*)} = \{n : n \in N \text{ and } D_{N\alpha}^{\#}(\{n\}) = 0_N\}$,

(b) $Sh_{N\alpha}^{(N*)} = \{n : n \in N \text{ and } Sh_{*N\alpha}(\{n\}) = 0_N\}$,

(c) $N\alpha < n >= CL_N\alpha(\{n\}) \cap nk_{\#}^{N\alpha}(\{n\})$.

Remark 2.8. Let $n_1,\ n_2 \in N$. Then

(a) $n_2 \in nk_{\#}^{N\alpha}(\{n_1\}) \Longleftrightarrow n_1 \in CL_N\alpha(\{n_2\})$.

(b) $n_2 \in Sh_{*N\alpha}(\{n_1\}) \Longleftrightarrow n_1 \in D_{N\alpha}^{\#}(\{n_1\})$.

Theorem 2.9. *Let $n_1,\ n_2 \in N$. Then the following conditions hold.*

(a) $n_2 \in CL_N\alpha(\{n_1\}) \Longrightarrow CL_N\alpha(\{n_2\}) \subseteq CL_N\alpha(\{n_1\})$.

(b) $n_2 \in nk_{\#}^{N\alpha}(\{n_1\}) \Longrightarrow nk_{\#}^{N\alpha}(\{n_2\}) \subseteq nk_{\#}^{N\alpha}(\{n_1\})$.

Proof. (a) Let $n_3 \in CL_N\alpha(\{n_2\})$. Then $\{n_3\}$ is not $\|wu\|\ N\alpha$-separated from $\{n_2\}$. So there exists an $N\alpha$-open set M containing n_3 such that $M \cap \{n_2\} \neq \phi$. Hence $n_2 \in M$ and by assumption $M \cap \{n_1\} \neq 0_N$. Hence $\{n_3\}$ is not $\|wu\|\ N\alpha$-separated from $\{n_1\}$. So $n_3 \in CL_N\alpha(\{n_1\})$. Therefore $CL_N\alpha(\{n_2\}) \subseteq CL_N\alpha(\{n_1\})$.

(b) Let $n_3 \in nk_{\#}^{N\alpha}(\{n_2\})$. Then $\{n_2\}$ is not $\|wu\|\ N\alpha$-separated from $\{n_3\}$. So $n_2 \in CL_N\alpha(\{n_3\})$. Hence $CL_N\alpha(\{n_2\}) \subseteq CL_N\alpha(\{n_3\})$. By assumption $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$ and then $n_1 \in CL_N\alpha(\{n_2\})$. So $CL_N\alpha(\{n_1\}) \subseteq CL_N\alpha(\{n_2\})$. Ultimately $CL_N\alpha(\{n_1\}) \subseteq CL_N\alpha(\{n_3\})$. Hence $n_1 \in CL_N\alpha(\{n_3\})$, that is $n_3 \in nk_{\#}^{N\alpha}(\{n_1\})$. Therefore $nk_{\#}^{N\alpha}(\{n_2\}) \subseteq nk_{\#}^{N\alpha}(\{n_1\})$.

□

Recall that a subset N of (N, τ_N) is called a degenerate set if N is either a null set or a singleton set.

Theorem 2.10. *Let $n_1,\ n_2 \in N$. Then,*

(a) *$\forall n_1 \in (N, \tau_N)$, $Sh_{*N\alpha}(\{x\})$ is degenerate $\Longleftrightarrow \forall n_1,\ n_2 \in (N, \tau_N)$, $n_1 \neq n_2$, $D_{N\alpha}^{\#}(\{n_1\}) \cap D_{N\alpha}^{\#}(\{n_2\}) = 0_N$,*

(b) *$\forall n_1 \in (N, \tau_N)$, $D_{N\alpha}^{\#}(\{n_1\})$ is degenerate $\Longleftrightarrow \forall n_1,\ n_2 \in (N, \tau_N)$, $n_1 \neq n_2$, $Sh_{*N\alpha}(\{n_1\}) \cap Sh_{*N\alpha}(\{n_1\}) = 0_N$.*

Proof. (a) Let $D_{N\alpha}^{\#}(\{n_1\}) \cap D_{N\alpha}^{\#}(\{n_2\}) \neq 0_N$. Then there exists a $n_3 \in (N, \tau_N)$ such that $n_3 \in D_{N\alpha}^{\#}(\{n_1\})$ and $n_3 \in D_{N\alpha}^{\#}(\{n_2\})$. Then $n_3 \neq n_2 \neq n_1$ and $n_3 \in CL_N\alpha(\{n_1\})$ and $n_3 \in CL_N\alpha(\{n_2\})$, that is $n_1,\ n_2 \in nk_{\#}^{N\alpha}(\{n_3\})$. Hence $nk_{\#}^{N\alpha}(\{n_3\})$ and so $Sh_{*N\alpha}(\{n_3\})$ is not a degenerate set.

Conversely, let $n_1,\ n_2 \in Sh_{*N\alpha}(\{n_3\})$. Then we get $n_1 \neq n_3$, $n_1 \in nk_{\#}^{N\alpha}(\{n_3\})$ and $n_2 \neq n_3$, $n_2 \in nk_{\#}^{N\alpha}(\{n_3\})$ and hence n_3 is an element of both $CL_N\alpha(\{n_1\})$ and $CL_N\alpha(\{n_2\})$, which is a contradiction.

(b) Obvious.

□

Theorem 2.11. *If $n_2 \in N\alpha < n_1 >$, then $N\alpha < n_1 >= N\alpha < n_2 >$.*

Proof. If $n_2 \in N\alpha < n_1 >$, then $n_2 \in CL_N\alpha(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_1\})$. Hence $n_2 \in CL_N\alpha(\{n_1\})$ and $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$ and so we have $CL_N\alpha(\{n_2\}) \subseteq CL_N\alpha(\{n_1\})$ and $nk_{\#}^{N\alpha}(\{n_2\}) \subseteq nk_{\#}^{N\alpha}(\{n_1\})$. Then $CL_N\alpha(\{n_2\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) \subseteq CL_N\alpha(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_1\})$. Hence $N\alpha < n_2 >\subseteq N\alpha < n_1 >$. The fact that $n_2 \in CL_N\alpha(\{n_1\})$ implies $n_1 \in nk_{\#}^{N\alpha}(\{n_2\})$ and $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$ implies $n_1 \in CL_N\alpha(\{n_2\})$. Then we have that $N\alpha < n_1 >\subseteq N\alpha < n_2 >$. So $N\alpha < n_1 >= N\alpha < n_2 >$.

□

Theorem 2.12. *For all $n_1,\ n_2 \in (N, \tau_N)$, either $N\alpha < n_1 > \cap N\alpha < n_2 >= 0_N$ or $N\alpha < n_1 >= N\alpha < n_2 >$.*

Proof. $N\alpha < n_1 > \cap N\alpha < n_2 >\neq 0_N$, $\exists n_3 \in (N, \tau_N)$ such that $n_3 \in N\alpha < n_1 >$ and $n_3 \in N\alpha < n_2 >$. So by the previous theorem, $N\alpha < n_3 >= N\alpha < n_1 >= N\alpha < n_2 >$. Hence the result.

□

Theorem 2.13. *For any two points $n_1,\ n_{\in}(N, \tau_N)$, the following are equivalent.*

(a) $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$.

(b) $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$.

Proof. $(a) \Longrightarrow (b)$ Let us assume $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$, $\exists n_3 \in nk_{\#}^{N\alpha}(\{n_1\})$ but $n_3 \notin nk_{\#}^{N\alpha}(\{n_2\})$. As $n_3 \in nk_{\#}^{N\alpha}(\{n_1\})$, $n_1 \in CL_N\alpha(\{n_3\})$ and $CL_N\alpha(\{n_1\}) \subseteq CL_N\alpha(\{n_3\})$. Also we have taken $n_3 \notin nk_{\#}^{N\alpha}(\{n_2\})$, $CL_N\alpha(\{n_3\}) \cap \{n_2\} = 0_N$, so $CL_N\alpha(\{n_1\}) \cap \{n_2\} = 0_N$ and so $\{n_2\}$ is $\|wu\|$ $N\alpha$-separated from $\{n_1\}$ and hence we get that $n_2 \notin CL_N\alpha(\{n_1\})$. Hence $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$.

$(b) \Longrightarrow (a)$ Suppose $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$, $\exists n_3 \in CL_N\alpha(\{n_1\})$ but $n_3 \notin CL_N\alpha(\{n_2\})$. So, we get an $N\alpha$-open set containing n_3 and n_1 but not n_2. That is $n_2 \notin nk_{\#}^{N\alpha}(\{n_1\})$. Hence $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$.

□

Definition 2.14. A NTS $(N,\ \tau_N)$ is said to be a $N\psi_{\alpha}^{\#0}$-space if every $N\alpha$-open set contains the $N\alpha$-closure of each of its singletons.

Definition 2.15. An NTS $(N,\ \tau_N)$ is said to be $N\psi_{\alpha}^{\#1}$-space if for $n_1,\ n_2$ in N with $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$, $\exists$ disjoint $N\alpha$-open sets N_1 and N_2 such that $CL_N\alpha(\{n_1\}$ is a subset of N_1 and $CL_N\alpha(\{n_2\}$ is a subset of N_2.

Lemma 2.16. *Let $(N,\ \tau_N)$ be a NTS and $n_1 \in N$. Then $n_2 \in nk_{\#}^{N\alpha}(\{n_1\}) \Longleftrightarrow n_1 \in CL_N\alpha(\{n_2\})$.*

Proof. Suppose that $n_2 \notin nk_{\#}^{N\alpha}(\{n_1\})$. Then $\exists$ $N\alpha$-open set N_2 containing n_1 such that $n_1 \notin N_2$. Therefore we've $n_1 \notin CL_N\alpha(\{n_2\})$. □

Theorem 2.17. *If a $NTS(N,\ \tau_N)$ is $N\psi_{\alpha}^{\#1}$-space, then the $NTS(N,\ \tau_N)$ is $N\psi_{\alpha}^{\#0}$-space.*

Proof. Let N_1 be an $N\alpha$-open and $n_1 \in N_1$. If $n_2 \notin N_1$, then since $n_1 \notin CL_N\alpha(\{n_2\})$, $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$. Hence, $\exists$ $N\alpha$-open set N_{n_2} such that $CL_N\alpha(\{n_2\}) \subset N_{n_2}$ and $N_{n_2}, \Longrightarrow n_2 \notin CL_N\alpha(\{n_1\})$. Thus $CL_N\alpha(\{n_1\}) \subset N_1$. Therefore $(N,\ \tau_N)$ is $N\psi_{\alpha}^{\#0}$-space. □

An $NTS(N,\ \tau_N)$ is $N\psi_{\alpha}^{\#1}$-space $\Longleftrightarrow$ for $n_1,\ n_2 \in N$; $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$, $\exists$ disjoint $N\alpha$-open sets N_1 and N_2 such that $CL_N\alpha(\{n_1\}) \subset N_1$ and $CL_N\alpha(\{n_2\}) \subset N_2$.

Theorem 2.18. *An $NTS(N, \tau_N)$ is a $N\psi_{\alpha}^{\#0}$-space $\Longleftrightarrow$ for any n_1 and n_2 in N, $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$ implies $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) = 0_N$.*

Proof. $Necessity$: Assume $(N,\ \tau_N)$ $N\psi_{\alpha}^{\#0}$-space and $n_1,\ n_2 \in N$ such that $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$. Then, $\exists$ $n_3 \in CL_N\alpha(\{n_1\})$ such that $n_3 \notin CL_N\alpha(\{n_2\})$ ($n_3 \in CL_N\alpha(\{n_2\})$) such that $n_3 \notin CL_N\alpha(\{n_1\})$. $\exists$ $N_2 \in N\alpha O(N,\ \tau_N)$ such that $n_2 \notin N_2$ and $n_1 \in N_2$; hence $n_1 \in N_2$. Therefore, we've $n_1 \notin CL_N\alpha(\{n_2\})$. Thus $n_1 \in N_1/CL_N\alpha(\{n_2\}) \in N\alpha O(N,\ \tau_N)$, $\Longrightarrow CL_N\alpha(\{n_1\}) \subset N_1/CL_N\alpha(\{n_2\})$

and $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) = 0_N$.
$Sufficiency$: Let $N_2 \in N\alpha O(N, \tau_N)$ and let $n_1 \in N_2$. We'll prove that $CL_N\alpha(\{n_1\}) \subset N_2$. Really, let $n_2 \notin N_2$, i.e.,$n_2 \in N/N_2$. Then $n_1 \neq n_2$ and $n_1 \notin CL_N\alpha(\{n_2\})$. This shows that $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$. By assumption, $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) = 0_N$. Hence $n_2 \notin CL_N\alpha(\{n_1\})$. Therefore $CL_N\alpha(\{n_1\}) \subset N_2$. □

Theorem 2.19. *A $NTS(N, \tau_N)$ is a $N\psi_\alpha^{\#0}$-space $\Longleftrightarrow$ for any points n_1 and n_2 in N_1, $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$ implies $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) = 0_N$.*

Proof. Suppose that (N, τ_N) is a $N\psi_\alpha^{\#0}$-space. For any points n_1 and n_2 in N if $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$ then $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$. Now we prove that $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) = 0_N$. Assume that $n_3 \in nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\})$. By $n_3 \in nk_{\#}^{N\alpha}(\{n_1\})$, it follows that $n_1 \in CL_N\alpha(\{n_3\})$. Since $n_1 \in CL_N\alpha(\{n_1\})$, by the above Theorem $CL_N\alpha(\{n_1\}) = CL_N\alpha(\{n_3\})$. Similarly, we have $CL_N\alpha(\{n_2\}) = CL_N\alpha(\{n_3\}) = CL_N\alpha(\{n_1\})$. This is a contradiction. Therefore, we have $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) = 0_N$.

Conversely, let (N, τ_N) be a NTS such that for any points n_1 and n_2 in N, $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$ implies $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) = 0_N$. If $CL_N\alpha(\{n_1\}) \neq CL_N\alpha(\{n_2\})$, $nk_{\#}^{N\alpha}(\{n_1\}) \neq nk_{\#}^{N\alpha}(\{n_2\})$. Therefore $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_2\}) = 0_N$ which implies $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) = 0_N$. Because $n_3 \in CL_N\alpha(\{n_1\})$ implies that $n_1 \in nk_{\#}^{N\alpha}(\{n_3\})$ and therefore $nk_{\#}^{N\alpha}(\{n_1\}) \cap nk_{\#}^{N\alpha}(\{n_3\}) \neq 0_N$. By hypothesis, therefore we have $nk_{\#}^{N\alpha}(\{n_1\}) = nk_{\#}^{N\alpha}(\{n_3\})$. Then $n_3 \in CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\})$ implies that $nk_{\#}^{N\alpha}(\{n_1\}) = nk_{\#}^{N\alpha}(\{n_3\}) = nk_{\#}^{N\alpha}(\{n_2\})$. This is a contradiction. Therefore, $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) = 0_N$ is a $N\psi_\alpha^{\#0}$-space. □

Theorem 2.20. *For a $NTS(N, \tau_N)$, the following properties are equivalent:*

(a) *(N, τ_N) is a $N\psi_\alpha^{\#0}$-space;*

(b) *For any nonempty NS N_1 and $H \in N\alpha O(N, \tau_N)$ such that $N_1 \cap H \neq 0_N$, $\exists\, F \in N\alpha C(N, \tau_N)$ such that $N_1 \cap F \neq 0_N$; and $F \subset H$;*

(c) *Any $H \in N\alpha O(N, \tau_N)$, $H = \bigcup\{F \in N\alpha C(N, \tau_N)/F \subset H\}$;*

(d) *Any $F \in N\alpha C(N, \tau_N)$, $F = \bigcap\{H \in N\alpha O(N, \tau_N)/F \subset H\}$;*

(e) *For any $n_1 \in N$; $CL_N\alpha(\{n_1\}) \subset nk_{\#}^{N\alpha}(\{n_1\})$.*

Proof. $(a) \Longrightarrow (b)$ Let N_1 be a nonempty NS of N and $H \in N\alpha O(N, \tau_N)$ such that $N_1 \cap H \neq 0_N$. $\exists\, n_1 \in N_1 \cap H$. Since $n_1 \in H \in N\alpha O(N, \tau_N)$, $CL_N\alpha(\{n_1\}) \subset H$. Set $F = CL_N\alpha(\{n_1\})$ then $F \in N\alpha C(N, \tau_N)$, $F \subset H$ and $N_1 \cap H \neq 0_N$.

$(b) \Longrightarrow (c)$ Let $H \in N\alpha O(N, \tau_N)$, then $H \supset \bigcup\{F \in N\alpha C(N, \tau_N)/F \subset H\}$. Let n_1 be any point of H. $\exists\, F \in N\alpha C(N, \tau_N)$ such that $n_1 \in F$ and $F \subset H$. Therefore, we have $n_1 \in F \subset \bigcup\{F \in N\alpha C(N, \tau_N)/F \subset H\}$ and hence $H = \bigcup\{F \in N\alpha C(N, \tau_N)/F \subset H\}$.

$(c) \Longrightarrow (d)$ This is obvious.

$(d) \Longrightarrow (e)$ Let n_1 be any point of N and $n_2 \notin nk_{\#}^{N\alpha}(\{n_1\})$. $\exists\, Q \in N\alpha O(N, \tau_N)$ such that $n_1 \in Q$ and $n_2 \notin Q$; hence $CL_N\alpha(\{n_1\}) \cap Q = 0_N$. By $(d)(\bigcap\{H \in N\alpha O(N, \tau_N)/CL_N\alpha(\{n_2\}) \subset H\} \cap Q = 0_N$ and $\exists\, H \in N\alpha O(N, \tau_N)$ such that $n_1 \notin H$ and $CL_N\alpha(\{n_2\}) \subset H$. Therefore, $CL_N\alpha(\{n_1\}) \cap G = 0_N$ and $n_2 \notin CL_N\alpha(\{n_1\})$. Consequently, we obtain $CL_N\alpha(\{n_1\}) \subset nk_{\#}^{N\alpha}(\{n_1\})$.

$(e) \Longrightarrow (a)$ Let $H \in N\alpha O(N, \tau_N)$ and $n_1 \in H$. Let $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$ then $n_1 \in CL_N\alpha(\{n_2\})$ and $n_2 \in H$. This implies that $nk_{\#}^{N\alpha}(\{n_1\}) \subset H$. Therefore, we obtain $n_1 \in CL_N\alpha(\{n_1\}) \subset nk_{\#}^{N\alpha}(\{n_1\}) \subset H$. This shows that (N, τ_N) is a $N\psi_\alpha^{\#0}$-space. □

Corollary 2.21. *For a $NTS(N, \tau_N)$, the following properties are equivalent:*
(a) *(N, τ_N) is a $N\psi_\alpha^{\#0}$-space.*
(b) *$CL_N\alpha(\{n_1\}) = nk_{\#}^{N\alpha}(\{n_1\})$ for all $n_1 \in N$.*

Proof. $(a) \implies (b)$ Suppose that $(N,\ \tau_N)$ is a $N\psi_\alpha^{\#0}$-space. By the above Theorem, $CL_N\alpha(\{n_1\}) \subset nk_{\#}^{N\alpha}(\{n_1\})$ for each $n_1 \in N$. Let $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$ then $n_1 \in CL_N\alpha(\{n_2\})$ and by Theorem $CL_N\alpha(\{n_1\}) = CL_N\alpha(\{n_2\})$. Therefore, $n_2 \in CL_N\alpha(\{n_1\})$ and hence $nk_{\#}^{N\alpha}(\{n_1\}) \subset CL_N\alpha(\{n_1\})$. This shows that $CL_N\alpha(\{n_1\}) = nk_{\#}^{N\alpha}(\{n_1\})$.

$(b) \implies (a)$ This is obvious by the above Theorem. □

Theorem 2.22. *For a $NTS(N,\ \tau_N)$, the following properties are equivalent:*
(a) $(N,\ \tau_N)$ *is a $N\psi_\alpha^{\#0}$-space.*
(b) $n_1 \in CL_N\alpha(\{n_2\}) \iff n_2 \in CL_N\alpha(\{n_1\})$.

Proof. $(a) \implies (b)$ Assume N is a $N\psi_\alpha^{\#0}$-space. Let $n_1 \in CL_N\alpha(\{n_2\})$ and N_1 be any $N\alpha$-open set such that $n_2 \in N_1$. Now by hypothesis, $n_1 \in N_1$. Therefore, every $N\alpha$-open set which contains n_2 contains n_1. Hence $n_2 \in CL_N\alpha(\{n_1\})$.

$(b) \implies (a)$ Let P be an $N\alpha$-open set and $n_1 \in P$. If $n_2 \notin P$, then $n_1 \notin CL_N\alpha(\{n_2\})$ and hence $n_2 \notin CL_N\alpha(\{n_1\})$. This implies that $CL_N\alpha(\{n_1\}) \subset P$. Hence $(N,\ \tau_N)$ is $N\psi_\alpha^{\#0}$-space. □

Remark 2.23. For a $NTS(N,\ \tau_N)$, $N\psi_\alpha^{\#0}$-space $\iff$ $N\alpha$-symmetric.

Theorem 2.24. *For a $NTS(N,\ \tau_N)$, the following properties are equivalent:*
(a) $(N,\ \tau_N)$ *is a $N\psi_\alpha^{\#0}$-space.*
(b) *If J is an $N\alpha$-closed, then $J = nk_{\#}^{N\alpha}(J)$.*
(c) *If J is an $N\alpha$-closed and $n_1 \in J$, then $nk_{\#}^{N\alpha}(\{n_1\}) \subset J$.*
(d) *If $n_1 \in N$, then $nk_{\#}^{N\alpha}(\{n_1\}) \subset CL_N\alpha(\{n_1\})$.*

Proof. $(a) \implies (b)$ Obvious.
$(b) \implies (c)$ In general, $N_1 \subset N_2$ implies $nk_{\#}^{N\alpha}(\{N_1\}) \subset nk_{\#}^{N\alpha}(\{N_2\})$. Therefore, it follows from (b) that $nk_{\#}^{N\alpha}(\{n_1\}) \subset nk_{\#}^{N\alpha}(J) = J$.

$(c) \implies (d)$ Since $n_1 \in CL_N\alpha(\{n_1\})$ and $CL_N\alpha(\{n_1\})$ is $N\alpha$-closed, by (c) $nk_{\#}^{N\alpha}(\{n_1\}) \subset CL_N\alpha(\{n_1\})$.

$(d) \implies (a)$ Let $n_1 \in CL_N\alpha(\{n_2\})$, $n_2 \in nk_{\#}^{N\alpha}(\{n_1\})$. Since $n_1 \in CL_N\alpha(\{n_1\})$ and $CL_N\alpha(\{n_1\})$ is an $N\alpha$-closed, by (d) we obtain $n_2 \in nk_{\#}^{N\alpha}(\{n_1\}) \subset CL_N\alpha(\{n_1\})$. Therefore $n_1 \in CL_N\alpha(\{n_2\})$ implies $n_2 \in CL_N\alpha(\{n_1\})$. The converse is obvious and $(N,\ \tau_N)$ is $N\psi_\alpha^{\#0}$-space. □

Lemma 2.25. *Let $(N,\ \tau_N)$ be a NTS and let n_1 and n_2 be any two points in N such that every net in N an $N\alpha$-converging to n_2 an $N\alpha$-converges to n_1. Then $n_1 \in CL_N\alpha(\{n_2\})$.*

Proof. Suppose that $n_{1n} = y$ for each $n \in N$. Then $\{n_{1n}\}_{n\in N}$ is a net in $CL_N\alpha(\{n_2\})$. By the fact that $\{n_{1n}\}_{n\in N}$ an $N\alpha$-converges to y, then $\{n_{1n}\}_{n\in N}$ an $N\alpha$-converges to n_1 and this means that $n_1 \in CL_N\alpha(\{n_2\})$. □

Theorem 2.26. *For a $NTS(N,\ \tau_N)$, the following properties are equivalent:*
(a) $(N,\ \tau_N)$ *is a $N\psi_\alpha^{\#0}$-space.*
(b) *If $n_1,\ n_2 \in N$ then $n_2 \in CL_N\alpha(\{n_1\}) \iff$ every net in N, $N\alpha$-converging to n_2 $N\alpha$-converging to n_1.*

Proof. $(a) \implies (b)$ Let $n_1,\ n_2 \in N$ such that $n_2 \in CL_N\alpha(\{n_1\})$. Let $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ be a net in N such that $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ an $N\alpha$-converges to n_2. Since $n_2 \in CL_N\alpha(\{n_1\})$, we have $CL_N\alpha(\{n_1\}) = CL_N\alpha(\{n_2\})$. Therefore $n_1 \in CL_N\alpha(\{n_2\})$. This means that $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ an $N\alpha$-converges to n_1. Conversely, let $n_1,\ n_2 \in N$ such that every net in N an $N\alpha$-converging to n_2 an $N\alpha$-converges to n_1. Then $n_1 \in CL_N\alpha(\{n_2\})$, we have $CL_N\alpha(\{n_1\}) = CL_N\alpha(\{n_2\})$. Therefore $n_2 \in CL_N\alpha(\{n_1\})$.

$(b) \implies (a)$ Assume that n_1 and n_2 are any two points of N such that $CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\}) \neq 0_N$. Let $n_3 \in CL_N\alpha(\{n_1\}) \cap CL_N\alpha(\{n_2\})$. So $\exists$ a net $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ in $CL_N\alpha(\{n_1\})$ such that $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ an $N\alpha$-converges to n_3. Since $n_3 \in CL_N\alpha(\{n_1\})$ then $\{n_{1_\alpha}\}_{\alpha\in\Lambda}$ an $N\alpha$-converges to n_2. It follows that $n_2 \in CL_N\alpha(\{n_1\})$. By the same token we obtain $n_1 \in CL_N\alpha(\{n_2\})$. Therefore $CL_N\alpha(\{n_1\}) = CL_N\alpha(\{n_2\})$ and $(N,\ \tau_N)$ is a $N\psi_\alpha^{\#0}$-space. □

References

[1] I.Arokiarani.I, R. Dhavaseelan, S. Jafari and M. Parimala, On Some New Notions and Functions in Neutrosophic Topological Spaces, Neutrosophic Sets and Systems 16, pp. 16-19, 2017

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20, pp. 87-96, 1986

[3] Broumi Said and Florentin Smarandache, Intuitionistic Neutrosophic Soft Set, Journal of Information and Computing Science 8(2), pp.130-140, 2013

[4] C. L. Chang, Fuzzy topological spaces, Journal of Mathematical Analysis and Applications 24, pp. 182-190, 1968

[5] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88(1), pp. 81-89, 1997

[6] D. Nagarajan, S. Broumi, F. Smarandache, and J. Kavikumar, Analysis of Neutrosophic Multiple Regression. Neutrosophic Sets and Systems 43, pp. 44-53, 2021

[7] A. A. Salama and S. A. Alblowi, Neutrosophic set and Neutrosophic topological spaces, IOSR Journal of Mathematics 3(4), pp. 31-35, 2012

[8] F. Smarandache, A unifying field in logics: Neutrosophic Logic. Neutrosophy, Neutrosophic set, Neutrosophic probability, American Research Press, Rehoboth, NM, 1999

[9] F. Smarandache, Neutrosophy and Neutrosophic logic. In:F. Smarandache(Ed.), Neutrosophy, neutrosophic logic, set, probability and statistics Proceeding of the International Conference, University of New Mexico, Gallup, NM 87301, USA, 2002

[10] Smarandache, Florentin, Said Broumi, Mamoni Dhar and Pinaki Majumdar, New Results of Intuitionistic Fuzzy Soft Set, International Journal of Information Engineering and Electronic Business, pp. 1-6, 2014

[11] A. Vadivel1, C. John Sundar, Neutrosophic δ-Open Maps and Neutrosophic δ-Closed Maps, International Journal of Neutrosophic Science 13(2), pp. 66-74, 2021

[12] Wadel Faris Al-omeri and Florentin Smarandache, New Neutrosophic Sets via Neutrosophic Topological Spaces, New Trends in Neutrosophic Theory and Applications 2, pp. 1-7, 2016

[13] L. A. Zadeh, Fuzzy sets, Information and Control 8, pp. 338-353, 1965